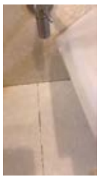
Chapter 2 Maxwell's Equations and Plane EM Waves

2-1 Dielectric and Conductor

Displacement vector: $\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon\vec{E}=\varepsilon_0(1+\chi_e)\vec{E}=\varepsilon_0\varepsilon_r\vec{E}$





Polarization vector: $\vec{P}=\lim\limits_{\Delta v\to 0}\dfrac{\sum_{k=1}^{n\Delta v}P_k}{\Delta v}$

$$V=\frac{1}{4\pi\varepsilon_0}\iiint_{v'}\frac{\vec{P}\cdot\hat{a}_R}{R^2}dv',$$

$$R^2=(x-x')^2+(y-y')^2+(z-z')^2,\quad \nabla'=\hat{x}\frac{\partial}{\partial x'}+\hat{y}\frac{\partial}{\partial y'}+\hat{z}\frac{\partial}{\partial z'}\Rightarrow\nabla'\left(\frac{1}{R}\right)=\frac{\hat{a}_R}{R^2}$$

$$\Rightarrow V=\frac{1}{4\pi\varepsilon_0}\iiint_{v'}\vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv'=\frac{1}{4\pi\varepsilon_0}\left[\iiint_{v'}\nabla'\cdot\left(\frac{\vec{P}}{R}\right)dv'-\iiint_{v'}\frac{\nabla'\cdot\vec{P}}{R}dv'\right]$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\oint\!\!\oint_{s'}\frac{\vec{P}\cdot\hat{a}_n'}{R}d\vec{S}'+\iiint_{v'}\frac{-\nabla'\cdot\vec{P}}{R}dv'\right]$$

Surface charge density: $\rho_{\mathrm{ps}}=\vec{P}\cdot\hat{a}_n$.

Volume charge density: $\rho_{\mathrm{p}}=-\nabla\cdot\vec{P}$

Total charge: $Q=\oint\!\!\oint_{s'}\vec{P}\cdot\hat{a}_n\,d\vec{S}'+\iiint_{v'}\nabla\cdot\vec{P}\,dv'=0$

$$\nabla\cdot\vec{E}=\frac{1}{\varepsilon_0}(\rho+\rho_p)\quad\Rightarrow\nabla\cdot\left(\varepsilon_0\vec{E}+\vec{P}\right)=\rho$$

Define $\vec{D}=\varepsilon_0\vec{E}+\vec{P}\quad\Rightarrow\nabla\cdot\vec{D}=\rho\quad\Leftrightarrow\quad\oint\!\!\oint_s\vec{D}\cdot d\vec{S}=Q$

Note: Generally, $\vec{D}=\overleftrightarrow{\varepsilon}\cdot\vec{E}$ or $\begin{bmatrix}D_x\\D_y\\D_z\end{bmatrix}=\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&\varepsilon_{13}\\\varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\\varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}\cdot\begin{bmatrix}E_x\\E_y\\E_z\end{bmatrix}$.

For biaxial dielectric, $\overleftrightarrow{\varepsilon}=\begin{bmatrix}\varepsilon_{11}&0&0\\0&\varepsilon_{22}&0\\0&0&\varepsilon_{33}\end{bmatrix}$

Eg. For an anisotropic medium characterized by $\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$**,** **find the value of the effective relative permittivity for (a)** $\vec{E} = \hat{z}E_0$ **, (b)** $\vec{E} = E_0(\hat{x} - 2\hat{y})$**, (c)** $\vec{E} = E_0(2\hat{x} + \hat{y})$**.**

(Sol.) (a) $\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} E_0 = \varepsilon_0 \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} E_0 = 9\varepsilon_0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} E_0$, ε_r =9

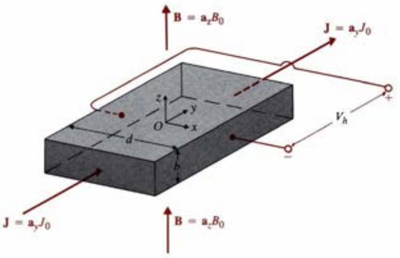
(b) $\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} E_0 = \varepsilon_0 \begin{bmatrix} 4 \\ -8 \\ 0 \end{bmatrix} E_0 = 4\varepsilon_0 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} E_0$, ε_r =4

(c) $\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} E_0 = \varepsilon_0 \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix} E_0 = 9\varepsilon_0 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} E_0$, ε_r =9

Hall Effect:

Current density: $\vec{J} = \hat{y}J_0 = Nq\vec{v}$

If the material is a conductor or an *n*-type semiconductor the charge carrier are electrons: $q < 0$



Hall field: $\vec{E}_h = -\vec{v} \times \vec{B} = -(\hat{y}v_0) \times (\hat{z}B_0) = -\hat{x}v_0 B_0$

Hall voltage: $V_h = -\int_0^d E_h dx = v_0 B_0 d$

Hall coefficient: $C_h = \dfrac{E_x}{J_y B_z} = \dfrac{1}{Nq} < 0$

If the material is a *p*-type semiconductor, the charge carries are holes: $q > 0$

Hall field: $\vec{E}_h = \hat{x}v_0 B_0$

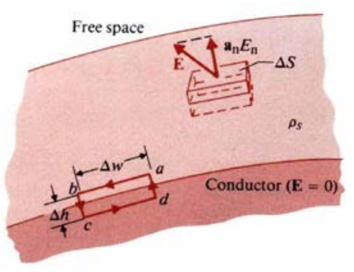
Hall voltage: $V_h = -v_0 B_0 d$

Hall coefficient: $C_h > 0$

2-2 Boundary Conditions of Electromagnetic Fields

Boundary conditions for electric fields:

Eg. Show that E_t=0 on the conductor plane.

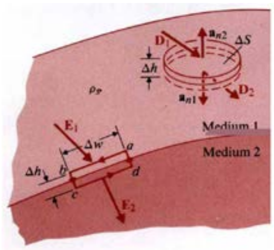


(Proof) ∵ The E-field inside a conductor is zero,

$$\therefore \oint \vec{E}\cdot d\vec{l} = E_t \Delta W = 0 \Rightarrow E_t = 0$$

$$\oiint \vec{E}\cdot d\vec{S} = E_n \Delta S = \frac{\rho_s \Delta S}{\varepsilon_0} \quad \Rightarrow E_n = \frac{\rho_s}{\varepsilon_0}$$

Eg. Show that E_{1t}= E_{2t} and $\hat{a}_{n2}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\rho_s$ on the interface between two dielectric.



(Proof) $\oint_{abcda} \vec{E}\cdot d\vec{l} = E_{1t}\Delta W - E_{2t}\Delta W = 0$, E_{1t}=E_{2t}

$$\oiint_s \vec{D}\cdot d\vec{S} = \left(\vec{D}_1\cdot\hat{a}_{n2}+\vec{D}_2\cdot\hat{a}_{n1}\right)\Delta S = \hat{a}_{n2}\cdot\left(D_1-D_2\right)\Delta S = \rho_s \Delta S$$

$\hat{a}_{n2}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\rho_s$ or D_{1n}-D_{2n}=ρ_s

If ρ_s=0, then D_{1n}=D_{2n} or $\varepsilon_1 E_{1n}$=$\varepsilon_2 E_{2n}$

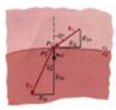
Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t}=0$	$E_{2t}=0$
$\mathbf{a}_{n2}\times\mathbf{H}_1=\mathbf{J}_s$	$H_{2t}=0$
$\mathbf{a}_{n2}\cdot\mathbf{D}_1=\rho_s$	$D_{2n}=0$
$B_{1n}=0$	$B_{2n}=0$

Boundary Conditions between Two Lossless Media

$$E_{1t}=E_{2t}\rightarrow\frac{D_{1t}}{D_{2t}}=\frac{\epsilon_1}{\epsilon_2}$$
$$H_{1t}=H_{2t}\rightarrow\frac{B_{1t}}{B_{2t}}=\frac{\mu_1}{\mu_2}$$
$$D_{1n}=D_{2n}\rightarrow\epsilon_1 E_{1n}=\epsilon_2 E_{2n}$$
$$B_{1n}=B_{2n}\rightarrow\mu_1 H_{1n}=\mu_2 H_{2n}$$

Eg. Two dielectric media are separated by a charge free boundary. The electric field intensity in media 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the electric field intensity at point P_2 in medium 2. [交大電子所]



(Sol.) $E_2\sin\alpha_2 = E_1\sin\alpha_1$, $\varepsilon_2 E_2\cos\alpha_2 = \varepsilon_1 E_1\cos\alpha_1 \Rightarrow \dfrac{\tan\alpha_2}{\tan\alpha_1}=\dfrac{\varepsilon_2}{\varepsilon_1}$

$$E_2=\sqrt{E_{2t}^2+E_{2n}^2}=\sqrt{(E_2\sin\alpha_2)^2+(E_2\cos\alpha_2)^2}$$

$$=\left[(E_1\sin\alpha_1)^2+\left(\frac{\varepsilon_1}{\varepsilon_2}E_1\cos\alpha_1\right)^2\right]^{1/2}=E_1\left[\sin^2\alpha_1+\left(\frac{\varepsilon_1}{\varepsilon_2}\cos\alpha_1\right)^2\right]^{1/2}$$

Eg. Assume that z=0 plane separates two lossless dielectric regions with ε_{r1}=2 and ε_{r2}=3. If $\vec{E}_1$ in region 1 is $\hat{x}2y-\hat{y}3x+\hat{z}(5+z)$, find $\vec{E}_2$ and $\vec{D}_2$ at z=0 in region 2.

