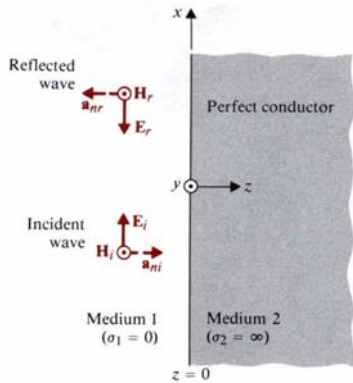# Chapter 3 Plane EM Waves and Lasers in Distinct Media

3-1 Normal Incidence at a Plane Conducting Boundary

$$\hat{a}_{ni} = +\hat{z}, \quad \hat{a}_{nr} = -\hat{z}$$

Incident plane wave:
$$\vec{E}_i(z) = \hat{x}E_{i0}e^{-j\beta_1 z}$$
$$\vec{H}_i(z) = \hat{y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1 z},$$

Reflective wave: $\vec{E}_r(z) = \hat{x}E_{r0}e^{j\beta_1 z}$,

Total field: $\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z)$

$\because$ Conducting boundary, $\therefore$ $\vec{E}_1(z=0) = 0 \Rightarrow E_{r0} = -E_{i0} \Rightarrow \vec{E}_r(z) = -\hat{x}\vec{E}_{i0}e^{+j\beta_1 z}$

$$\vec{H}_r(z) = \frac{1}{\eta_1}\hat{a}_{nr} \times \vec{E}_r(z) = \hat{y}\frac{E_{i0}}{\eta_1}e^{+j\beta_1 z} \Rightarrow \begin{cases} \vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = -\hat{x}j2E_{10}\sin\beta_1 z \\ \vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \hat{y}2\frac{E_{i0}}{\eta_1}\cos\beta_1 z \end{cases}$$

(a) E_1 versus z.

(b) H_1 versus z.

Standing wave:

Zero (null) of E_1 at $\beta_1 z = -n\pi$

Maximum of H_1 at $z = -n\lambda/2$

Zero (null) of H_1 at $\beta_1 z = -(2n+1)\pi/2$

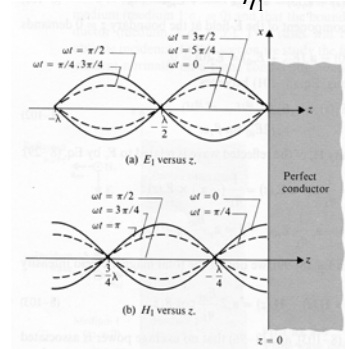Maximum of E_1 at $z = -(2n+1)\lambda/4$

Eg. A *y*-polarized plane wave, *f*=100*MHz*, amplitude=6*mV/m*, propagates +*x* direction and impinges normally on a conducting plane at *x*=0. (a) Write $\vec{E}_i$, $\vec{H}_i$, $\vec{E}_r$, $\vec{H}_r$, $\vec{E}_1$, and $\vec{H}_1$.

(b) Determine the location nearest to the conducting plane when $\vec{E}_1$=0.

(Sol.) $\omega = 2\pi f = 2\pi \times 10^8$, $\beta_1 = \frac{\omega}{c} = \frac{2\pi}{3}$, $\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\Omega$, $\hat{a}_{ni} = \hat{x}$, $\hat{a}_{nr} = -\hat{x}$

(a) $\vec{E}_i(x) = \hat{y}6\times 10^{-3}e^{-j\frac{2\pi}{3}x}$, $\vec{H}_i(x) = \frac{1}{\eta_1}\hat{a}_{ni}\times\vec{E}_i(x) = \hat{z}\frac{10^{-4}}{2\pi}e^{-j\frac{2\pi}{3}x}$, $\vec{E}_r(x) = -\hat{y}6\times 10^{-3}e^{+j\frac{2\pi}{3}x}$,

$\vec{H}_r(x) = \hat{z}\frac{10^{-4}}{2\pi}e^{+j\frac{2\pi}{3}x}$, $\vec{E}_1(x) = \vec{E}_i(x) + \vec{E}_r(x) = -\hat{y}j12\times 10^{-3}\sin(\frac{2\pi}{3}x)$,

$\vec{H}_1(x) = \vec{H}_i(x) + \vec{H}_r(x) = \hat{z}\frac{10^{-4}}{\pi}\cos(\frac{2\pi}{3}x)$ (b) $\frac{2\pi x}{3} = -\pi$, $x = -\frac{3}{2}$

Eg. A right-hand circularly plane wave represented by $E(z) = E_0(\hat{x} - j\hat{y})e^{-j\beta z}$

impinges normally on a perfectly conducting wall at $z=0$. (a) Determine the polarization of the reflected wave. (b) Find the induced current on the conducting wall.

(Sol.) (a) $\vec{E}_r(z) = (\hat{x}E_{rx} + \hat{y}E_{ry})e^{j\beta z}$, $\vec{E}_i(0) + \vec{E}_r(0) = 0 \Rightarrow E_{rx}=-E_0, E_{ry}=jE_0,$

$\therefore$ $\vec{E}_r(z) = (-\hat{x}E_0 + \hat{y}jE_0)e^{j\beta z}$ is left-hand circularly-polarized and $-z$-propagated.

(b) $\vec{J} = \hat{a}_n \times (\vec{H}_1 - \vec{H}_2) = \hat{a}_{ni} \times \vec{H}_1 \ (\because \vec{H}_2 = 0)$, $\hat{a}_{ni} = \hat{z}$, $\hat{a}_{nr} = -\hat{z}$

$\vec{H}_i(z) = \dfrac{1}{\eta_0}\hat{a}_{ni} \times \vec{E}_i = \dfrac{E_0}{\eta_0}(\hat{x}j + \hat{y})e^{-j\beta z}$, $\vec{H}_r(z) = \dfrac{1}{\eta_0}\hat{a}_{nr} \times \vec{E}_r = \dfrac{E_0}{\eta_0}(\hat{x}j + \hat{y})e^{j\beta z} \Rightarrow \vec{H}_1(z) = \dfrac{E_0}{\eta_0}(\hat{x}j + \hat{y})(e^{-j\beta z} + e^{j\beta z})$

$\vec{J}(z) = \dfrac{E_0}{\eta_0}(-\hat{x} + \hat{y}j)(e^{-j\beta z} + e^{j\beta z})$

3-2 Oblique Incidences at a Plane Conducting Boundary
Case 1 Perpendicular Polarization (TE):
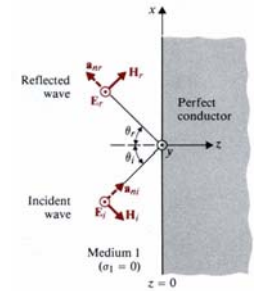$\hat{a}_{ni} = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i$, $\hat{a}_{nr} = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r$

Incidence plane wave: $\vec{E}_i(x,z) = \hat{y}E_{i0}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$

$\vec{H}_i(x,z) = \dfrac{1}{\eta_1}[\hat{a}_{ni} \times \vec{E}_i(x,z)] = \dfrac{E_{i0}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$

$\because$ $\vec{E}_1(x,0) = \vec{E}_i(x,0) + \vec{E}_r(x,0) = 0$, $\therefore$ $E_{r0}=-E_{i0}, \theta_r=\theta_i$

$\vec{E}_r(x,z) = -\hat{y}E_{i0}e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$, $\vec{H}_r(x,z) = \dfrac{1}{\eta_1}[\hat{a}_{nr} \times \vec{E}_r(x,z)] = \dfrac{E_{i0}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$
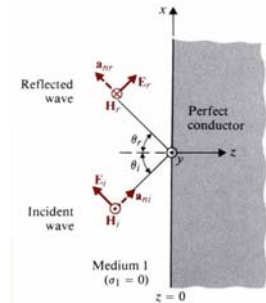
Case 2 Parallel Polarization (TM):
$\hat{a}_{ni} = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i$, $\hat{a}_{nr} = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r$

Incident plane wave: $\vec{H}_i(x,z) = \hat{y}\dfrac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$

$\vec{E}_i(x,z) = -\eta_1[\hat{a}_{ni} \times \vec{H}_i(x,z)] = E_{i0}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$

$\because$ $\vec{E}_1(x,0) = \vec{E}_i(x,0) + \vec{E}_r(x,0) = 0$, $\therefore$ $E_{r0}=-E_{i0}, \theta_r=\theta_i$

$\vec{E}_r(x,z) = -E_{i0}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$, $\vec{H}_r(x,z) = \hat{y}\dfrac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$

Eg. A uniform sinusoidal plane wave $E_i(x,y) = \hat{y}10e^{-j(6x+8z)}$ **in air is incident on a perfectly conducting plane at z=0. (a) Find the frequency and wavelength of the wave. (b) Determine the incident angle. (c) Find** $\bar{E}_r(x,z)$ **and** $\bar{H}_r(x,z)$ **of the reflected wave.**
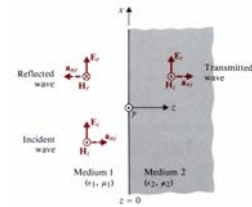
(Sol.) TE case:

(a) $\beta^2 = 6^2 + 8^2 \Rightarrow \beta = \dfrac{\omega}{v_p} \Rightarrow \omega = 3\times10^9,\ f = \dfrac{3\times10^9}{2\pi},\ \lambda = \dfrac{\pi}{5}.$

(b) $\tan\theta_i = \dfrac{3}{4} \Rightarrow \theta_i = \tan^{-1}(\dfrac{3}{4}) \approx 37°$

(c) $E_r = -10 \Rightarrow \bar{E}_r(x,z) = -\hat{y}10e^{-j(6x-8z)},\ \ \bar{H}_r = \dfrac{1}{\eta_0}\hat{a}_{nr} \times \bar{E}_r(x,z) = -\dfrac{1}{12\pi}(\dfrac{\hat{x}4}{5} + \dfrac{\hat{z}3}{5})e^{-j(6x-8z)}$