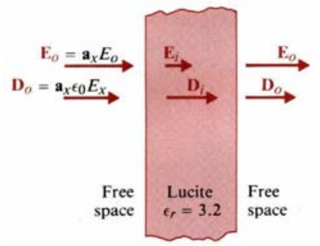
(Sol.) $\vec{E}_1=\hat{x}2y-\hat{y}3x+\hat{z}5$, $\vec{E}_{1t}(z=0)=\vec{E}_{2t}(z=0)=\hat{x}2y-\hat{y}3x$,

$$\vec{D}_{1n}(z=0)=\vec{D}_{2n}(z=0)\quad\Rightarrow 2\vec{E}_{1n}(z=0)=3\vec{E}_{2n}(z=0)$$

$$\vec{E}_{2n}(z=0)=\frac{2}{3}\left(\hat{z}5\right)=\hat{z}\frac{10}{3},\ \therefore\ \vec{E}_2(z=0)=\hat{x}2y-\hat{y}3x+\hat{z}\frac{10}{3}$$

$$\vec{D}_2(z=0)=\left(\hat{x}2y-\hat{y}3x+\hat{z}\frac{10}{3}\right)3\varepsilon_0$$

Eg. A lucite sheet (ε_r=3.2) is introduced perpendicularly in a uniform electric field $\vec{E}_0=\hat{x}E_0$ in free space. Determine $\vec{E}_i$, $\vec{D}_i$ and $\vec{P}_i$ inside the lucite. [中央地球物理所]

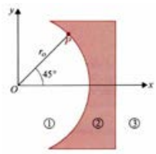


(Sol.) $\vec{D}_i=\hat{x}D_i=\hat{x}D_0=\hat{x}\varepsilon_0E_0$

$$\vec{E}_i=\frac{1}{\varepsilon}\vec{D}_i=\frac{1}{\varepsilon_0\varepsilon_r}\vec{D}_i=\hat{x}\frac{E_0}{3.2}$$

$$\vec{P}_i=\vec{D}_i-\varepsilon_0\vec{E}_i=\hat{x}\left(1-\frac{1}{3.2}\right)\varepsilon_0E_0=\hat{x}\frac{11}{16}\varepsilon_0E_0\quad(C/m)$$

Eg. Dielectric lenses can be used to collimate electromagnetic fields. The left surface of the lens is that of a circular cylinder, and right surface is a plane. If $\vec{E}_1$ at point $P(r_0,45°,z)$ in region 1 is $\hat{a}_r5-\hat{a}_\phi3$, what must be the dielectric constant of the lens in order that $\vec{E}_3$ in region 3 is parallel to the x-axis?

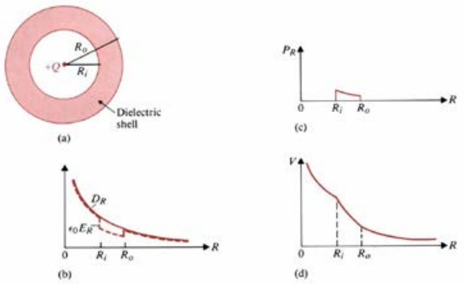


(Sol.) Assume $\vec{E}_2=\hat{a}_rE_{2r}+\hat{a}_\phi E_{2\phi}$, $\because\ E_{1t}=E_{2t}=E_\phi\Rightarrow E_{2\phi}=-3$

For $\vec{E}_3$ // $x-axis\Rightarrow\vec{E}_2$ // $x-axis\Rightarrow E_{2\phi}=-E_{2r}\quad\Rightarrow E_{2r}=3$

$$\hat{a}_n\cdot\vec{D}_1=\hat{a}_n\cdot\vec{D}_2\Rightarrow\varepsilon_1E_{r1}=\varepsilon_2E_{r2},\ \varepsilon_05=\varepsilon_0\varepsilon_{r2}3\quad\Rightarrow\varepsilon_{r2}=\frac{5}{3}$$

Eg. A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ε_r. Determine $\vec{E}$, V, $\vec{D}$, and $\vec{P}$ as functions of the radial distance R. [高考]



(Sol.) $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$

$R>R_o$:

$$\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0 R^2}, \quad V = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\vec{D} = \hat{a}_R \frac{Q}{4\pi R^2} \quad \text{and} \quad \vec{P} = 0$$

$R_i<R<R_o$:

$$\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2} = \hat{a}_R \frac{Q}{4\pi\varepsilon R^2}, \quad \vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}, \quad \vec{P} = \hat{a}_R\left(1-\frac{1}{\varepsilon_r}\right)\frac{Q}{4\pi R^2}$$

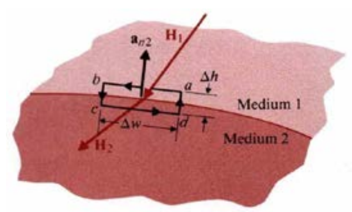
$$V = -\int_{\infty}^{R_o} \frac{Q}{4\pi\varepsilon_0 R^2} dR - \int_{R_o}^{R} \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2} dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_o} + \frac{1}{\varepsilon_r R}\right]$$

$R<R_i$: $\vec{E} = \hat{a}_R \frac{Q}{4\pi\varepsilon_0 R^2}, \quad \vec{D} = \hat{a}_R \frac{Q}{4\pi R^2}, \quad \vec{P} = 0,$

$$V = V\Big|_{R=R_i} - \int_{R_i}^{R} \frac{Q}{4\pi\varepsilon_0 R^2} dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_o} - \left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]$$

Boundary conditions for magnetic fields:

Eg. Show that $\mu_1 H_{1n} = \mu_2 H_{2n}$ and $\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}$.



(Proof) $\oiint \vec{B} \cdot d\vec{S} = 0 \Rightarrow B_{1n}\Delta S - B_{2n}\Delta S = 0$, $B_{1n} = B_{2n}$

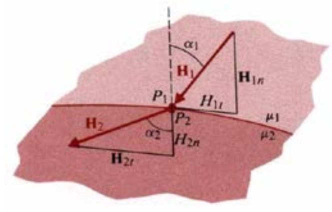
$\Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$

$$\oint \vec{H} \cdot d\vec{l} = I \Rightarrow \oint_{abcda} \vec{H} \cdot d\vec{l} = H_1 \cdot \Delta w + H_2 \cdot (-\Delta w) = J_{sw}\Delta w$$

$$\Rightarrow H_{1t} - H_{2t} = J_{sw} \Rightarrow \hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{J}$$

If $J=0$, then $H_{1t} = H_{2t}$

Eg. Two magnetic media with permeabilities μ_1 and μ_2 have a common boundary. The magnetic field intensity in medium 1 at the point P_1 has a magnitude H_1 and makes an angle α_1 with the normal. Determine the magnitude and the direction of the magnetic field intensity at point P_2 in medium 2.



(Sol.) $\begin{cases} \mu_2 H_2 \cos\alpha_2 = \mu_1 H_1 \cos\alpha_1 \\ H_2 \sin\alpha_2 = H_1 \sin\alpha_1 \end{cases} \Rightarrow \dfrac{\tan\alpha_2}{\tan\alpha_1} = \dfrac{\mu_2}{\mu_1}$

$$\Rightarrow \alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right)$$

$$H_2 = \sqrt{H_{2t}^2 + H_{2n}^2} = \sqrt{(H_2\sin\alpha_2)^2 + (H_2\cos\alpha_2)^2} = H_1\left[\sin^2\alpha_1 + \left(\frac{\mu_1}{\mu_2}\cos\alpha_1\right)^2\right]^{1/2}$$

Eg. Consider a plane boundary (y=0) between air (region 1, μ_{r1}=1) and iron (region 2, μ_{r2}=5000). (a) Assuming $\vec{B}_1 = 0.5\hat{x} - 10\hat{y}$ (mT), find $\vec{B}_2$ and the angle that $\vec{B}_2$ makes with the interface. (b) Assuming $\vec{B}_2 = 10\hat{x} + 0.5\hat{y}$ (mT), find $\vec{B}_1$ and the angle that $\vec{B}_1$ makes with the normal to the interface.

(Sol.)

(a) $\vec{B}_1 = 0.5\hat{x} - 10\hat{y}$, $\vec{B}_2 = B_{2x}\hat{x} + B_{2y}\hat{y}$, $H_{2x} = \dfrac{B_{2x}}{5000\mu_o} = H_{1x} = \dfrac{0.5}{\mu_o} \Rightarrow B_{2x} = 2500$

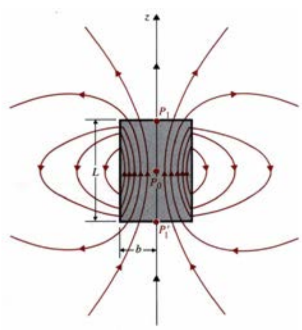
$B_{2y} = B_{1y} = -10 \quad \Rightarrow \vec{B}_2 = 2500\hat{x} - 10\hat{y}$, $\tan\alpha_2 = \dfrac{\mu_2}{\mu_1}\tan\alpha_1 = 500\dfrac{B_{1x}}{B_{1y}} = 25$

(b) $\vec{B}_2 = 10\hat{x} + 0.5\hat{y}$, $\vec{B}_1 = B_{1x}\hat{x} + B_{1y}\hat{y}$, $H_{1x} = \dfrac{B_{1x}}{\mu_1} = H_{2x} = \dfrac{B_{2x}}{\mu_2}$

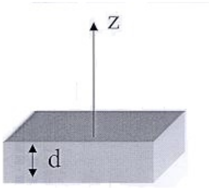
$\Rightarrow B_{1x} = \dfrac{1}{\mu_{BE}}B_{2x} = \dfrac{10}{5000} = 0.002$, $B_{1y} = B_{2y} = 0.5$, $\therefore \ \vec{B}_1 = 0.002\hat{x} + 0.5\hat{y}$,

$\tan\alpha_1 = \dfrac{B_{1x}}{B_{1y}} = \dfrac{0.002}{0.5} = 0.004$

Magnetic flux lines round a cylindrical bar magnet:



Eg. A very large slab of material of thickness *d* lies perpendicularly to uniform magnetic field intensity $\vec{H}_0 = \hat{z}H_0$. Ignoring edge effect, determine the magnetic field intensity in the slab: (a) if the slab material has a permeability *μ*, (b) if the slab permanent magnet having a magnetization vector $\vec{M}_i = \hat{z}M_i$. [台大物研]



(Sol.)

(a) $\vec{B}_2 = \mu_2 \vec{H}_2$, $B_{2z} = B_{1z} \Rightarrow \mu\, H_2 = \mu_o H_o$, $\vec{H}_2 = \hat{z}H_2 = \hat{z}\dfrac{\mu_o}{\mu}H_o$

(b) $\vec{B}_2 = \mu_o(\vec{H}_2 + \vec{M}_i)$, $B_{2z} = B_{1z} \Rightarrow \mu_o(H_2 + M_i) = \mu_o H_o \Rightarrow \vec{H}_2 = \hat{z}(H_o - M_i)$

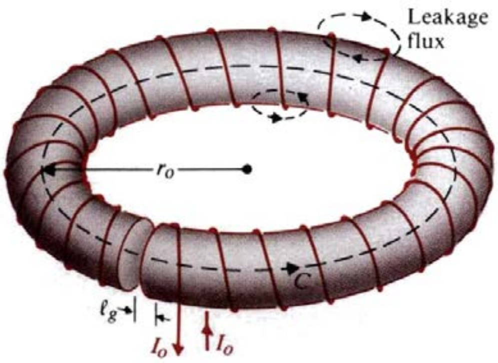
Eg. Assume that *N* turns of wire are wound around a toroidal core of a ferromagnetic material with permeability *μ*. The core has a mean radius r_0, a circular cross section of radius *a* ($a << r_0$), and a narrow air gap of length l_g, as shown in Figure. A steady current I_0 flows in the wire. Determine (a) the magnetic flux density B_f in the ferromagnetic core; (b) the magnetic field intensity H_f in the core; and, (c) the magnetic field intensity H_g in the gap. [台大電研]

(Sol.)

$$\oint_C \vec{H}\cdot d\vec{l} = NI_o,\quad \vec{B}_f = \vec{B}_g = \hat{a}_\phi B_f,\quad \frac{B_f}{\mu}(2\pi r_o - l_g) + \frac{B_f}{\mu_o} l_g = NI_o$$

$$B_f = \frac{\mu_o \mu N I_o}{\mu_o(2\pi r_o - l_g) + \mu l_g} \Rightarrow \vec{H}_f = \hat{a}_\phi \frac{\mu_o N I_o}{\mu_o(2\pi r_o - l_g) + \mu l_g}$$

$$\vec{H}_g = \hat{a}_\phi \frac{\mu N I_o}{\mu_o(2\pi r_o - l_g) + \mu l_g}$$



2-3 Steady-state Currents

Differential current: $\Delta I=\dfrac{\Delta Q}{\Delta t}=\dfrac{Nq\vec{v}\cdot\hat{a}_n\Delta s\Delta t}{\Delta t}=Nq\vec{v}\cdot\Delta\vec{s}$

Current density: $\vec{J}=Nq\vec{v}=\rho\vec{v}$ (A/m^2), $I=\iint_s \vec{J}\cdot d\vec{S}$ (A)

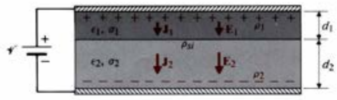
Let $\vec{v}=\mu\vec{E},\ \vec{J}=\rho\vec{v}=-\mu\rho\vec{E}=\sigma\vec{E}$

μ : mobility σ : conductivity

$$\sigma=-\rho_e u_e+\rho_h u_h$$

electrons holes

Eg. An *emf* *V* is applied across a parallel-plate capacitor of area *S*. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ε_1 and ε_2, and conductivities σ_1 and σ_2, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics. [高考]



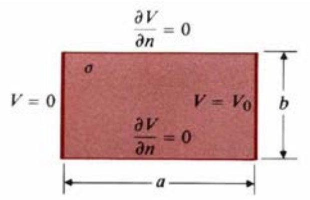
(Sol.)

$$V=(R_1+R_2)I=\left(\frac{d_1}{\sigma_1 S}+\frac{d_2}{\sigma_2 S}\right)I$$

$$J=\frac{I}{S}=\frac{V}{(d_1/\sigma_1)+(d_2/\sigma_2)}=\frac{\sigma_1\sigma_2 V}{\sigma_2 d_1+\sigma_1 d_2}$$

$$V=E_1d_1+E_2d_2,\quad J=\sigma_1E_1=\sigma_2E_2,\quad E_1=\frac{\sigma_2 V}{\sigma_2 d_1+\sigma_1 d_2},\quad E_2=\frac{\sigma_1 V}{\sigma_2 d_1+\sigma_1 d_2}$$

Eg. Assume a rectangular conducting sheet of conductivity σ, width *a*, and height *b*. A potential difference is applied to the side edges. Find (a) the potential distribution, (b) the current density everywhere within the sheet. [台科大電子所]



(Sol.)

(a) $V(x)=Cx,\ V(a)=Ca=V_0\Rightarrow V(x)=\dfrac{V_o}{a}x$

(b) $\vec{E}=-\nabla V(x)=-\hat{x}\dfrac{V_o}{a}\Rightarrow\vec{J}=\sigma\vec{E}=-\hat{x}\dfrac{\sigma V_o}{a}$

Equation of continuity: $I = \oiint_s \vec{J}\cdot d\vec{S} = -\frac{dQ}{dt} = -\frac{d}{dt}\iiint_{v'} \rho dv = \iiint \nabla\cdot\vec{J}dv \Rightarrow \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

If $\vec{J} = \sigma\vec{E},\ \sigma\nabla\cdot\vec{E} + \frac{\partial p}{\partial t} = \sigma\frac{\rho}{\varepsilon} + \frac{\partial\rho}{\partial t} = 0$, $\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$

Eg. Lightning strikes a lossy dielectric sphere ε_r=1.2, σ=10*S*/*m*, of radius 0.1*m* at time *t*=0, depositing uniformly in the sphere a total charge 1*mC*. Determine for all *t* for (a) the electric field intensity both inside and outside the sphere, (b) the current density in the sphere, (c) calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value, (d) calculate the charge in the electrostatic energy stored in the sphere as the charge density diminished from the initial value to 1% of the value. What happens to this energy? (e) determine the electrostatic energy stored in the space outside the sphere. Does this energy charge with time?

(Sol.) $\rho_0 = \frac{Q}{\frac{4}{3}\pi b^3} = 0.239\left(C/m^3\right),\ \ \rho = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$

$$\begin{cases} (a)\ R<b: \vec{E}_i = \hat{a}_R \dfrac{\frac{4\pi}{3}R^3\rho}{4\pi\varepsilon R^2} = \hat{a}_R \dfrac{\rho_0 R}{3\varepsilon} e^{-\frac{\sigma}{\varepsilon}t} = \hat{a}_R 7.5\times10^9 R\cdot e^{-7.42\times10^{10}t}\left(V/m\right) \\ ....R>b: \vec{E}_0 = \hat{a}_R \dfrac{Q}{4\pi\varepsilon_0 R^2} = \hat{a}_R \dfrac{Q}{R^2}\times10^6\left(V/m\right) \\ (b)\ R<b: \vec{J}_i = \sigma\vec{E}_i = \hat{a}_R 7.5\times10^{10}\ \mathrm{Re}^{-7.42\times10^{10}t} \\ ....R>b: \vec{J}_0 = 0 \end{cases}$$

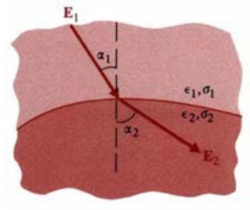
$$\begin{cases} (c)\ e^{-\frac{\sigma}{\varepsilon}t} = \dfrac{\rho}{\rho_0} = 0.01 \Rightarrow t = \dfrac{\ell n100}{(\sigma/\varepsilon)} = 4.88\times10^{-12}(s) \\ (d)\ W_i = \dfrac{\varepsilon}{2}\iiint_{v'} E_i^2 dv' \propto e^{-2\left(\frac{\xi}{\varepsilon}\right)t} \Rightarrow \dfrac{W_i}{(W_i)_0} = (0.01)^2 = 10^{-4} \\ (e)\ W_0 = \dfrac{\varepsilon_0}{2}\int_0^\infty E_0^2 4\pi R^2 dR = 4.5\times10^3(J) \end{cases}$$

Boundary conditions for current densities:

$$\begin{cases} \nabla\cdot\vec{J} = 0 \Rightarrow \\ \nabla\times\left(\dfrac{\vec{J}}{\sigma}\right) = 0 \Rightarrow \end{cases} \begin{cases} J_{1n} = J_{2n} \\ \dfrac{J_{1t}}{J_{2t}} = \dfrac{\sigma_1}{\sigma_2} \end{cases}$$

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla\cdot\vec{J} = 0$	$\oint_S \vec{J}\cdot d\vec{s} = 0$
$\nabla\times\left(\frac{\vec{J}}{\sigma}\right) = 0$	$\oint_C \frac{1}{\sigma}\vec{J}\cdot d\vec{\ell} = 0$

Eg. Two lossy dielectric media with permittivities and conductivities (ε_1, σ_1) and (ε_2, σ_2) are in contact. An electric field with a magnitude $\vec{E}_1$ is incident from medium 1 upon the interface at an angle α_1 and measured from the common normal, as in Figure.



(a) Find the magnitude and direction of $\vec{E}_2$ in medium 2.

(b) Find the surface charge density at the interface.

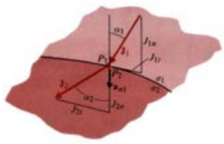
(Sol.)

(a) $E_{1t} = E_{2t}$ => $E_1 \sin\alpha_1 = E_2 \sin\alpha_2$, $J_{1n} = J_{2n}$ => $\sigma_1 E_1 \cos\alpha_1 = \sigma_2 E_2 \cos\alpha_2$

$$=> \begin{cases} E_2 = E_1\sqrt{\sin^2\alpha_1 + (\dfrac{\sigma_1}{\sigma_2}\cos\alpha_1)^2} \\ \tan\alpha_2 = \dfrac{\sigma_2}{\sigma_1}\tan\alpha_1 \end{cases}$$

(b) $D_{2n} - D_{1n} = \rho_s$ => $\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_S$, $\rho_S = \left(\dfrac{\sigma_1}{\sigma_2}\varepsilon_2 - \varepsilon_1\right)E_1\cos\alpha$

Eg. Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. The steady current density in medium 1 at point P_1 has a magnitude J_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the current density at point P_2 in Medium 2. [台大電研]



(Sol.)

$$J_1\cos\alpha_1 = J_2\cos\alpha_2,\ \sigma_2 J_1\sin\alpha_1 = \sigma_1 J_2\sin\alpha_2 \Rightarrow \frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\sigma_2}{\sigma_1}$$

$$J_2 = \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2\sin\alpha_2)^2 + (J_2\cos\alpha_2)^2} = \left[\left(\frac{\sigma_2}{\sigma_1}J_1\sin\alpha_1\right)^2 + (J_1\cos\alpha_1)^2\right]^{1/2}$$

$\begin{cases} J_{1n} = J_{2n} \Rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \\ D_{1n} - D_{2n} = \rho_s \Rightarrow \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \end{cases} \Rightarrow \rho_s = \left(\varepsilon_1\dfrac{\sigma_2}{\sigma_1} - \varepsilon_2\right)E_{2n} = \left(\varepsilon_1 - \varepsilon_2\dfrac{\sigma_1}{\sigma_2}\right)E_{1n}$. If

$\sigma_2 >> \sigma_1 \Rightarrow \rho_s = \varepsilon_1 E_{1n} = D_{1n}$.