3-3 Normal Incidence at a Plane Dielectric Boundary

Incident plane wave: $\begin{cases} \bar{E}_i(z) = \hat{x}E_{io}e^{-j\beta_1 z} \\ \bar{H}_i(z) = \hat{y}\dfrac{E_{io}}{\eta_1}e^{-j\beta_1 z} \end{cases}$, Reflected wave:

$\begin{cases} \bar{E}_r(z) = \hat{x}E_{ro}e^{j\beta_1 z} \\ \bar{H}_r(z) = -\hat{y}\dfrac{E_{ro}}{\eta_1}e^{j\beta_1 z} \end{cases}$ Transmitted wave: $\begin{cases} \bar{E}_t(z) = \hat{x}E_{to}e^{-j\beta_2 z} \\ \bar{H}_t(z) = \hat{y}\dfrac{E_{to}}{\eta_2}e^{-j\beta_2 z} \end{cases}$

Continuity of tangential field component at z=0

$\Rightarrow \begin{cases} E_i(0) + E_r(0) = E_t(0) \\ H_i(0) + H_r(0) = H_t(0) \end{cases} \Rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ \dfrac{1}{\eta_1}(E_{i0} - E_{r0}) = \dfrac{E_{t0}}{\eta_2} \end{cases} \Rightarrow \begin{cases} E_{r0} = \dfrac{\eta_2 - \eta_1}{\eta_1 + \eta_2}E_{i0} \\ E_{t0} = \dfrac{2\eta_2}{\eta_2 + \eta_1}E_{i0} \end{cases}$

Reflection coefficient: $\Gamma = \dfrac{E_{r0}}{E_{i0}} = -\dfrac{H_{r0}}{H_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

Transmission coefficient: $\tau = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ and $1+\Gamma = \tau \Rightarrow \begin{cases} E_t(z) = \hat{x}\tau E_{i0}e^{-j\beta_2 z} \\ H_t(z) = \hat{y}\dfrac{\tau E_{i0}e^{-i\beta_2 z}}{\eta_2} \end{cases}$

Case 1 $\Gamma > 0$ ($\eta_2 > \eta_1$): Maximum at $2\beta_1 z_{max} = -2n\pi$ or $z_{max} = -n\lambda_1/2 = -n\pi/\beta_1$
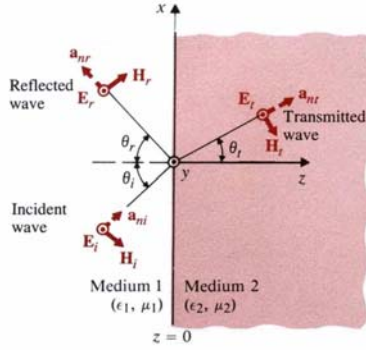Minimum at $2\beta_1 z_{min} = -(2n+1)\pi$ or $z_{min} = -(2n+1)\lambda_1/4 = -(2n+1)\pi/2\beta_1$
Case 2 $\Gamma < 0$ ($\eta_2 < \eta_1$): Max at $z_{max} = -(2n+1)\lambda_1/4$. Min at $z_{min} = -n\lambda_1/2$

Standing wave ratio (SWR): $S = \dfrac{|E|_{max}}{|E|_{min}} = \dfrac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \dfrac{S-1}{S+1}$

Eg. What condition $\left|\vec{E}_r\right| = \left|\vec{E}_t\right|$ **occurs in normal incident case? And Standing wave ratio =?**

(Sol.) $\left|\dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right| = \left|\dfrac{2\eta_2}{\eta_2 + \eta_1}\right| \Rightarrow \begin{cases} \eta_1 = -\eta_2 \, (not \; reasonable) \\ \eta_1 = 3\eta_2 \Rightarrow \Gamma = -\dfrac{1}{2}, \; \left|\Gamma\right| = \dfrac{1}{2} \end{cases}, \; S = \dfrac{1 + \left|\Gamma\right|}{1 - \left|\Gamma\right|} = 3$

3-4 Oblique Incidences at a Plane Dielectric Boundary

Case 1 Perpendicular Polarization (TE):

$$\hat{a}_{ni} = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i,$$

$$\hat{a}_{nr} = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r,$$

$$\hat{a}_{nt} = \hat{x}\sin\theta_t + \hat{z}\cos\theta_t.$$

Incident plane wave:

$$\begin{cases} \vec{E}_i(x,z) = \hat{y}E_{io}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\ \vec{H}_i(x,z) = \dfrac{E_{io}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \end{cases},$$

Reflected wave:
$$\begin{cases} \vec{E}_r(x,z) = \hat{y}E_{ro}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \vec{H}_r(x,z) = \dfrac{E_{ro}}{\eta_1}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \end{cases},$$

Transmitted (refractive) wave:
$$\begin{cases} \vec{E}_t(x,z) = \hat{y}E_{to}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \\ \vec{H}_t(x,z) = \dfrac{E_{to}}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases},$$

Continuity of tangential field components at $z=0 \Rightarrow \begin{cases} E_{iy}(x,0) + E_{ry}(x,0) = E_{ty}(x,0) \\ H_{ix}(x,0) + H_{rx}(x,0) = H_{tx}(x,0) \end{cases}$

$$\Rightarrow \begin{cases} E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_i} = E_{t0}e^{-j\beta_2\sin\theta_t} \\ \dfrac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}) = -\dfrac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t} \end{cases}$$

$$\Rightarrow \begin{cases} \beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t,\ E_{i0} + E_{r0} = E_{t0} \\ \dfrac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \dfrac{E_{t0}}{\eta_2}\cos\theta_t \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} = \dfrac{n_1}{n_2} = \dfrac{v_{p2}}{v_{p1}} = \dfrac{\eta_2}{\eta_1},\ \theta_i = \theta_r \\ \Gamma_\perp = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i},\ \tau_\perp = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \end{cases}$$

and $1+\Gamma_\perp=\tau_\perp$.

Note: Snell's Law holds only in case of lossless media.

Brewster angle: No reflection occurs when $\sin\theta_{B\perp} = \sqrt{\dfrac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-(\mu_1/\mu_2)^2}} = \dfrac{1}{\sqrt{1+(\mu_1/\mu_2)}}$

if $\varepsilon_1=\varepsilon_2,\ \mu_1\neq\mu_2$

(Proof) $\Gamma_\perp=0 \Rightarrow \eta_2 \cos\theta_i = \eta_1 \cos\theta_t$,

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_i} = \frac{\eta_2}{\eta_1}\cos\theta_i = \frac{\sqrt{\mu_1/\varepsilon_1}}{\sqrt{\mu_2/\varepsilon_2}}\cos\theta_i \Rightarrow \sin\theta_{B\perp} = \sqrt{\frac{1-\mu_1\varepsilon_2/\mu_2\varepsilon_1}{1-(\mu_1/\mu_2)^2}}.$$

Eg. A uniform plane wave $\vec{E}_i(x,z) = \hat{y}6e^{-j(4x+3z)}$ in medium 1 ($4\varepsilon_0,\mu_0$) is incident on a plane of medium 2 ($64\varepsilon_0/9,\mu_0$) at $z=0$. (a) Is it in perpendicular polarization or parallel polarization? (b) Find the frequency and wavelength of the wave in medium 1. (c) Write down the reflected angle and the refractive angle of the transmitted wave. (d) Write down the unit propagation vectors of the incident wave $\hat{a}_{ni}$, the reflected wave $\hat{a}_{nr}$, and the transmitted wave $\hat{a}_{nt}$, respectively. (e) Compute the reflection and the transmission coefficients. (f) Find $\vec{E}_r(x,z)$ and $\vec{H}_r(x,z)$ of the reflected wave. (g) Find $\vec{E}_t(x,z)$ and $\vec{H}_t(x,z)$ of the transmitted wave.

(Sol.) (a) $\because \hat{y} \perp xz-plane$, $\therefore$ Perpendicular polarization (TE)

(b) $\beta_1 = \sqrt{4^2+3^2} = 5 = \frac{2\pi}{\lambda} = \omega\sqrt{\mu_0 4\varepsilon_0} = \frac{\omega 2}{c} \Rightarrow \lambda = \frac{2\pi}{5}$, $f = \frac{15}{4\pi}\times 10^8$

(c) $\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{4}{3} \Rightarrow \sin\theta_t = \frac{3}{4}\sin\theta_i = \frac{3}{5}$, $\cos\theta_t = \frac{4}{5}$

(d) $\hat{a}_{ni} = \frac{4\hat{x}+3\hat{z}}{5}$, $\hat{a}_{nr} = \frac{4\hat{x}-3\hat{z}}{5}$, $\hat{a}_{nt} = \frac{3\hat{x}+4\hat{z}}{5}$

(e) $\eta_1 = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = 60\pi$, $\eta_2 = \sqrt{\frac{\mu_0}{(64\varepsilon_0/9)}} = 45\pi$

$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = -\frac{7}{25}$, $\tau_\perp = \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = \frac{18}{25}$

(f) $\vec{E}_r(x,z) = \Gamma_\perp \cdot \hat{y}6e^{-j(4x-3z)} = -\hat{y}\frac{42}{25}e^{-j(4x-3z)}$

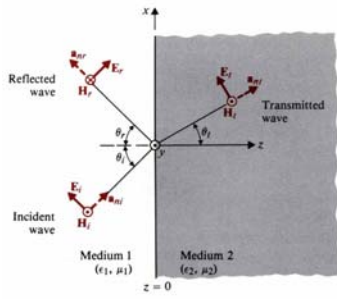$\vec{H}_r(x,z) = \frac{1}{\eta_1}(\hat{a}_{nr}\times\vec{E}_r) = \frac{-7}{1250\pi}(3\hat{x}+4\hat{z})e^{-j(4x-3z)}$

(g) In medium 2, $\beta_2 = \frac{15}{2}\times 10^8 \cdot \sqrt{\mu_0 \cdot \frac{64}{9}\varepsilon_0} = \frac{20}{3}$

$\vec{E}_t(x,z) = \tau_\perp \cdot \hat{y}6e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} = \hat{y}\frac{108}{25}e^{-j(4x+\frac{16}{3}z)}$