2-4 Maxwell's Equations and Plane EM Waves

Maxwell's Equations

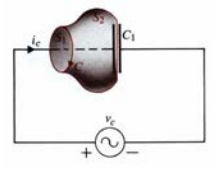
Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E}\cdot d\ell = -\dfrac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H}\cdot d\ell = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t}\cdot ds$	Ampère's circuital law
$\nabla\cdot\mathbf{D} = \rho$	$\oint_S \mathbf{D}\cdot ds = Q$	Gauss's law
$\nabla\cdot\mathbf{B} = 0$	$\oint_S \mathbf{B}\cdot ds = 0$	No isolated magnetic charge

Note: $\dfrac{\partial \vec{D}}{\partial t}$ is equivalent to a current density, called the displacement current density.

Eg. A voltage source $V_0\sin(\omega t)$, is connected across a parallel-plate capacitor C. Find the displacement current in the capacitor.

(Sol.) $i_C = C\dfrac{dv_C}{dt} = CV_0\omega\cos\omega t = \varepsilon\dfrac{A}{d}V_0\omega\cos\omega t$

$\vec{D} = \varepsilon\vec{E} \Rightarrow D = \varepsilon\dfrac{V_0}{d}\sin\omega t$,



$$i_D = \iint_A \frac{\partial\vec{D}}{\partial t}\cdot d\vec{S} = \varepsilon\frac{A}{d}V_0\omega\cos\omega t = i_C$$

Lorentz condition: $\nabla\cdot\vec{A} + \mu\varepsilon\dfrac{\partial V}{\partial t} = 0$

$$\nabla\times\vec{B} = \mu\vec{J} + \mu\frac{\partial\vec{D}}{\partial t} = \nabla\times\nabla\times\vec{A} = \mu\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}(-\nabla V - \frac{\partial\vec{A}}{\partial t}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - \nabla(\mu\varepsilon\frac{\partial V}{\partial t}) - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \nabla(\nabla\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t})$$

If Lorentz Condition holds, we have $\nabla^2\vec{A} - \mu\varepsilon\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$

$\because$ $\nabla\cdot\vec{D} = \rho = \nabla\cdot\varepsilon\vec{E} = \rho = \nabla\cdot\varepsilon(-\nabla V - \dfrac{\partial\vec{A}}{\partial t}) \Rightarrow \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla\cdot\vec{A}) = \nabla^2 V + \dfrac{\partial}{\partial t}(-\mu\varepsilon\dfrac{\partial V}{\partial t}) = -\dfrac{\rho}{\varepsilon}$

$\therefore$ $\nabla^2 V - \mu\varepsilon\dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\varepsilon}$

Effective permittivity:

$$\nabla\times\vec{H} = \vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega(\varepsilon + \frac{\sigma}{j\omega})\vec{E} = j\omega\varepsilon_C\vec{E}$$

$\Rightarrow \varepsilon_C = \varepsilon - j\dfrac{\sigma}{\omega} = \varepsilon' - j\varepsilon'' \Rightarrow \sigma = \omega\varepsilon''$. Similarly, $\mu = \mu' - j\mu''$

Loss tangent: $\tan\delta_C = \dfrac{\varepsilon''}{\varepsilon'} = \dfrac{\sigma}{\omega\varepsilon}$

Eg. A sinusoidal electric intensity of amplitude 250V/m and frequency 1GHz exists in a lossy dielectric medium that has a relative permittivity of 2.5 and loss tangent of 0.001. Find the average power dissipated in the medium per cubic meter.

$$\tan\delta_\varepsilon = 0.001 = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r},$$

$$\sigma = 0.001(2\pi 10^9)(\frac{10^{-9}}{36\pi})(2.5) = 1.39\times10^{-4}(S/m)$$

$$p = \frac{1}{2}JE = \frac{1}{2}\sigma E^2 = \frac{1}{2}\times(1.39\times10^{-4})\times250^2 = 4.34(W/m^3)$$

(Sol.)

Maxwell's Equations in the source-free regions:

$$\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t},\quad \nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t},\quad \nabla\cdot\vec{E}=0,\quad \nabla\cdot\vec{H}=0$$

Phasor representations: Eg. $\hat{x}Ae^{-j(\beta z+\theta)}$**,** $(\hat{x}\frac{3}{5}+\hat{y}\frac{4}{5})e^{-j(\beta z+\theta)}$**, etc.**

Instantaneous representations: Eg. $\hat{x}A\cos(\omega t-\beta z+\theta)=\mathrm{Re}[\hat{x}Ae^{-j(\beta z+\theta)}\cdot e^{j\omega t}]$**, etc.**

In case $\vec{E}$ and $\vec{H}$ are proportional to $e^{j\omega t}$, we have $\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}=-j\omega\mu\vec{H}$ and

$\nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}=j\omega\varepsilon\vec{E}$.

Eg. Given that $\vec{H}=\hat{y}2\cos(15\pi x)\sin(6\pi 10^9 t-\beta z)$ **in air, find** $\vec{E}$ **and** $\boldsymbol{\beta}$**.**

(Sol.) *Phasor*: $\vec{H}=\hat{y}2\cos(15\pi x)e^{-j\beta z}$, $(15\pi)^2+\beta^2=\omega^2\mu_0\varepsilon_0=400\pi^2\Rightarrow\beta=13.2\pi$

$$\vec{E}=\frac{1}{j\omega\varepsilon_0}\nabla\times\vec{H}=[\hat{x}158\pi\cos(15\pi x)+\hat{z}j180\pi\sin(15\pi x)]e^{-j\beta z}$$

$$\vec{E}(x,z,t)=\mathrm{Re}[\vec{E}(x,z)e^{j\omega t}]$$

Eg. Given that $\vec{E}=\hat{y}0.1\cos(10\pi x)\sin(6\pi 10^9 t-\beta z)$ **in air, find** $\vec{H}$ **and** $\boldsymbol{\beta}$**.**

(Sol.) *Phasor*: $\vec{E}=\hat{y}0.1\cos(10\pi x)e^{-j\beta z}$, $(10\pi)^2+\beta^2=\omega^2\mu_0\varepsilon_0=400\pi^2\Rightarrow\beta=10\sqrt{3}\pi$

$$\vec{H}=-\frac{1}{j\omega\mu_0}\nabla\times\vec{E}=\frac{j}{\omega\mu_0}[\hat{x}0.1\beta\cos(10\pi x)+\hat{z}0.1(10\pi)\cos(10\pi x)]e^{-j\beta z}$$

$$\vec{H}(x,z,t)=\mathrm{Re}[\vec{H}(x,z)e^{j\omega t}]$$

Eg. The electric field intensity of a spherical wave in free space is $\vec{E}=\hat{a}_R\frac{E_0}{R}\sin\theta\cos(\omega t-kR)$**. Determine the magnetic field intensity.**

(Sol.) Phasor: $\vec{E}=\hat{a}_R\frac{E_0}{R}\sin\theta\cdot e^{-jkR}$

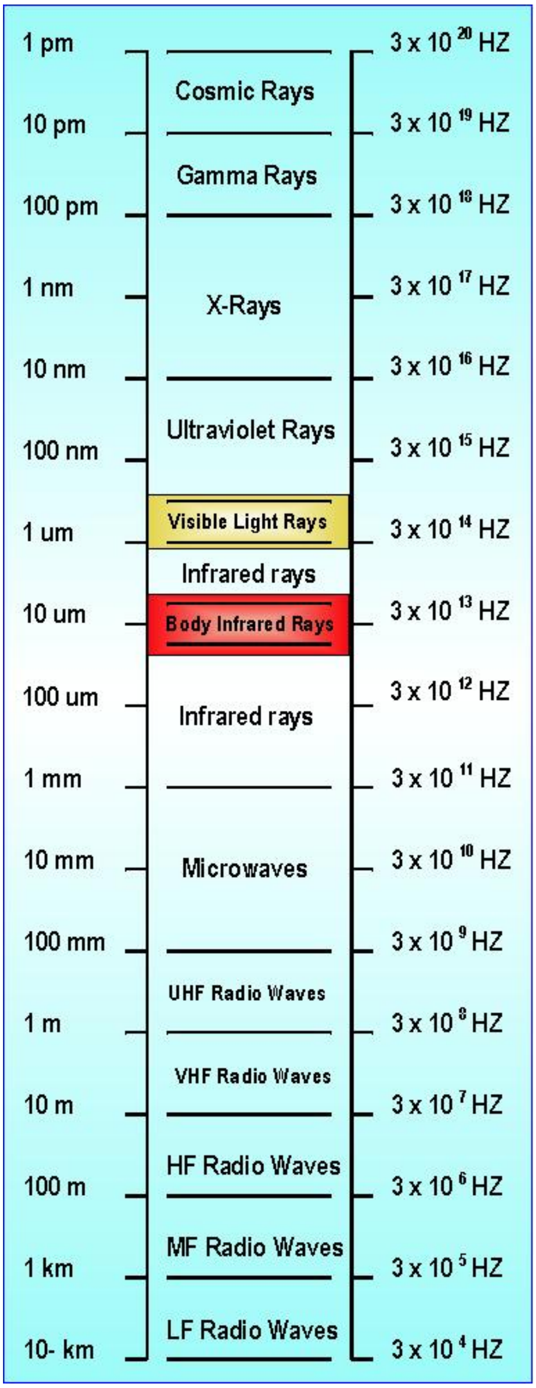
$$-j\omega\mu_0\vec{H}=\nabla\times\vec{E}=\hat{a}_\phi\frac{1}{R}\frac{\partial}{\partial R}(RE_0)=\hat{a}_\phi(-jk)\frac{E_0}{R}\sin\theta\cdot e^{-jkR}\Rightarrow\vec{H}=\hat{a}_\phi\frac{E_0}{R}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sin\theta\cdot e^{-jkR}$$

Plane EM waves excited by a current sheet:

Given $\vec{J}(t) = -\hat{x}J(t)$ at z=0, the field components of the EM plane wave excited by the current density are $\vec{E}(z,t) = \hat{x}\frac{\eta}{2}J(t \mp \frac{z}{v_p})$ and $\vec{H}(z,t) = \pm\hat{y}\frac{1}{2}J(t \mp \frac{z}{v_p})$, respectively. If it is a sinusoidal EM plane wave, $\vec{J}(t) = -\hat{x}J_0 \cos(\omega t)$ at z=0.

We have $\vec{E}(z,t) = \hat{x}\frac{\eta J_0}{2}\cos(\omega t \mp kz)$, $\vec{H}(z,t) = \pm\hat{y}\frac{J_0}{2}\cos(\omega t \mp kz)$.

Electromagnetic wave spectrum:



2-5 Plane EM waves in a simple, nonconducting and source-free region

In a simple, nonconducting and source-free region:

$$\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t},\quad \nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t},\quad \nabla\cdot\vec{E}=0,\quad \nabla\cdot\vec{H}=0$$

$$\nabla\times\nabla\times E=-\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})=-\mu\varepsilon\frac{\partial^2 E}{\partial t^2}=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}=-\nabla^2\vec{E}\Rightarrow\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0.$$

Velocity of the plane EM wave**:** $v=\dfrac{1}{\sqrt{\mu\varepsilon}}$

In vacuum, $\mu_0=4\pi\times10^{-7}$, $\varepsilon_0=\dfrac{1}{36\pi}\times10^{-9}\Rightarrow c=\dfrac{1}{\sqrt{\mu_0\varepsilon_0}}\approx3\times10^{8}(m/s)$.

Wave number: $k=\omega/v=\omega\sqrt{\mu\varepsilon}=\dfrac{2\pi}{v/f}=\dfrac{2\pi}{\lambda}$

Assume $\vec{E}\propto e^{j\omega t}\Rightarrow\nabla^2\vec{E}+k^2\vec{E}=0$ (drop $e^{j\omega t}$ factor)

Suppose $\vec{E}=\vec{E}(z)\Rightarrow\dfrac{d^2\vec{E}(z)}{dz^2}+k^2\vec{E}=0\Rightarrow\vec{E}(z)=E_0^+e^{-jkz}+E_0^-e^{jkz}$

Traveling wave in +z-direction:

$$E_0^+(z,t)=\mathrm{Re}[E_0^+e^{-jkz}\cdot e^{j\omega t}]=E_0^+\cos(\omega t-kz)$$

Let ωt-kz=constant$\Rightarrow$ **Phase velocity:** $v_p=\dfrac{dz}{dt}=\dfrac{\omega}{k}$

If $\vec{E}=\hat{x}E_x^+(z),\nabla\times\vec{E}=-j\omega\mu(\hat{x}H_x^++\hat{y}H_y^++\hat{z}H_z^+)$

$\Rightarrow H_x^+=H_z^+=0,\ H_y^+(z)=-\dfrac{1}{j\omega\mu}\cdot(-jk)E_x^+(z)=\dfrac{1}{\eta}E_x^+(z)$, where $\eta=\dfrac{\omega\mu}{k}=\sqrt{\dfrac{\mu}{\varepsilon}}$,

and $\eta_0=120\pi\approx377\Omega$ in free space.

TEM waves (Transverse electromagnetic waves): $\vec{E}$ and $\vec{H}\perp$ direction of propagation ($\hat{a}_n$)

$$\vec{E}(\vec{R})=\vec{E}(x,y,z)=\vec{E}_0e^{-jk_xx-jk_yy-jk_zz}=\vec{E}_0e^{-j\vec{k}\cdot\vec{R}}=\vec{E}_0e^{-jk\hat{a}_n\cdot\vec{R}},\quad\text{where}\quad\vec{R}=\hat{x}x+\hat{y}y+\hat{z}z,$$

$\vec{k}=\hat{a}_nk$, and $k_x^2+k_y^2+k_z^2=\omega^2\mu\varepsilon$

$$\because\nabla\cdot\vec{E}=0=\nabla\cdot(\vec{E}_0e^{-jk\hat{a}_n\cdot\vec{R}})=e^{-jk\hat{a}_n\cdot\vec{R}}\nabla\cdot\vec{E}_0+\vec{E}_0\cdot(\nabla e^{-jk\hat{a}_n\cdot\vec{R}})=\vec{E}_0\cdot(\nabla e^{-jk\hat{a}_n\cdot\vec{R}})$$
$$=\vec{E}_0\cdot-j(\hat{x}k_x+\hat{y}k_y+\hat{z}k_z)e^{-jk\hat{a}_n\cdot\vec{R}}=-jk(\vec{E}_0\cdot\hat{a}_n)e^{-jk\hat{a}_n\cdot\vec{R}}$$

$\hat{a}_n\cdot\vec{E}_0=0\Rightarrow\vec{E}_0\perp\hat{a}_n$ (**TE**). Similarly, $\nabla\cdot\vec{H}=0\Rightarrow\vec{H}_0\perp\hat{a}_n$ (**TM**)

Relation between E-field and H-field of the plane EM wave:

$\vec{E}(\vec{R})=\frac{1}{j\omega\varepsilon}\nabla\times\vec{H}(\vec{R})=\frac{1}{j\omega\varepsilon}(-jk)\hat{a}_n\times\vec{H}(\vec{R})\Rightarrow\vec{E}(\vec{R})=-\eta\hat{a}_n\times\vec{H}(\vec{R})$, where $\eta=\frac{\omega\mu}{k}=\sqrt{\frac{\mu}{\varepsilon}}$

$$\vec{H}(\vec{R})=-\frac{1}{j\omega\mu}\nabla\times\vec{E}(\vec{R})=\frac{1}{\eta}\hat{a}_n\times\vec{E}(\vec{R})\Rightarrow\vec{H}(\vec{R})=\frac{1}{\eta}\hat{a}_n\times\vec{E}(R)\Rightarrow\vec{H}\perp\hat{a}_n$$

Eg. The instantaneous expression for the magnetic field intensity of a uniform plane wave propagating in the +*y* direction in air is given by $\vec{H}=\hat{z}4\times10^{-6}\cos(10^7\pi t-k_0y+\frac{\pi}{4})$ *A/m*. (a) Determine k_0 and the location where H_z vanishes at *t*=3*ms*. (b) Write the instantaneous expression for $\vec{E}$.

(Sol.) $\omega=10^7\pi\Rightarrow k_0=\frac{\omega}{c}=\frac{10^7\pi}{3\times10^8}=\frac{\pi}{30}$, $\hat{a}_n=\hat{y}$

(a) $\cos[(2n+1)\frac{\pi}{2}]=0\Rightarrow10^7\pi\times3\times10^{-3}-\frac{\pi}{30}y+\frac{\pi}{4}=\frac{2n+1}{2}\pi\Rightarrow y=30(3\times10^4-\frac{1}{4}-n)$

(b) $\vec{E}(z,t)=-\eta_0\hat{a}_n\times\vec{H}(z,t)$, $\vec{E}(z,t)=-\hat{x}480\pi10^{-6}\cos(10^7\pi t-\frac{\pi}{30}y+\frac{\pi}{4})$

Eg. A 100M*Hz* uniform plane wave $\vec{E}=\hat{x}E_x$ propagates in the +*z* direction. Suppose ε_r=4, μ_r=1, σ=0, and it has a maximum value of 10^{-4}*V/m* at *t* =0 and *z*=0.125*m*. (a) Write the instantaneous expressions for $\vec{E}$ and $\vec{H}$. (b) Determine the location where $\vec{E}$ is a positive maximum when *t*=10^{-8}*sec*.

(Sol.) $k=\omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}=\frac{4\pi}{3}$, $\hat{a}_n=\hat{z}$, $\eta=\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}=60\pi$

(a) $\vec{E}(z,t)=\hat{x}E_x=\hat{x}10^{-4}\cos(2\pi\times10^8t-kz+\theta)$ has the maximum in case of

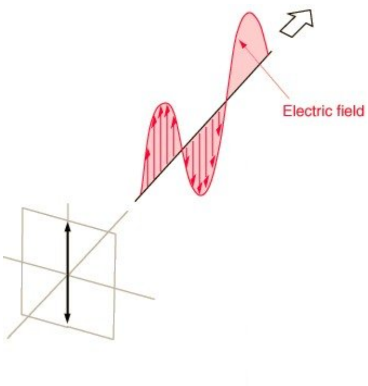
$2\pi\times10^8t-kz+\theta=0\Rightarrow\theta=\frac{\pi}{6}\Rightarrow\vec{E}(z,t)=\hat{x}10^{-4}\cos(2\pi10^8t-\frac{4\pi}{3}z+\frac{\pi}{6})$,

$$\vec{H}(z,t)=\frac{1}{\eta}\hat{a}_n\times\vec{E}(z,t)=\hat{y}\frac{10^{-4}}{60\pi}\cos(2\pi10^8t-\frac{4\pi}{3}z+\frac{\pi}{6})$$

(b) $\cos(2n\pi)=1$, $2\pi10^8(10^{-8})-\frac{4\pi}{3}z_{\max}+\frac{\pi}{6}=2n\pi\Rightarrow z_{\max}=\frac{13}{8}\pm\frac{3n}{2}$

Polarization of the EM wave: The direction of electric field of the EM wave.

In the following text, we assume all EM waves to be ***z*-propagated** if we do not specify them.