$\vec{H}_t(x,z) = \frac{1}{\eta_2}(\hat{a}_{nt}\times\vec{E}_t) = \frac{27}{125\pi}(-\frac{4}{5}\hat{x}+\frac{3}{5}\hat{z})e^{-j(4x+\frac{16}{3}z)}$

Case 2 Parallel Polarization (TM):

$$\hat{a}_{ni} = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i, \quad \hat{a}_{nr} = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r$$

$$\hat{a}_{nt} = \hat{x}\sin\theta_t + \hat{z}\cos\theta_t$$

Incident plane wave:

$$\begin{cases} \vec{H}_i(x,z) = \hat{y}\dfrac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \\[2mm] \vec{E}_i(x,z) = E_{i0}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \end{cases},$$

Reflected wave:
$$\begin{cases} \vec{E}_r(x,z) = E_{ro}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\[2mm] \vec{H}_r(x,z) = -\hat{y}\dfrac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \end{cases},$$

Transmitted wave:
$$\begin{cases} \vec{E}_t(x,z) = E_{to}(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \\[2mm] \vec{H}_t(x,z) = \hat{y}\dfrac{E_{t0}}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases}.$$

Continuity of tangential field components at $z=0 \Rightarrow \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} = \dfrac{v_{p2}}{v_{p1}} = \dfrac{\eta_2}{\eta_1}$

$$\begin{cases} (E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t \\[2mm] \dfrac{1}{\eta_1}(E_{i0} - E_{r0}) = \dfrac{E_{t0}}{\eta_2} \end{cases} \Rightarrow \begin{cases} \Gamma_\| = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \\[3mm] \tau_\| = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \end{cases}, \text{ and } 1 + \Gamma_\| = \tau_\| \cdot \dfrac{\cos\theta_t}{\cos\theta_i}$$

Brewster angle: No reflection occurs when $\sin\theta_{B\|} = \sqrt{\dfrac{1 - \mu_2\varepsilon_1/\mu_1\varepsilon_2}{1 - (\varepsilon_1/\varepsilon_2)^2}} = \dfrac{1}{\sqrt{1 + (\varepsilon_1/\varepsilon_2)}}$

If $\mu_1 = \mu_2 \Rightarrow \theta_{B\|} = \tan^{-1}\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \tan^{-1}(\dfrac{n_1}{n_2})$.

Eg. A uniform plane wave $\vec{E}_i(x,z) = (-3\hat{x} + 4\hat{z})e^{-j(8x+6z)}$ **in air is incident on a plane of medium ($16\varepsilon_0/9, \mu_0$) at z=0. (a) Is it in perpendicular polarization or parallel polarization? (b) Find the frequency and wavelength of the wave in medium 1. (c) Write down the refractive angle θ_t of the transmitted wave. Is θ_i equal to the Brewster angle? (d) Write down the unit propagation vectors of the incident wave $\hat{a}_{ni}$, the reflected wave $\hat{a}_{nr}$, and the transmitted wave $\hat{a}_{nt}$, respectively. (e) Compute the reflection and the transmission coefficients. (f) Find** $\vec{E}_r(x,z)$ **and** $\vec{H}_r(x,z)$ **of the reflected wave. (g) Find** $\vec{E}_t(x,z)$ **and** $\vec{H}_t(x,z)$ **of the transmitted wave.**

(Sol.) (a) $\vec{H}_i = \dfrac{1}{\eta_0}(\hat{a}_{ni} \times \vec{E}) = -\hat{y}5e^{-j(8x+6z)}$, $\therefore$ Parallel polarization (*TM*)

(b) $\beta_1 = \sqrt{8^2 + 6^2} = 10 = \dfrac{2\pi}{\lambda} = \omega\sqrt{\mu_0\varepsilon_0} \Rightarrow \lambda = \dfrac{\pi}{5}m$

$\omega = 3 \times 10^9 \Rightarrow f = \dfrac{15}{\pi} \times 10^8$

(c) $\theta_i = \sin^{-1}(\dfrac{4}{5})$, $\sin\theta_B = \dfrac{1}{\sqrt{1 + (\dfrac{\varepsilon_1}{\varepsilon_2})}} = \dfrac{4}{5}$, $\therefore \theta_i = \theta_B$

$\dfrac{\sin\theta_i}{\sin\theta_t} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \dfrac{4}{3} \Rightarrow \sin\theta_t = \dfrac{4}{5} \times \dfrac{3}{4} = \dfrac{3}{5} \Rightarrow \theta_t = \sin^{-1}(\dfrac{3}{5})$, $\cos\theta_t = \dfrac{4}{5}$

(d) $\hat{a}_{ni} = \dfrac{4\hat{x} + 3\hat{z}}{5}$, $\hat{a}_{nr} = 0$, $\hat{a}_{nt} = \dfrac{3\hat{x} + 4\hat{z}}{5}$

(e) $\eta_1 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 120\pi$, $\eta_2 = \sqrt{\dfrac{\mu_0}{(16\varepsilon_0/9)}} = 90\pi \Rightarrow \Gamma_{\parallel} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = 0$,

$\tau_{\parallel} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \dfrac{3}{4}$

(f) $\vec{E}_r(x,z) = \vec{H}_r(x,z) = 0$

(g) In medium 2, $\beta_2 = \omega\sqrt{\mu_0 \cdot \dfrac{16}{9}\varepsilon_0} = \dfrac{40}{3}$

$\vec{E}_t(x,z) = \tau_{\parallel} \cdot (-3\hat{x} + 4\hat{z})e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} = (-\dfrac{9}{4}\hat{x} + 3\hat{z})e^{-j(8x + \frac{32}{3}z)}$

$\vec{H}_t(x,z) = \dfrac{1}{\eta_2}(\hat{a}_{nt} \times \vec{E}_t) = \dfrac{1}{90\pi}(-\dfrac{18}{5}\hat{y})e^{-j(8x+\frac{32}{3}z)} = -\hat{y}\dfrac{1}{25\pi}e^{-j(8x+\frac{32}{3}z)}$

Snell's Law: $\dfrac{\sin\theta_t}{\sin\theta_i}=\dfrac{\beta_1}{\beta_2}=\dfrac{n_1}{n_2}=\dfrac{v_{p2}}{v_{p1}}=\dfrac{\eta_2}{\eta_1}=\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}}=\sqrt{\dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}}$ and $\theta_r=\theta_i$
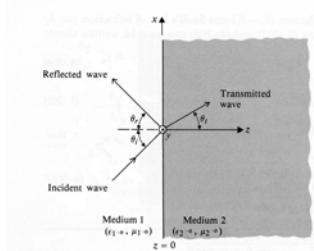
The reflection coefficient: $\Gamma=\dfrac{E_{r0}}{E_{i0}}$ and **the transmission coefficient:** $\tau=\dfrac{E_{t0}}{E_{i0}}$

$$\begin{cases}\Gamma_\perp=\dfrac{\eta_2/\cos\theta_t-\eta_1/\cos\theta_i}{\eta_2/\cos\theta_t+\eta_1/\cos\theta_i}=\dfrac{n_1\cos\theta_i-n_2\cos\theta_t}{n_1\cos\theta_i+n_2\cos\theta_t}=\dfrac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)}\\[4mm]\tau_\perp=\dfrac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t+\eta_1/\cos\theta_i}=\dfrac{2n_1\cos\theta_i}{n_1\cos\theta_i+n_2\cos\theta_t}=\dfrac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)}\end{cases}$$

for perpendicular polarization (TE)

$$\begin{cases}\Gamma_\parallel=\dfrac{\eta_2\cos\theta_t-\eta_1\cos\theta_i}{\eta_2\cos\theta_t+\eta_1\cos\theta_i}=\dfrac{n_1/\cos\theta_i-n_2/\cos\theta_t}{n_1/\cos\theta_i+n_2/\cos\theta_t}=\dfrac{\tan(\theta_t-\theta_i)}{\tan(\theta_t+\theta_i)}\\[4mm]\tau_\parallel=\dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t+\eta_1\cos\theta_i}=\dfrac{2n_1/\cos\theta_t}{n_1/\cos\theta_i+n_2/\cos\theta_t}=\dfrac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}\end{cases}$$

for parallel polarization (TM)

The media are called right-handed materials in case ε, μ, and n are positive. All the natural media are right-handed. However, some artificial materials, such as a few photonic crystals, may be left-handed. It implies that the effective values of ε, μ, and n are negative. The refractive angle θ_t, as well as θ_i, is positive in case of obliquely-incident light passing the interface between the distinct right-handed media. That is, the incident light and the transmitted light are the opposite sides of the normal line of the interface.

Example of an obliquely-incident laser beam passing a slab made of left-handed medium:
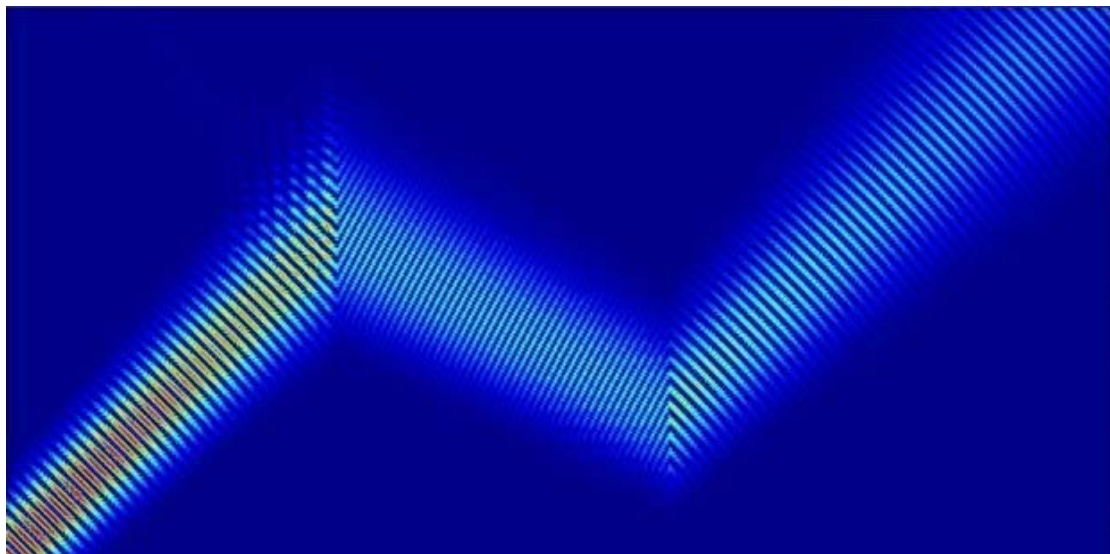
Eg. A light ray is incident from air obliquely on a transparent sheet of thickness d with an index of refraction as shown in the figure. The angle of incidence is θ_i. Find (a) θ_t, (b) the distance l_1 at the point of exit, and (c) the amount of the lateral displacement l_2 of the emerging ray.