Linear polarizations in the *x* and the *y*-direction, respectively: $\vec{E} = \hat{x}E_x e^{-j(kz+\theta)}$, $\vec{E} = \hat{y}E_y e^{-j(kz+\theta)}$

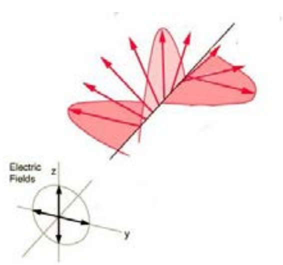
Linear polarization in general case: $\vec{E} = \hat{x}E_x e^{-j(kz+\theta)} + \hat{y}E_y e^{-j(kz+\theta)}$, where E_x and E_y are in phase (we can assume the both to be real).

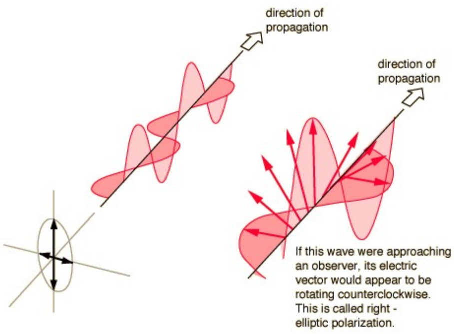
Right–hand circular polarization:

$$\vec{E} = \hat{x}E_0 e^{-j(kz+\theta)} - \hat{y}jE_0 e^{-j(kz+\theta)}$$

Left–hand circular polarization:

$$\vec{E} = \hat{x}E_0 e^{-j(kz+\theta)} + \hat{y}jE_0 e^{-j(kz+\theta)}$$





Right–hand elliptical polarization:

$$\vec{E} = \hat{x}E_{10} e^{-j(kz+\theta)} - \hat{y}jE_{20} e^{-j(kz+\theta)} \quad (E_{10} \neq E_{20})$$

Left–hand elliptical polarization:

$$\vec{E} = \hat{x}E_{10} e^{-j(kz+\theta)} + \hat{y}jE_{20} e^{-j(kz+\theta)} \quad (E_{10} \neq E_{20})$$

We can receive/transmit linearly-polarized EM waves by a linear dipole antenna.




We can receive/transmit circularly-polarized EM waves by a circular reflector antenna.

Instantaneous Expression for $\vec{E}$ of right–hand elliptical–polarization (drop phase factor $e^{-j\theta}$):

$$\vec{E}(z,t)=\mathrm{Re}\{[\hat{x}E_{10}e^{-jkz}-\hat{y}jE_{20}e^{-jkz}]e^{j\omega t}\}=\hat{x}E_{10}\cos(\omega t-kz)+\hat{y}E_{20}\sin(\omega t-kz)$$

$$=\hat{x}E_1(z,t)+\hat{y}E_2(z,t)$$

$$\Rightarrow \cos(\omega t)=\frac{E_1(0,t)}{E_{10}},\ \ \sin(\omega t)=\frac{E_2(0,t)}{E_{20}}\Rightarrow[\frac{E_1(0,t)}{E_{10}}]^2+[\frac{E_2(0,t)}{E_{20}}]^2=1,\ \ \omega t=\tan^{-1}\frac{E_2(0,t)}{E_1(0,t)}$$

1. $\hat{x}E_x=\frac{1}{2}(\hat{x}E_x-\hat{y}jE_y)+\frac{1}{2}(\hat{x}E_x+\hat{y}jE_y)$: A linearly polarized plane wave can be resolved into a right –hand and left–hand elliptically- or circularly-polarized waves.

2. $\hat{x}E_0-\hat{y}jE_0=(\hat{x}\frac{E_0+E_1}{2}-\hat{y}j\frac{E_1-E_0}{2})+(\hat{x}\frac{E_0-E_1}{2}+\hat{y}j\frac{E_0+E_1}{2})$:

A circularly–polarized plane wave can be resolved into two opposite elliptically–polarized waves.

3. $\hat{x}E_1-\hat{y}jE_2=(\hat{x}\frac{E_1+E_2}{2}-\hat{y}j\frac{E_1+E_2}{2})+(\hat{x}\frac{E_1-E_2}{2}+\hat{y}j\frac{E_1-E_2}{2})$:

An elliptically–polarized plane wave can be resolved into two opposite circularly–polarized waves.

Eg. The $\vec{E}$ field of a uniform plane wave propagating in a dielectric medium is given by $E(t,z)=\hat{x}2\cos(10^8t-\frac{z}{\sqrt{3}})-\hat{y}\sin(10^8t-\frac{z}{\sqrt{3}})$ *V/m*. (a) Determine the frequency and wavelength of the wave. (b) What is the dielectric constant of the medium? (c) Describe the polarization of the wave. (d) Find the corresponding $\vec{H}$ field.

(Sol.) Phasor: $\vec{E}=\hat{x}2e^{-jz/\sqrt{3}}+\hat{y}je^{-jz/\sqrt{3}}$

(a) $\omega=10^8\Rightarrow f=1.59\times10^7\,Hz$, $k=\frac{1}{\sqrt{3}}\Rightarrow\lambda=\frac{2\pi}{k}=2\sqrt{3}\pi$

(b) $v=\frac{\omega}{\beta}=\sqrt{3}\times10^8=1/\sqrt{\mu_0\varepsilon_0\varepsilon_r}\Rightarrow\varepsilon_r=3$

(c) It is the left–hand elliptically-polarized wave propagating along +z direction.

(d) $\eta=\sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}}=\frac{120\pi}{\sqrt{3}}$, $\hat{a}_n=\hat{z}$

$$\vec{H}=\frac{1}{\eta}\hat{a}_n\times\vec{E}=\frac{1}{\eta}\hat{z}\times(\hat{x}2e^{-jz/\sqrt{3}}+\hat{y}je^{-jz/\sqrt{3}})=\frac{\sqrt{3}}{120\pi}(\hat{y}2e^{-jz/\sqrt{3}}-\hat{x}je^{-jz/\sqrt{3}})$$

$$\Rightarrow\vec{H}(z,t)=\mathrm{Re}[\vec{H}(z)e^{j\omega t}]=\frac{\sqrt{3}}{120\pi}[\hat{x}\sin(10^8t-\frac{z}{\sqrt{3}})+\hat{y}2\cos(10^8t-\frac{z}{\sqrt{3}})]$$

Eg. Write down the instantaneous expression for the electric- and magnetic-field intensities of sinusoidal time-varying uniform plane wave propagating in free space and having the following characteristics: (1) *f*=10GHz; (2) direction of propagation is the +*z* direction; (3) left-hand circular polarization; (4) the initial condition is the electric field in the *z*=0 plane and *t*=0 having an *x*-component equal to E_0 and a *y*-component equal to $\sqrt{3}E_0$. [台大電研]

(Sol.) $\omega = 2\pi \times 10^{10}$, $v = \dfrac{\omega}{k} = c = 3\times 10^{8} \Rightarrow k = \dfrac{2\pi}{3}\times 10^{2}$

Phasor: $\vec{E} = \hat{x}Ae^{-j(kz+\theta)} + \hat{y}jAe^{-j(kz+\theta)}$ for the left-hand circular polarization

$\Rightarrow \vec{E}(z,t) = \mathrm{Re}\{[\hat{x}Ae^{-(jkz+\theta)} + \hat{y}jAe^{-j(kz+\theta)}]e^{j\omega t}\} = \hat{x}A\cos(\omega t - kz + \theta) - \hat{y}A\sin(\omega t - kz + \theta)$

z=0 and *t*=0, $\vec{E}(0,0) = \hat{x}A\cos(\theta) - \hat{y}A\sin(\theta) = \hat{x}E_0 + \hat{y}\sqrt{3}E_0 \Rightarrow \theta=\tan^{-1}(-\sqrt{3})$, $A=2E_0$

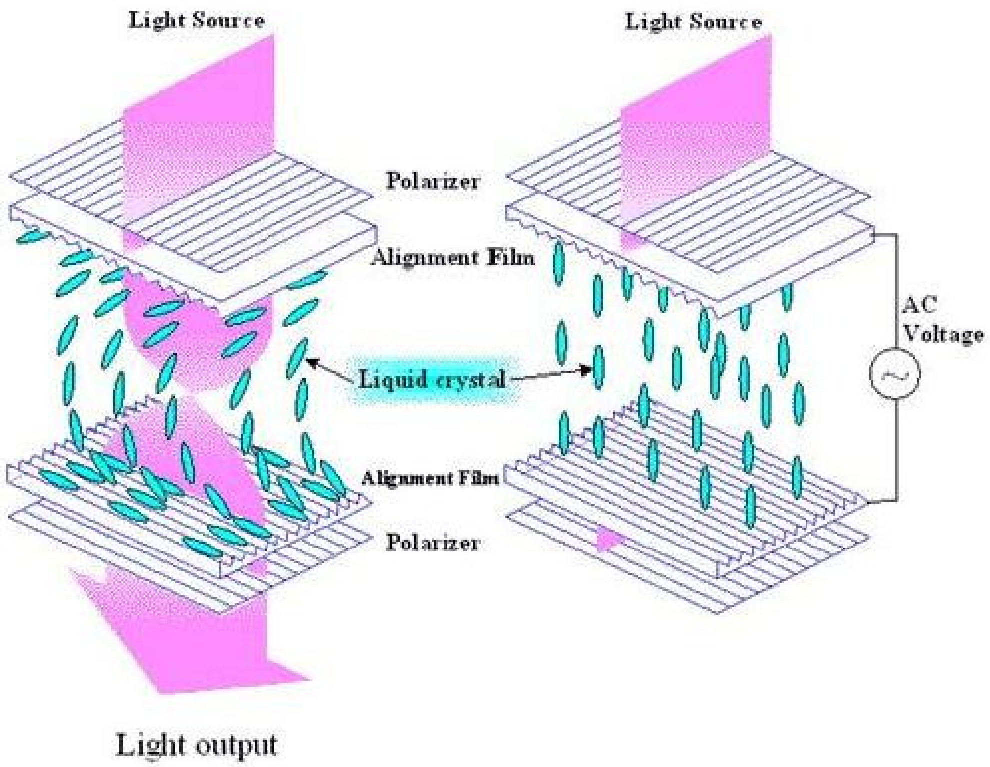
$$\vec{E}(z,t) = \hat{x}2E_0\cos[2\pi 10^{10}t - \frac{2\pi}{3}10^{2}z + \tan^{-1}(-\sqrt{3})] - \hat{y}2E_0\sin(2\pi 10^{10}t - \frac{2\pi}{3}10^{2}z + \tan^{-1}(-\sqrt{3})]$$

$$\vec{H} = \frac{1}{\eta_0}\hat{a}_n \times \vec{E} = \frac{1}{\eta_0}\hat{z}\times[\hat{x}Ae^{-j(kz+\theta)} - \hat{y}jAe^{-j(kz+\theta)}] = \frac{2E_0}{120\pi}[\hat{y}e^{-j(kz+\theta)} + \hat{x}je^{-j(kz+\theta)}]$$

$$\Rightarrow \vec{H}(z,t) = \mathrm{Re}[\vec{H}(z)e^{j\omega t}]$$

Application of polarization: Liquid Crystal Display (LCD)

The polarizations of incident lights are synchronized by the rotations of molecules of liquid crystal, which were controlled by an AC voltage. And then the output polarizer can block the orthogonally-polarized lights to control the output optical intensities.



Poynting vector: $\vec{P} = \vec{E}\times\vec{H}$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t},\quad \nabla\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t} \quad \Rightarrow \nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}$$

$$= -\vec{H}\cdot\frac{\partial(\mu\vec{H})}{\partial t} - \vec{E}\cdot\frac{\partial(\varepsilon\vec{E})}{\partial t} - \vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}(\frac{1}{2}\mu|\vec{H}|^2) - \frac{\partial}{\partial t}(\frac{1}{2}\varepsilon|\vec{E}|^2) - \sigma|\vec{E}|^2$$

$$\therefore\ \oint\!\!\oint_s(\vec{E}\times\vec{H})\cdot d\vec{S} = \iiint_v\nabla\cdot(\vec{E}\times\vec{H})dv = -\frac{\partial}{\partial t}\iiint_v(\frac{\varepsilon}{2}|\vec{E}|^2 + \frac{\mu}{2}|\vec{H}|^2)dv - \iiint_v\sigma|\vec{E}|^2\,dv$$

$\Rightarrow \vec{P} = \vec{E}\times\vec{H}$ is the electromagnetic power flow per unit area.

Instantaneous power density: $\vec{P}(z,t) = \text{Re}[\vec{E}(z)e^{j\omega t}]\times\text{Re}[\vec{H}(z)e^{j\omega t}]$

Set $\vec{E}(z) = \hat{x}E_x(z) = \hat{x}E_0e^{-(\alpha+j\beta)z} \Rightarrow \vec{H}(z) = \frac{1}{\eta}[\hat{a}_n\times\vec{E}(z)] = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cdot e^{-j(\beta z+\theta_\eta)}$,

$$\therefore\ \vec{E}(z,t) = \text{Re}[\vec{E}(z)e^{j\omega t}] = \hat{x}E_0e^{-\alpha z}\cos(\omega t-\beta z)$$

and $\vec{H}(z,t) = \text{Re}[H(z)e^{j\omega t}] = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t-\beta z-\theta_\eta)$

$$\Rightarrow \vec{P}(z,t) = \vec{E}(z,t)\times\vec{H}(z,t) = \text{Re}[\vec{E}(z)e^{j\omega t}]\times\text{Re}[\vec{H}(z)e^{j\omega t}]$$

$$= \hat{z}\frac{|E_0|^2}{2|\eta|}e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t-2\beta z-\theta_\eta)] \propto |E_0|^2$$

Average power density: $\vec{P}_{av} = \frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)$

$\vec{P}_{av} = \frac{1}{T}\int_0^T\vec{P}(z,t)dt = \hat{z}\frac{|E_0|^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta$, where T is the period. And it can be proved that $\vec{P}_{av} = \frac{1}{2}\text{Re}(\vec{E}\times\vec{H}^*)$.

Eg. Show that $\vec{P}(z,t)$ of a circularly–polarized plane wave propagating in a lossless medium is a constant.

(Sol.) Assuming right–hand circularly–polarized plane wave, $\hat{a}_n = \hat{z}$

$$\vec{E}(z,t) = E_0[\hat{x}\cos(\omega t-\beta z) + \hat{y}\sin(\omega t-\beta z)]$$

$$\vec{H}(z,t) = \frac{1}{\eta}(\hat{a}_n\times\vec{E}) = \frac{E_0}{\eta}[-\hat{x}\sin(\omega t-\beta z) + \hat{y}\cos(\omega t-\beta z)]$$

$$\vec{P}(z,t) = \vec{E}(z,t)\times\vec{H}(z,t) = \hat{z}\frac{{E_0}^2}{\eta}$$

Eg. The radiation electric field intensity of an antenna system is $\vec{E} = \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi$, find the expression for the average outward power flow per unit area.

(Sol.) $\hat{a}_n = \hat{a}_r$, $\vec{H} = \dfrac{1}{\eta}(\hat{a}_n \times \vec{E}) = (-\hat{a}_\theta \dfrac{E_\phi}{\eta} + \hat{a}_\phi \dfrac{E_\theta}{\eta})$

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}\mathrm{Re}[(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) \times (-\hat{a}_\theta \frac{E_\phi{}^*}{\eta} + \hat{a}_\phi \frac{E_\theta{}^*}{\eta})] = \frac{1}{2\eta}\hat{a}_r(|E_\theta|^2 + |E_\phi|^2)$$

Eg. Find $\vec{P}$ on the surface of a long, straight conducting wire of radius *b* and conductivity σ that carries a direct current *I*. Verify Poynting's theorem.

(Sol.) $\vec{J} = \hat{z}\dfrac{I}{\pi b^2} \Rightarrow \vec{E} = \dfrac{\vec{J}}{\sigma} = \hat{z}\dfrac{I}{\sigma \pi b^2}$, $\vec{H} = \hat{a}_\phi \dfrac{I}{2\pi b} \Rightarrow \vec{P} = \vec{E} \times \vec{H} = -\hat{a}_r \dfrac{I^2}{2\sigma \pi^2 b^3}$

$$-\oint\!\!\!\oint_s \vec{P} \cdot d\vec{S} = -\oint\!\!\!\oint_s \vec{P} \cdot \hat{a}_r dS = \frac{I^2}{2\sigma \pi^2 b^2} \cdot 2\pi b \ell = I^2 (\frac{\ell}{\sigma \pi b^2}) = I^2 R$$

2-6 Plane EM Wave in a Lossy Media

$$\nabla \times H = \vec{J} + j\omega\varepsilon\vec{E} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\vec{E} = j\omega\varepsilon_c\vec{E}\ ,\ \varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''.$$

Similarly, $\mu_c = \mu' - j\mu''$

Complex wave number: $k_c = \omega\sqrt{\mu\varepsilon_c}$. **Loss tangent:** $\tan\delta_c \approx \varepsilon''/\varepsilon' = \frac{\sigma}{\omega\varepsilon}$

Propagation constant: $\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}(1+\frac{\sigma}{j\omega\varepsilon})^{1/2}$

$$E \propto e^{-\gamma z} = e^{-jk_c z} = e^{-\alpha z}\cdot e^{-j\beta z}$$

If the medium is lossless, α=0; else if the medium is lossy, α>0.

Phase constant: $\beta = \frac{2\pi}{\lambda}$

$$\Rightarrow \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2}-1]^{1/2},\quad \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2}+1]^{1/2}$$

***Case 1 Low-loss Dielectric*:** $\frac{\sigma}{\omega\varepsilon} << 1 \Rightarrow \alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$, $\beta \approx \omega\sqrt{\mu\varepsilon}[(1+\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Intrinsic impedance: $\eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}(1+j\frac{\sigma}{2\omega\varepsilon})$

Phase velocity: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon_c}} \approx \frac{1}{\sqrt{\mu\varepsilon}}[1-\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

***Case 2 Good Conductor*:** $\frac{\sigma}{\omega\varepsilon} >> 1 \Rightarrow \alpha = \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$,

and $\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \approx (\sqrt{\frac{j\omega\mu}{\sigma}}) = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$

Phase velocity: $v_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$

Skin Depth (depth of penetration): $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}$.

For a good conductor, $\delta = \frac{1}{\alpha} \approx \frac{1}{\beta} = \frac{\lambda}{2\pi}$

Eg. $\vec{E}(t,z) = \hat{x}100\cos(10^7\pi t)$ ***V/m*** **at** ***z*****=0 in seawater:** ε_r**=72,** μ_r**=1, σ=4*****S/m*****. (a) Determine** α**,** β**,** v_p**, and** η_c**. (b) Find the distance at which the amplitude of** ***E*** **is 1% of its value at** ***z*****=0. (c) Write** ***E*****(*****z,t*****) and** ***H*****(*****z,t*****) at** ***z*****=0.8*****m*****, suppose it propagates in the +z direction.**

(Sol.) $\omega = 10^7\pi$, f=5×10^6*Hz*, $\sigma/\omega\varepsilon_0\varepsilon_r$=200>>1, ∴ Seawater is a good conductor in this case.