(Sol.) (a) $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{1}{n} \Rightarrow \theta_t = \sin^{-1}\left(\dfrac{\sin\theta_i}{n}\right)$

(b) $\ell_1 = d\tan\theta_t = \dfrac{d\sin\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}$
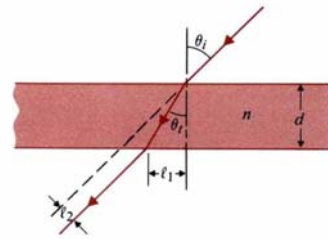
(c) $\ell_2 = \sqrt{\ell_1^2 + d^2}\cdot\sin(\theta_i - \theta_t) = \sqrt{\dfrac{d^2\sin^2\theta_i}{n^2 - \sin^2\theta_i} + d^2}\cdot[\sin\theta_i\cos\theta_t - \cos\theta_i\sin\theta_t] = d\sin\theta_i - \dfrac{d\sin\theta_i\cos\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}$

Rigid Body (sings).

Gin a body meet a body
Flyin' through the air,
Gin a body hit a body,
Will it fly? and where?
Ilka impact has its measure,
Ne'er a ane hae I,
Yet a' the lads they measure me,
Or, at least, they try.

Gin a body meet a body
Altogether free,
How they travel afterwards
We do not always see.
Ilka problem has its method
By analytics high;
For me, I ken na ane o' them,
But what the waur am I?

by James Clerk Maxwell

3-5 Total Reflection and Critical Angle (θ_c)

In case of $\varepsilon_1 > \varepsilon_2$ (or $n_1 > n_2$):

Define critical angle as $\theta_c \equiv \sin^{-1}(\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}) = \sin^{-1}(\dfrac{n_2}{n_1})$,

While $\theta_i > \theta_c \Rightarrow \sin\theta_t = \sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i > 1$

$\Rightarrow \cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \pm j\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$

$\hat{a}_{nt} = \hat{x}\sin\theta_t + \hat{z}\cos\theta_t$, $\vec{E}_t$ and $\vec{H}_t \propto e^{-j\beta_2 z\cos\theta_t}\cdot e^{-j\beta_2 x\sin\theta_t} = e^{-\alpha_2 z}\cdot e^{-j\beta_{2s}x} \to 0$ as

$z \to \infty$, where $\alpha_2 = \beta_2\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}\sin^2\theta_t - 1}$ and $\beta_{2x} = \beta_2\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\sin\theta_i$
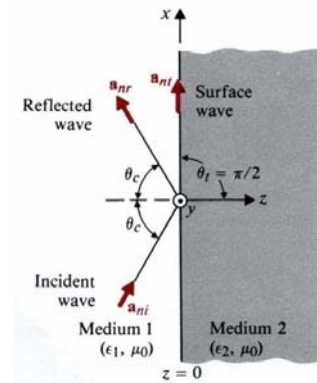
Eg. An electromagnetic wave from an underwater source with perpendicular polarization is incident on a water-air interface at $\theta_i=20°$. Using $\mu_r=1$ and $\varepsilon_r=81$ for fresh water, find (a) critical angle θ_c, (b) reflection coefficient $\Gamma_\perp$, (c) transmission coefficient $\tau_\perp$, and (d) attenuation in dB for each wavelength into the air.

(Sol.) $\eta_2 = \eta_0 = 120\pi$, $\eta_1 = \dfrac{40\pi}{3}$, $\theta_i = 20°$, $\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - (9\sin 20°)^2}$

(a) $\theta_c = \sin^{-1}(\sqrt{\dfrac{1}{81}}) = 6.38°$, (b) $\Gamma_\perp = \dfrac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = e^{j0.66}$, (c) $\tau_\perp = 1 + \Gamma_\perp = 1.89e^{j0.33}$,

(d) $\alpha_2 = \beta_2\cdot\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$.

Attenuation per wavelength is $-20\log e^{-2\alpha_2\lambda_2} = 159$dB, where $\lambda_2 = \dfrac{\lambda_0}{\sqrt{81}} = \dfrac{\lambda_0}{9}$

Eg. ε of water at optical frequency $=1.75\varepsilon_0$, find d at a distance under water yields an illuminated circular area of a radius $r=5m$.

(Sol.) $\theta_c = \sin^{-1}\sqrt{\dfrac{1}{1.75}} = 49.2°$, $d\tan 49.2° = 5 \Rightarrow d = 4.32m$
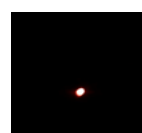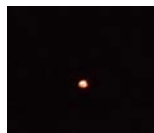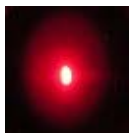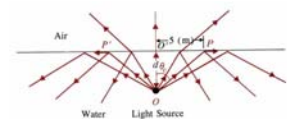
Eg. For preventing interference of waves in neighboring fibers and for mechanical protection, individual optical fibers are usually cladded by a material of a lower refractive index, as shown in the figure, where $n_1 > n_2$. (a) Express the maximum angle of incident θ_a in terms of n_0, n_1 and n_2 for meridional rays incident on the core's end face to be trapped inside the core by total internal reflection. (Meridional rays are those that pass through the fiber axis. The angle θ_a is called the acceptance angle, $\sin(\theta_a)$ is the numerical aperture (NA) of the fiber.)
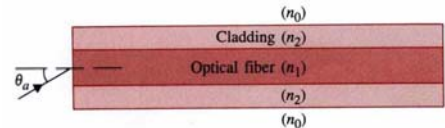
(Sol.) $\sin\phi = \dfrac{n_0}{n_1}\sin\theta_a$, $\varphi = \dfrac{\pi}{2} - \phi > \theta_c = \sin^{-1}(\dfrac{n_2}{n_1})$

$\sin\theta_c = \dfrac{n_2}{n_1} < \sin\varphi = \cos\phi = \sqrt{1 - \dfrac{n_0^2 \sin^2\theta_a}{n_1^2}}$

$\dfrac{n_2^2}{n_1^2} < 1 - \dfrac{n_0^2 \sin^2\theta_a}{n_1^2}$, $\sin\theta_a < \dfrac{\sqrt{n_1^2 - n_2^2}}{n_0}$, where $n_0=1$

Eg. Determine ε_{r1} such that light is confined within the rod.
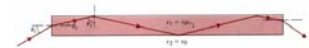
(Sol.) $\sin\theta_1 \geq \sin\theta_c = \sqrt{\dfrac{1}{\varepsilon_{r1}}}$, $\theta_1 = \dfrac{\pi}{2} - \theta_t \Rightarrow \cos\theta_t \geq \sqrt{\dfrac{1}{\varepsilon_{r1}}}$ ...(1)

$\sin\theta_t = \sqrt{\dfrac{1}{\varepsilon_{r1}}} \cdot \sin\theta_i$ ...(2) $\Rightarrow \sqrt{1 - \dfrac{\sin^2\theta_i}{\varepsilon_{r1}}} \geq \sqrt{\dfrac{1}{\varepsilon_{r1}}} \Rightarrow \varepsilon_{r1} \geq 1 + \sin^2\theta_i$

Eg. Assuming $\varepsilon_r=4$ for glass, calculate the percentage of the incident light power reflected back by the prism.

(Sol.) $\theta_c = \sin^{-1}\sqrt{\dfrac{1}{4}} = 30° < 45°$, $\eta_{air} = 120\pi$, $\eta_{prism} = 60\pi$

$\tau_{air-prism} = \dfrac{2 \times 60\pi}{120\pi + 60\pi} = \dfrac{2}{3}$, $\tau_{prism-air} = \dfrac{2 \times 120\pi}{120\pi + 60\pi} = \dfrac{4}{3}$

$\tau_{total} = \left|\dfrac{2}{3}\right|^2 \cdot 1^2 \cdot 1^2 \cdot \left|\dfrac{4}{3}\right|^2 = \dfrac{64}{81} \approx 79\%$
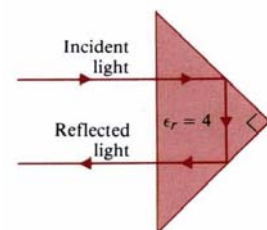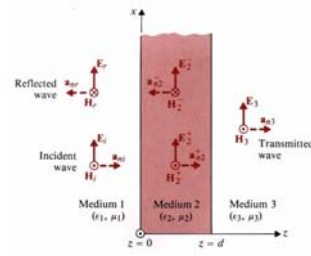

Note: The **depth of focus (DOF)** of an optical imaging system is usually defined as **DOF=$0.5\lambda^2/NA^2$.**

3-6 Normal Incidence at Three-layer Dielectric Interfaces

In medium 1,

$$\begin{cases} \bar{E}_1 = \hat{x}(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z}) \\ \bar{H}_1 = \hat{y}\dfrac{1}{\eta_1}(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z}) \end{cases}, \quad E_{r0} = \Gamma_0 \cdot E_{i0}$$

In medium 2,

$$\begin{cases} \bar{E}_2 = \hat{x}(E_2{}^+ e^{-j\beta_2 z} + E_2{}^- e^{j\beta_2 z}) \\ \bar{H}_2 = \hat{y}\dfrac{1}{\eta_2}(E_2{}^+ e^{-j\beta_2 z} - E_2{}^- e^{j\beta_2 z}) \end{cases} \quad \text{In medium 3,} \quad \begin{cases} \bar{E}_3 = \hat{x}E_3{}^+ e^{-j\beta_3 z} \\ \bar{H}_3 = \hat{y}\dfrac{E_3{}^+}{\eta_3} e^{-j\beta_3 z} \end{cases}$$

$\because$ Continuity of tangential field components at z=0 & z=d,

$\therefore$ $E_1(0)=E_2(0)$, $H_1(0)=H_2(0)$, $E_2(d)=E_3(d)$, $H_2(d)=H_3(d)$

Define $Z_2(0) = \eta_2 \cdot \dfrac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d} = \eta_2 \cdot \dfrac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}$

$\Rightarrow$ **Effective reflection coefficient** $\Gamma_0 = \dfrac{E_{r0}}{E_{i0}} = -\dfrac{H_{r0}}{H_{i0}} = \dfrac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1}$

Eg. If no reflection occurs, find the relation among d, η_1, η_2, and η_3.

$$\Gamma_0 = 0 = \dfrac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1} \Rightarrow \eta_2(\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d) = \eta_1\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d)$$

(Sol.)

$$\Rightarrow \begin{cases} \eta_3 \cos\beta_2 d = \eta_1 \cos\beta_2 d \\ \eta_2{}^2 \sin\beta_2 d = \eta_1\eta_3 \sin\beta_2 d \end{cases} \Rightarrow \begin{cases} \eta_2 = \sqrt{\eta_1\eta_3} \\ \cos\beta_2 d = 0 \end{cases} or \begin{cases} \eta_1 = \eta_3 \\ \sin\beta_2 d = 0 \end{cases}$$

Case 1:

$$\begin{cases} \eta_2 = \sqrt{\eta_1\eta_3} \ or \ n_2 = \sqrt{n_1 n_3} \\ d = \dfrac{(2n+1)}{4}\lambda_2, \ n = 0,1,2,... \end{cases}$$

Case 2:

$$\begin{cases} \eta_1 = \eta_3 \ or \ n_1 = n_3 \\ d = \dfrac{n\lambda_2}{2}, \ n = 0,1,2,... \end{cases}$$

(Quarter-wave impedance transformer)   (Half-wave impedance transformer)

Eg. A transparent dielectric coating is applied to glass (μ_r=1, ε_r=4) to eliminate the reflection of red light λ_0=0.75μm. Determine the required dielectric constant and thickness of the coating.

(Sol.) (a) $\eta_1 = \eta_0 = 120\pi$, $\eta_3 = 60\pi$

$\eta_2 = \sqrt{\eta_1\eta_3} = \dfrac{120\pi}{\sqrt{2}} = \dfrac{120\pi}{\sqrt{\varepsilon_r}} \Rightarrow \varepsilon_r = 2 \Rightarrow n_2 = \sqrt{2}$, $d = \dfrac{(2n+1)}{4}\lambda_2 = \dfrac{(2n+1)}{4} \cdot \dfrac{0.75}{\sqrt{2}}$, n=0,1,2,...

3-7 Optical Theory of Multi-Layer Films
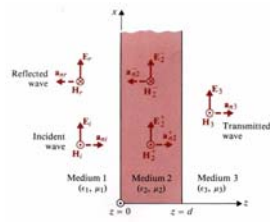
Normal incidence: At $z=0$:
$$\begin{cases} E_i + E_r = E_2 + E_2{}' \\ n_1 E_i - n_1 E_r = n_2 E_2 - n_2 E_2{}' \end{cases}, \quad z=d:$$

$$\begin{cases} E_2 e^{-jk_2 d} + E_2{}' e^{jk_2 d} = E_3 \\ n_2 E_2 e^{-jk_2 d} - n_2 E_2{}' e^{jk_2 d} = n_3 E_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 \\ n_1 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_1 \end{bmatrix} \cdot \left(\frac{E_r}{E_i}\right) = \begin{bmatrix} \cos(k_2 d) & \dfrac{-j\sin(k_2 d)}{n_2} \\ -jn_2 \sin(k_2 d) & \cos(k_2 d) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ n_3 \end{bmatrix} \cdot \left(\frac{E_3}{E_i}\right)$$

Or, $\begin{bmatrix} 1 \\ n_1 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_1 \end{bmatrix} \cdot \Gamma = M \cdot \begin{bmatrix} 1 \\ n_3 \end{bmatrix} \cdot \tau$, where $k_2 = 2\pi n_2/\lambda$, $\Gamma = E_r/E_i$, $\tau = E_3/E_i$

Optical transfer matrix: $M = \begin{bmatrix} \cos(k_2 d) & \dfrac{-j\sin(k_2 d)}{n_2} \\ -jn_2 \sin(k_2 d) & \cos(k_2 d) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

$$\Rightarrow \Gamma = \frac{M_{11} n_1 + M_{12} n_3 n_1 - M_{21} - M_{22} n_3}{M_{11} n_1 + M_{12} n_3 n_1 + M_{21} + M_{22} n_3} \quad \text{and} \quad \tau = \frac{2 n_1}{M_{11} n_1 + M_{12} n_3 n_1 + M_{21} + M_{22} n_3}$$

Oblique incidence: $M = \begin{bmatrix} \cos[k_2 d \cos(\theta_i)] & \dfrac{-j\sin[k_2 d \cos(\theta_i)]}{p} \\ -jp\sin[k_2 d \cos(\theta_i)] & \cos[k_2 d \cos(\theta_i)] \end{bmatrix}$, where $p = n_2 \cos\theta_i$

(TE), or $p = n_2/\cos\theta_i$ (TM)

Anti-reflecting film: $\Gamma = 0$ and $d = \lambda/4n_2 \Rightarrow M_{11} = M_{22} = 0$, $M_{12} = -j/n_2$, $M_{21} = -jn_2$

$\Rightarrow M_{12} n_3 n_1 = -j n_3 n_1/n_2 = M_{21} = -jn_2 \Rightarrow n_2 = \sqrt{n_1 n_3}$

Effective optical transfer matrix of n-layer film: $M_t = M_1 M_2 M_3 \ldots M_n$.
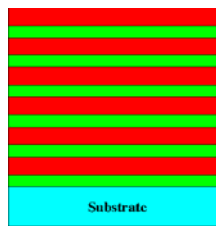
High-reflectance film: A stack of N alternate quarter-wave layers of high index n_h and low index n_l materials

$$M = \begin{bmatrix} 0 & \dfrac{-j}{n_l} \\ -jn_l & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dfrac{-j}{n_h} \\ -jn_h & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dfrac{-j}{n_l} \\ -jn_l & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \dfrac{-j}{n_h} \\ -jn_h & 0 \end{bmatrix} \cdots = \begin{bmatrix} \left(\dfrac{-n_h}{n_l}\right)^N & 0 \\ 0 & \left(\dfrac{-n_l}{n_h}\right)^N \end{bmatrix}$$

$$\Rightarrow \Gamma = \frac{(n_h/n_l)^{2N} - 1}{(n_h/n_l)^{2N} + 1} \to 1.$$

Eg. Determine the effective reflectances of an eight-layer stack (*N*=4) and thirty-layer stack (*N*=15) of ZnS (*n_h*=2.3) and MgF₂ (*n_f*=1.35).
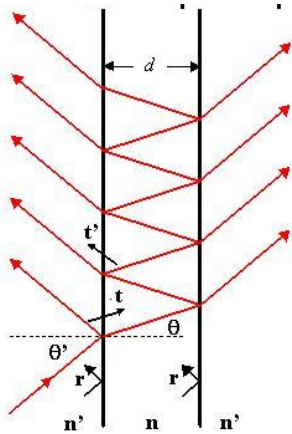
(Sol.) (a) $|\Gamma|^2 = [\dfrac{(2.3/1.35)^8 - 1}{(2.3/1.35)^8 + 1}]^2 = 0.9409$, (b) $|\Gamma|^2 = [\dfrac{(2.3/1.35)^{30} - 1}{(2.3/1.35)^{30} + 1}]^2 = 0.999$

Eg. The relation between the reflectivity and the incidence angle.
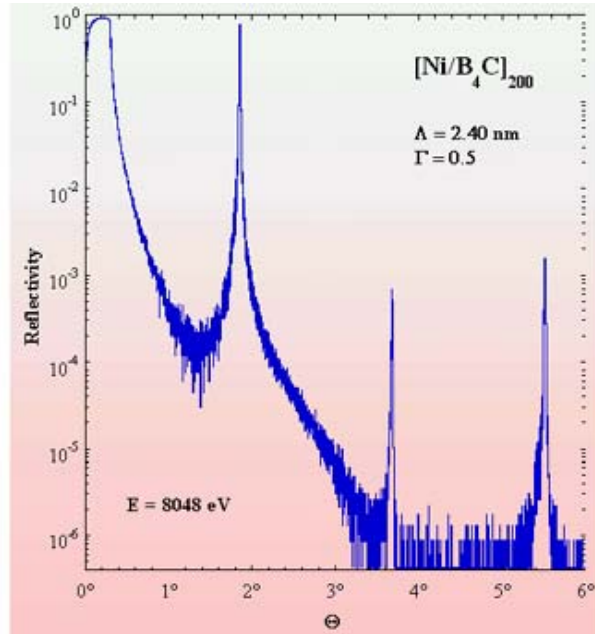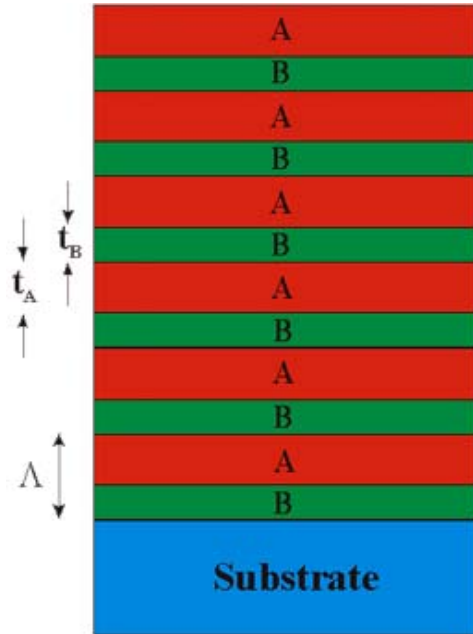
3-8 Fabry-Perot Resonators

Fabry-Perot resonator: Lightwave is resonant between two parallel plates.