(a) $\alpha = \sqrt{\pi f\mu\sigma} = 8.89Np/m = \beta$, $\eta_c = (1+j)\sqrt{\dfrac{\pi f\mu}{\sigma}}$

$$v_p = \frac{\omega}{\beta} = 3.53\times10^6 m/s,\ \lambda = \frac{2\pi}{\beta} = 0.707m,\ \delta = \frac{1}{\alpha} = 0.112m$$

(b) $e^{-\alpha z} = 0.01 \Rightarrow z = \dfrac{1}{\alpha}\ln(100) = 0.518m$

(c) $E(z,t) = \mathrm{Re}[E(z)e^{j\omega t}] = \hat{x}100e^{-\alpha z}\cos(\omega t - \beta z)$

$$z = 0.8m \Rightarrow E(0.8,t) = \hat{x}100e^{-0.8\alpha}\cos(\omega t - 0.8\beta) = \hat{x}0.082\cos(10^7\pi t - 7.11)$$

$$\vec{H}(0.8,t) = \frac{1}{\eta}\hat{a}_n\times\vec{E}(0.8,t),\quad H(0.8,t) = \hat{y}\,\mathrm{Re}[\frac{E_x(0.8)}{\eta_c}e^{j\omega t}] = \hat{y}0.026\cos(10^7\pi t - 1.61)$$

Eg. The magnetic field intensity of a linearly polarized uniform plane wave propagating in the +***y*** **direction in seawater** ε_r**=80,** μ_r**=1, σ=4*****S/m*** **is** $\vec{H} = \hat{x}0.1\sin(10^{10}\pi t - \dfrac{\pi}{3})$ ***A/m*****. (a) Determine the attenuation constant, the phase constant, the intrinsic impedance, the phase velocity, the wavelength, and the skin depth. (b) Find the location at which the amplitude of** ***H*** **is 0.01** ***A/m*****. (c) Write the expressions for** ***E*****(*****y,t*****) and** ***H*****(*****y,t*****) at** ***y*****=0.5*****m*** **as function of** ***t*****.**

(Sol.) (a) $\sigma/\omega\varepsilon$=0.18<<1, ∴ Seawater is a low-loss dielectric in this case.

$$\Rightarrow \alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = 83.96Np/m \qquad \eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}(1+j\frac{\sigma}{2\omega\varepsilon}) = 41.8e^{j0.0283\pi}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}[(1+\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2] = 300\pi,\ v_p = \frac{\omega}{\beta} = 3.33\times10^7 m/s,\ \delta = \frac{1}{\alpha} = 1.19\times10^{-2}m,$$

$$\lambda = \frac{2\pi}{\beta} = 6.67\times10^{-3}m$$

(b) $e^{-\alpha y} = \dfrac{0.01}{0.1} \Rightarrow y = \dfrac{1}{\alpha}\ln 10 = 2.74\times10^{-2}m$

(c) $H(y,t) = \hat{x}0.1e^{-\alpha y}\sin(10^{10}\pi t - \beta y - \dfrac{\pi}{3})$, $y = 0.5, \beta = 300\pi$

$$\Rightarrow \vec{H}(0.5,t) = \hat{x}5.75\times10^{-20}\sin(10^{10}\pi t - \frac{\pi}{3})$$

$$\hat{a}_n = \hat{y} \Rightarrow \vec{E}(0.5,t) = -\eta_c\hat{a}_n\times\vec{H}(0.5,t) = \hat{z}2.41\times10^{-18}\sin(10^{10}\pi t - \frac{\pi}{3} + 0.0283\pi)$$

Eg. Given that the skin depth for graphite at 100 *MHz* is 0.16*mm*, determine (a) the conductivity of graphite, and (b) the distance that a 1*GHz* wave travels in graphite such that its field intensity is reduced by 30*dB*.

(Sol.) (a) $\delta = \dfrac{1}{\sqrt{\pi f \mu \sigma}} = 0.16\times10^{-3} \Rightarrow \sigma = 0.99\times10^{5}\, S/m$

(b) At f=10^9*Hz*, $\alpha = \sqrt{\pi f \mu \sigma} = 1.98\times10^{4}\, Np/m$

$$-30(dB) = 20\log_{10} e^{-\alpha z} \Rightarrow z = \frac{1.5}{\alpha \log_{10} e} = 1.75\times10^{-4}\, m$$

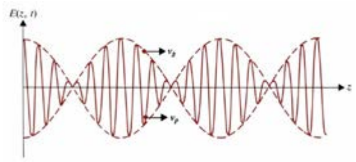
Eg. Determine and compare the intrinsic impedance, attenuation constant, and skin depth of copper σ_{cu}=5.8×10^7*S/m*, silver σ_{ag}=6.15×10^7*S/m*, and brass σ_{br}=1.59×10^7*S/m* at following frequencies: 60*Hz* and 1*GHz*.

(Sol.) $\alpha = \sqrt{\pi f \mu \sigma}$, $\delta = \dfrac{1}{\alpha}$, $f \uparrow \Rightarrow \delta \downarrow$, $\eta_c = (1+j)\dfrac{\alpha}{\sigma}$

Copper: 60*Hz*$\Rightarrow \eta_c = 2.02(1+j)\times10^{-6}\Omega$, $\alpha = 1.17\times10^{2}\, Np/m$, $\delta = 8.53\times10^{-3}\, m$

1*GHz*$\Rightarrow \eta_c = 8.25(1+j)\times10^{-3}\Omega$, $\alpha = 4.79\times10^{5}\, Np/m$, $\delta = 2.09\times10^{-6}\, m$

Group velocity: $v_g = \dfrac{d\omega}{d\beta} = \dfrac{1}{d\beta/d\omega}$



$$\bar{E}(t,z) = E_0 \cos[(\omega+\Delta\omega)t - (\beta+\Delta\beta)z] + E_0 \cos[(\omega-\Delta\omega)t - (\beta-\Delta\beta)z]$$

$$= 2E_0 \cos(t\Delta\omega - z\Delta\beta)\cos(\omega t - \beta z)$$

Let $t\Delta\omega - z\Delta\beta$ =constant $\Rightarrow v_g = \dfrac{dz}{dt} = \dfrac{\Delta\omega}{\Delta\beta} = \dfrac{1}{\Delta\beta/\Delta\omega} = \dfrac{d\omega}{d\beta} = \dfrac{1}{d\beta/d\omega}$

Eg. Show that $v_g = v_p + \beta \dfrac{dv_p}{d\beta}$ **and** $v_g = v_p - \lambda \dfrac{dv_p}{d\lambda}$

(Proof) $v_p = \dfrac{\omega}{\beta}$, $\omega = v_p \beta$, $v_g = \dfrac{d\omega}{d\beta} = v_p + \beta \dfrac{dv_p}{d\beta}$

$\because$ $\beta = \dfrac{2\pi}{\lambda}$, $\beta\lambda = 2\pi$, $\lambda d\beta + \beta d\lambda = 0 \Rightarrow \dfrac{\beta}{d\beta} = -\dfrac{\lambda}{d\lambda}$, $v_g = v_p - \lambda \dfrac{dv_p}{d\lambda}$

An example of longitudinal v_p>0 but longitudinal v_g=0 in barber's pole.

Eg. A 3*GHz*, *y*-polarized uniform plane wave propagates in the +*x* direction in a nonmagnetic medium having a dielectric constant 2.5 and a loss tangent 10^{-2}. (a) Determine the distance over which the amplitude of the propagating wave will be cut in half. (b) Determine the intrinsic impedance, the wavelength, the phase velocity, and the group velocity of the wave in the medium. (c) Assuming $\vec{E}=\hat{y}50\sin(6\pi 10^9 t+\frac{\pi}{3})$ *V/m* at *x*=0, write the instantaneous expression for $\vec{H}$ for all *t* and *x*.

(Sol.) $10^{-2}=\dfrac{\sigma}{\omega\varepsilon}<<1 \Rightarrow \sigma=10^{-2}\times 2\pi\times 3\times 10^9\times\dfrac{1}{36\pi}\times 10^{-9}\times 2.5=4.166\times 10^{-3}$

It is a low–loss dielectric material: $\beta=\omega\sqrt{\mu\varepsilon}[1+\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]=99.34 rad/m$

$$\eta_c\approx\sqrt{\frac{\mu}{\varepsilon}}(1+j\frac{\sigma}{2\omega\varepsilon})=238\angle 0.29^\circ\,\Omega$$

(a) $\alpha\approx\dfrac{\sigma}{2}\sqrt{\dfrac{\mu}{\varepsilon}}=0.497,\ e^{-0.497d}=\dfrac{1}{2}\Rightarrow d=1.395m$

(b) $v_p=\dfrac{\omega}{\beta}=1.8973\times 10^8 m/s,\ v_g=\dfrac{d\omega}{d\beta}=\dfrac{1}{(d\beta/d\omega)}=1.8975\times 10^8 m/s$

(c) $\vec{E}=\hat{y}50e^{-0.497x}\cdot e^{j\frac{\pi}{3}}\Rightarrow\vec{H}=\dfrac{1}{\eta_t}\hat{a}_n\times\vec{E}=\hat{z}0.21e^{-0.497x}\cdot e^{j(\frac{\pi}{3}-0.0016\pi)}$

$$\Rightarrow\vec{H}(x,t)=\hat{z}0.21e^{-0.497x}\cdot\sin(6\pi 10^9 t-99.34x+0.332\pi)\quad A/m$$

Plasma: Ionized gasses with equal electron and ion densities.

Ionosphere: 50~500 *Km* in altitude

Simple model of plasma: An electron of charge –*e*, mass *m*, position $\vec{x}$

$-e\vec{E}=m\dfrac{d^2\vec{x}}{dt^2}=-m\omega^2\vec{x}\Rightarrow\vec{x}=\dfrac{e}{m\omega^2}\vec{E}\Rightarrow$ Electric dipole $\vec{p}=-e\vec{x}=\dfrac{-e^2}{m\omega^2}\vec{E}$

$\therefore$ Total electric dipole moment: $\vec{P}=N\vec{p}=-\dfrac{Ne^2}{m\omega^2}\vec{E}$

$\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon_0(1-\dfrac{Ne^2}{m\omega^2\varepsilon_0})\vec{E}=\varepsilon_0(1-\dfrac{\omega_p{}^2}{\omega^2})\vec{E}$, where $\omega_p=\sqrt{\dfrac{Ne^2}{m\varepsilon_0}}$ is the plasma angular frequency, and the effective permittivity is $\varepsilon=\varepsilon_0(1-\dfrac{\omega_p{}^2}{\omega^2})=\varepsilon_0(1-\dfrac{f_p{}^2}{f^2})$.

Propagation constant: $\gamma=j\omega\sqrt{\mu\varepsilon_0}\cdot\sqrt{1-(\dfrac{f_p{}^2}{f^2})}$

Intrinsic impedance of the plasma: $\eta_c = \dfrac{\eta_0}{\sqrt{1-(\dfrac{f_p}{f})^2}}$ where $\eta_0 = 120\pi(\Omega)$

Case 1 $f<f_p$: γ is real, η_c is pure imaginary $\Rightarrow$ Attenuation $\Rightarrow$ EM wave is in cutoff.

Case 2 $f>f_p$: γ is pure imaginary, η_c is real $\Rightarrow$ EM wave can propagate through the plasma.