Path difference between two successive rays:

$$\frac{d}{\cos\theta} + \frac{d}{\cos\theta} \cdot \cos 2\theta = 2d\cos\theta$$

Total output E-field:

$$E_t = E_i tt' + E_i tt' rr' e^{j\delta} + E_i tt' r^2 r'^2 e^{j2\delta} + \cdots = \frac{E_i tt'}{1 - rr' e^{j\delta}},$$

where $\delta = 2kd\cos\theta = \dfrac{4\pi nd\cos\theta}{\lambda_0}$ is the **optical phase difference**, and d is the thickness.

Total output intensity Fabry-Perot resonator:

$$I_t = \frac{I_i T^2}{\left|1 - Re^{j\delta}\right|^2} = \frac{I_i T^2}{(1-R)^2 + 4R\sin^2(\delta/2)} \quad, \quad \text{where} \quad R = rr', \quad T = tt', \quad \text{and}$$

$R + T + A(\text{absorption}) = 1$.
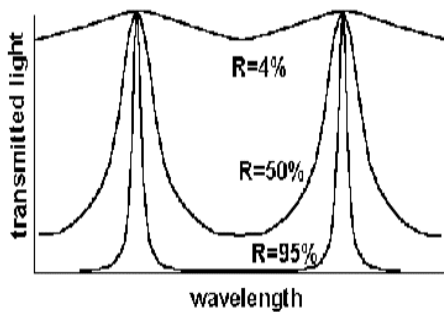
Transmittance of Fabry-Perot resonator:

$$\frac{I_t}{I_i} = \frac{T^2}{(1-R)^2} \cdot \frac{1}{1 + F\sin^2(\delta/2)} \quad, \quad \text{where}$$

$F = \dfrac{4R}{(1-R)^2}$ is called the **coefficient of finesse.**

It is a measurement of the sharpness of the interference fringes. Note that F becomes larger as well as R increases.

$$\left(\frac{I_t}{I_i}\right)_{max} = \frac{T^2}{(1-R)^2} \text{ occurs if } \delta = \frac{4\pi nd\cos\theta}{\lambda_0} = 2N\pi \quad \left(\text{Or}, f = \frac{c}{\lambda_0} = \frac{N}{2nd\cos\theta}\right).$$

$$\left(\frac{I_t}{I_i}\right)_{min} = \frac{T^2}{(1+R)^2} \text{ occurs if } \delta = \frac{4\pi nd\cos\theta}{\lambda_0} = N\pi \quad \left(\text{Or}, f = \frac{c}{\lambda_0} = \frac{N}{4nd\cos\theta}\right).$$

$\therefore$ Fabry-Perot Resonator can be used as an optical spectral analyzer.

Free spectral range in frequency: $f_{N+1} - f_N = \dfrac{c}{2nd\cos\theta}$

Natural cleavage planes: Semiconductor lasers usually utilize the natural cleavage planes as the both parallel mirrors of the Fabry-Perot resonator.

Miller indices: If a plane crossing $(k,0,0)$, $(0,h,0)$ and $(0,0,l)$, then (k,h,l) is called Miller indices of the plane. **Note:** $(\bar{k},h,l)$ denotes $(-k,h,l)$, $(k,\bar{h},l)$ denotes $(k,-h,l)$, etc.
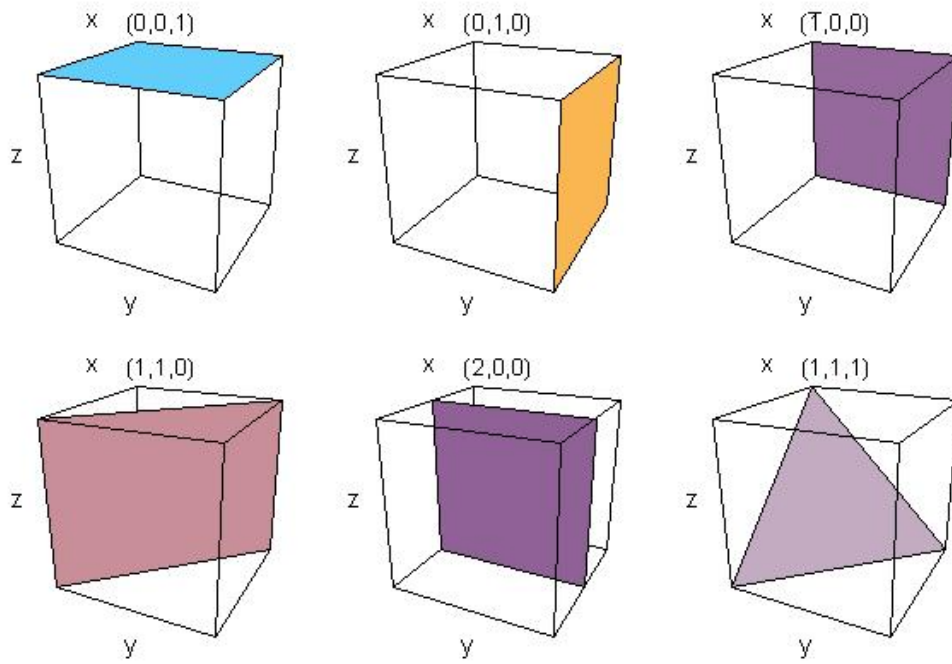
Note: **GaAs**'s natural cleavage plane is (1,1,0)-plane. **Si**'s and **Ge**'s natural cleavage plane are (1,1,1)-plane.
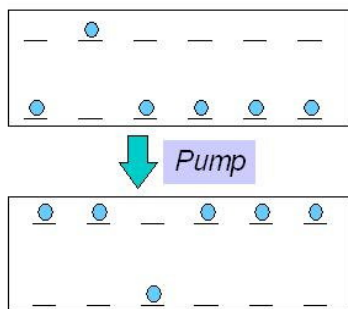
3-9 Introduction to Lasers
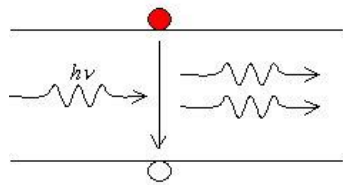
Stimulated emission: A photon induces an electron to fall down from a higher-energy level to a lower-energy level. And then it generates another photon with the same wavelength and the same phase.
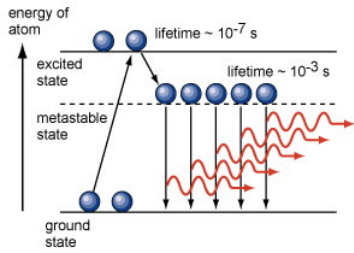

Basic characteristics of lasers:

1. Alignment.
2. Small broadening angle.
3. Single wavelength.
4. High Coherence.

Lasing conditions:

1. **Population inversion:** A certain higher-energy level has more electrons than a lower-energy level.

 Eg. The energy-level system of a ruby laser: The population inversion occurs between the metastable state and the ground state.

2. Pumping systems: Utilizing current driving or other methods to pump electrons from a lower-energy level into a higher-energy level.

3. **Optical resonators:** The laser light traverses back and forth within an optical resonator to generate the stimulated emission.
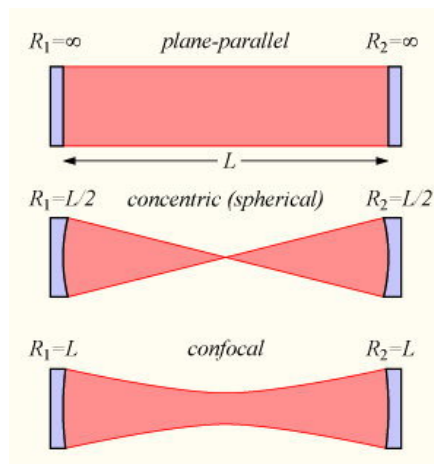
Application of lasers:

Laser Pointer

Laser Pickup Head

Laser Sight/Collimation

Laser knife in surgery   Laser acupuncture   Laser acupuncture machine
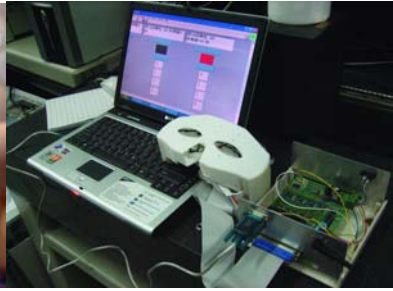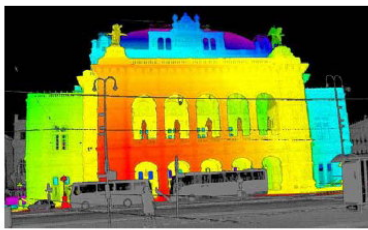
維也納歌劇院
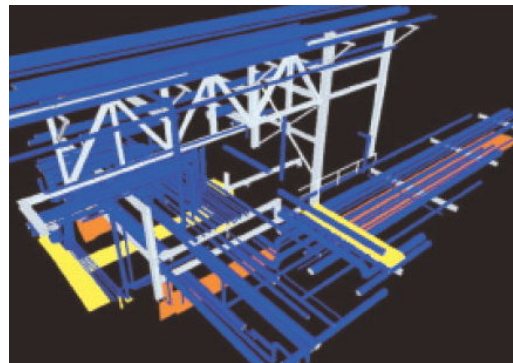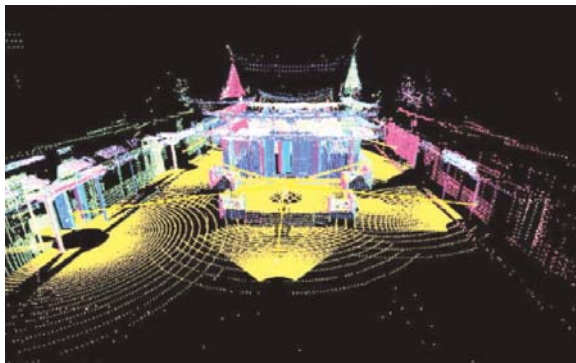
Laser scanner and its applications

3-10 General Optical Resonators and Laser Modes
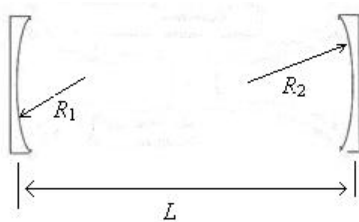
General laser resonator: Two mirrors with radii R_1 and R_2, separation L.

Stability condition: $0 \le (1 - \dfrac{L}{R_1})(1 - \dfrac{L}{R_2}) \le 1$

If a laser medium is situated within an unstable resonator, no stable laser can emit.
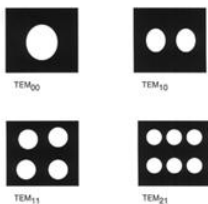
Laser modes: TEM_{mn} mode if $m+1$ spots in the horizontal direction and $n+1$ spots in the vertical direction.

Gaussian beam (TEM₀₀ mode): Fundamental laser mode in an optical resonator or a lenslike medium.

R and w variations with z
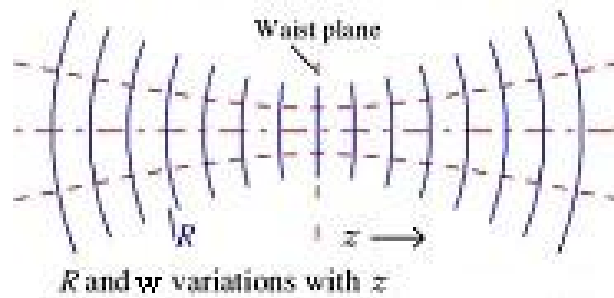
$$\nabla^2 \vec{E} + k^2 \vec{E} = 0 \Rightarrow (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + k^2)\vec{E} = (\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k^2)E = 0.$$

Let $E=\psi(x,y,z)e^{-jkz}$ be a scalar field $\Rightarrow e^{-jkz}\nabla_t^2\psi + \frac{\partial}{\partial z}[\psi'e^{-jkz} - jk\psi e^{-jkz}) + k^2\psi e^{-jkz} = 0$,

where $\psi'=d\psi/dz$. Assume $k\psi'>>\psi''<<k^2\psi \Rightarrow \nabla_t^2\psi - 2jk\psi'=0$.

Let $\psi=E_0\exp\{-j[P(z)+kr^2/2q(z)]\}$, where $r^2=x^2+y^2$.
Substitute $\psi=E_0\exp\{-j[P(z)+kr^2/2q(z)]\}$ into $\nabla_t^2\psi - 2jk\psi'=0$
$\Rightarrow -(k/q)^2r^2 - 2j(k/q) - k^2r^2(1/q)' - 2kP'=0$
$\Rightarrow (1/q)^2+(1/q)'=0$ and $P'=-j/q$.

Let $1/q=s'/s \Rightarrow s=az+b$, $q=q_0+z$, and $P'=-j/(z+q_0) \Rightarrow P=-j\ln(1+z/q_0)$.
Set $q_0=j\pi w_0^2 n/\lambda$ and $k=2\pi n/\lambda \Rightarrow \psi=\exp\{-j[-j\ln(1+z/q_0)+kr^2/2(z+q_0)]\} \Rightarrow$

$$E(x,y,z)= E_0\frac{w_0}{w(z)}\cdot\exp\left\{-j[kz-\eta(z)]-(x^2+y^2)\cdot[\frac{1}{w^2(z)}+\frac{jk}{2R(z)}]\right\}, \text{ where } R(z)=z\left\{1+[\frac{\pi w_0^2 n}{\lambda z}]^2\right\},$$

$$w(z)=w_0\cdot\left\{1+[\frac{\lambda z}{\pi w_0^2 n}]^2\right\}, \quad \eta(z)=\tan^{-1}(\frac{\lambda z}{\pi w_0^2 n}), \text{ and } \frac{1}{q(z)}=\frac{1}{R(z)}-\frac{j\lambda}{\pi n w^2(z)}.$$

At **z=0**, waist: $w(0)=w_0$, $R(0)=\infty$, and $\eta(0)=0 \Rightarrow$ **$E(x,y,z=0)=E_0\exp[-(x^2+y^2)/w_0^2]$**

Beam divergent angle of Gaussian beam: **$\theta_{beam}= \tan^{-1}[\frac{\lambda}{\pi w_0 n}]$**.
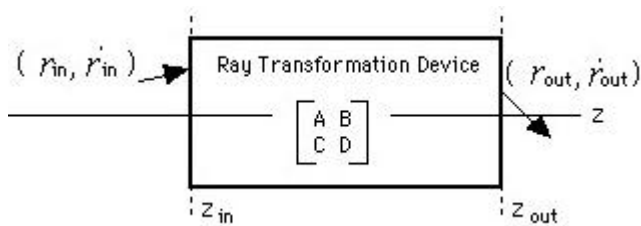
Beam Shaper: Reshape laser beam and intensity distribution.

Transformation of the Gaussian beam propagating through an optical element—

**ABCD law:** $q_2 = \dfrac{Aq_1 + B}{Cq_1 + D}$

ABCD matrix: It can be utilized to describe a ray propagating through an optical element

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

For cascade of n optical elements, $\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \cdots \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$

Case 1 Ray passes *a planar dielectric interface*: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{bmatrix}$
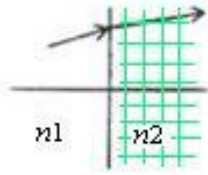
(Proof) $r_{out} = r_{in}$, $r'_{out}/r'_{in} = \tan\theta_{out}/\tan\theta_{in} \doteq \sin\theta_{out}/\sin\theta_{in} = n_1/n_2$

$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{bmatrix}$
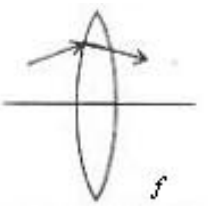
Case 2 Ray passes *a dielectric of thickness d*: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d/n \\ 0 & 1 \end{bmatrix}$
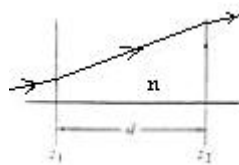
(Proof) $r_{out} \doteq r_{in} + (d/n)\tan\theta_{in} = r_{in} + (d/n)r'_{in}$, $r'_{out} = r'_{in}$

$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d/n \\ 0 & 1 \end{bmatrix}$. If $n=1$, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$.

Case 3 Ray passes *a thin lens of focal length f*: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$
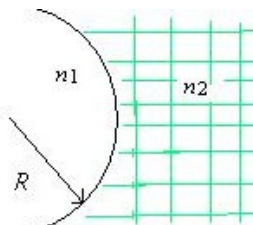
(Proof) $r_{out} = r_{in}$, $r'_{out} = \tan\theta_{out} \doteq \tan\theta_{in} - r_{in}/f = r'_{in} - r_{in}/f$

$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$

Case 4 Ray passes *a spherical dielectric interface*: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{bmatrix}$, where

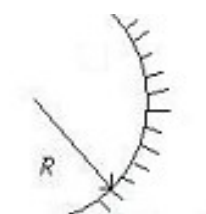$R>0$ if the surface is concave; $R<0$ if the surface is convex.

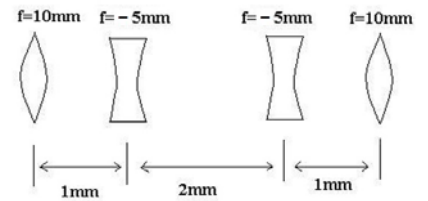(Proof) $r_{out} = r_{in}$, $r'_{out} = \tan\theta_{out} \doteq \sin\theta_{out}$

$\doteq (n_1/n_2)\sin\theta_{in} + (1 - n_1/n_2)r_{in}/R \doteq (n_1/n_2)\tan\theta_{in} + (1 - n_1/n_2)r_{in}/R$

$= (n_1/n_2)r'_{in} + (1 - n_1/n_2)\, r_{in}/R \Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{n_2 - n_1}{n_2 R} & \dfrac{n_1}{n_2} \end{bmatrix}$

Case 5 Ray is *reflected by a spherical mirror*: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{2}{R} & 1 \end{bmatrix}$
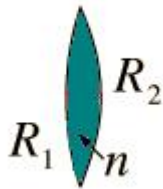
Eg. Obtain the effective focal length of the thin lens system.

f=10mm f=−5mm   f=−5mm   f=10mm

|⟶| |⟵   ⟶| |⟵|
1mm   2mm   1mm

(Sol.) $M_1 = M_7 = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{10} & 1 \end{bmatrix}$, $M_2 = M_6 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $M_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$,

$M_3 = M_5 = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{5} & 1 \end{bmatrix}$

$\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = M_7 M_6 M_5 M_4 M_3 M_2 M_1 \Rightarrow C_t = -1/f, f = 6.3376mm$

Eg. Show that lens-maker's formula $\dfrac{1}{f} = (n-1)\left(\dfrac{-1}{R_1} + \dfrac{1}{R_2}\right)$, **where R_1 or $R_2 > 0$ if the surface is convex to right; otherwise, R_1 or $R_2 < 0$.**

R_2

R_1 n

(Proof) $\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1-n}{R_2} & n \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{n-1}{nR_1} & \dfrac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1-n}{R_2} - \dfrac{1-n}{R_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$

$\Rightarrow \dfrac{1}{f} = (n-1)\left(\dfrac{-1}{R_1} + \dfrac{1}{R_2}\right)$

Eg. Consider a system of two closely-contacted thin lenses, of focal lengths f_1 and f_2, respectively. The ABCD matrix is $\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\left(\dfrac{1}{f_1} + \dfrac{1}{f_2}\right) & 1 \end{bmatrix}$.

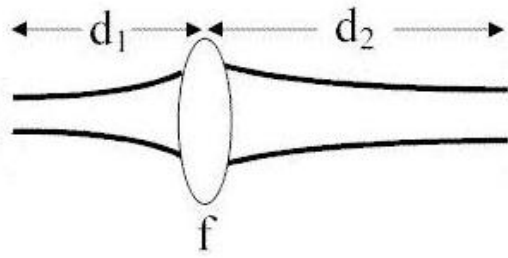**Hence the effective focal length f is expressed by** $\dfrac{1}{f} = \dfrac{1}{f_1} + \dfrac{1}{f_2}$.

Eg. A Gaussian beam propagated a distance d_1 in free space and passed a thin lens of focal length f. And then it traversed a distance d_2 in free space. Determine the ABCD matrix.

(Sol.)
$$\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{d_2}{f} & d_1+d_2-\dfrac{d_1 d_2}{f} \\ -\dfrac{1}{f} & 1-\dfrac{d_1}{f} \end{bmatrix}$$
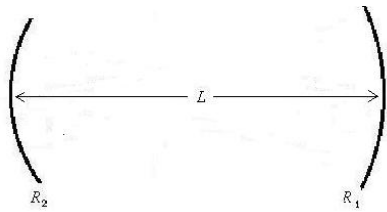
Theorem **The stable condition of an optical resonator, composed of two spherical mirrors of curvature radii R_1 and R_2, separation L, is**
$$0 \le (1-\frac{L}{R_1})(1-\frac{L}{R_2}) \le 1.$$

(Proof) One round trip:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{2}{R_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\dfrac{2}{R_1} & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{2L}{R_1} & L(2-\dfrac{2L}{R_1}) \\ -\dfrac{2}{R_2}-\dfrac{2}{R_1}(1-\dfrac{2L}{R_2}) & -\dfrac{2L}{R_2}+(1-\dfrac{2L}{R_1})(1-\dfrac{2L}{R_2}) \end{bmatrix}$$

Characteristic polynomial is $\lambda^2-(A+D)\lambda+AD-BC=\lambda^2-(A+D)\lambda+1=0$.

Stable resonator if $|\lambda|=1$ and λ is imaginary: $(A+D)^2-4 \le 0 \Rightarrow -2 \le 2-\dfrac{4L}{R_1}-\dfrac{4L}{R_2}+\dfrac{4L^2}{R_1 R_2} \le 2$

$\Rightarrow 0 \le 4-\dfrac{4L}{R_1}-\dfrac{4L}{R_2}+\dfrac{4L^2}{R_1 R_2} \le 4 \Rightarrow 0 \le 1-\dfrac{L}{R_1}-\dfrac{L}{R_2}+\dfrac{L^2}{R_1 R_2} \le 1 \Rightarrow 0 \le (1-\dfrac{L}{R_1})(1-\dfrac{L}{R_2}) \le 1.$

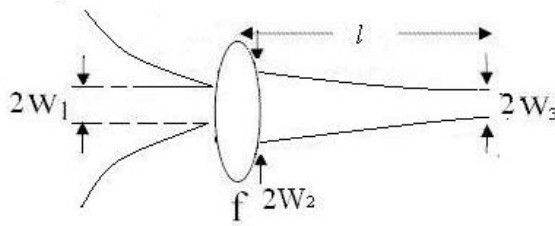**Eg. Show that the Fabry-Perot resonator is a stable resonator.**

(Sol.) $R_1=R_2=\infty$, $(1-\dfrac{L}{R_1})(1-\dfrac{L}{R_2})=1$ fulfills the stable condition.

Eg. Consider two resonators: (a) R_1=5cm, R_2=10cm, L=20cm. (b) R_1=5cm, R_2=10cm, L=3cm. Which is a stable resonator?

(Sol.) (a) $(1-\dfrac{L}{R_1})(1-\dfrac{L}{R_2})$=(-3)(-1)=3>1, $\therefore$ It is not a stable resonator!

(b) $0<(1-\dfrac{L}{R_1})(1-\dfrac{L}{R_2})$=(0.4)(0.7)=0.28<1, $\therefore$ It is a stable resonator!

Eg. A Gaussian beam is focused by a lens. The waist of Gaussian beam is $2w_1$, which is located on the surface of a lens of focal length f. Determine the dimension of the waist $2w_3$ of the output beam and its location l.

(Sol.) At plane 1, $R_1=\infty$, $\dfrac{1}{q_1}=\dfrac{1}{R_1}-\dfrac{j\lambda}{\pi n w_1^2}=-\dfrac{j\lambda}{\pi n w_1^2}$

At plane 2, $q_2=\dfrac{A_1 q_1+B_1}{C_1 q_1+D_1}$ and $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$ $\Rightarrow q_2=\dfrac{a+jb}{a^2+b^2}$, where $a=1/f$,

$b=\lambda/\pi w_1^2 n$. At plane 3, $q_3=\dfrac{A_2 q_2+B_2}{C_2 q_2+D_2}$ and $\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}=\begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$

$\Rightarrow q_3=q_2+l\Rightarrow\dfrac{1}{q_3}=\dfrac{1}{R_3}-\dfrac{j\lambda}{\pi n w_3^2}=\dfrac{\dfrac{-a}{a^2+b^2}+l-\dfrac{jb}{a^2+b^2}}{[\dfrac{-a}{a^2+b^2}+l]^2+(\dfrac{b}{a^2+b^2})^2}=-\dfrac{j\lambda}{\pi n w_3^2}$

$\Rightarrow l=\dfrac{a}{a^2+b^2}=\dfrac{f}{1+(\dfrac{f\lambda}{\pi w_1^2 n})^2}$ <f and $w_3=\dfrac{f\lambda/\pi w_1 n}{\sqrt{1+(f\lambda/\pi w_1^2 n)^2}}\neq 0$
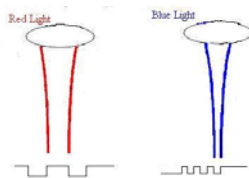

$1°$ $\because$ $l<f$, $\therefore$ the realistic focal length of a Gaussian beam is less than the ideal focal length of the lens.

$2°$ $\because$ In case of very short λ, $w_3=\dfrac{f\lambda/\pi w_1 n}{\sqrt{1+(f\lambda/\pi w_1^2 n)^2}}\propto\lambda$, $\therefore$

we must develop the short-λ laser to read data stored in the high-density optical disk. Otherwise, the long-λ laser can not be focused within a small range to detect a small pit. Thus conventional VCD/DVD utilizes red (650nm) but blu-ray DVD utilizes blue light (405nm). By the same reason, we must utilize short-wavelength light to fabricate the masks of the IC patterns comprising tiny devices.
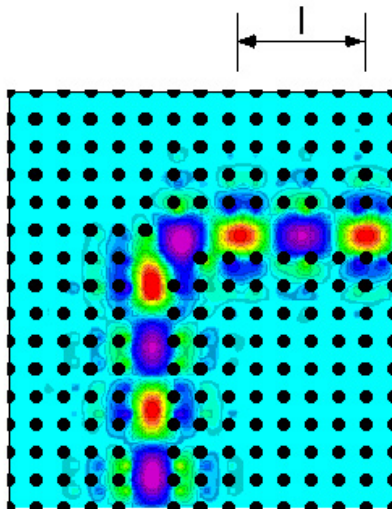
3-11 Gratings and Photonic Crystals
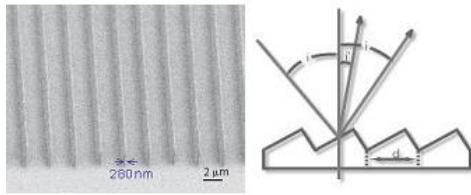
Grating: Periodical structure for optical diffraction.

Photonic Crystals: Arrays of optical scatterers.
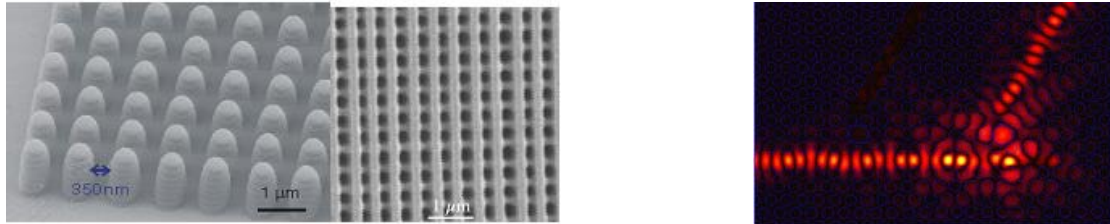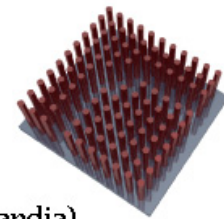

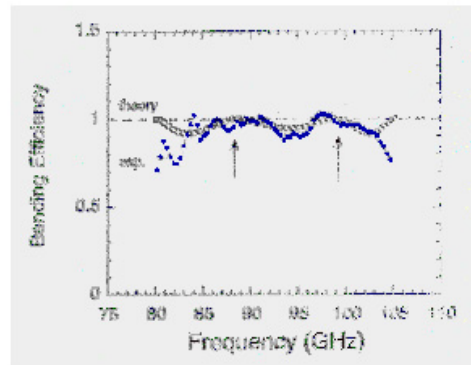## Sharp Bends in PBG Waveguides

Maps onto the problem of 1D resonant electron scattering

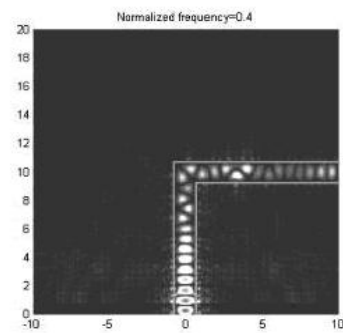Sharp 90 degree bend
100% transmission!

90° Bend

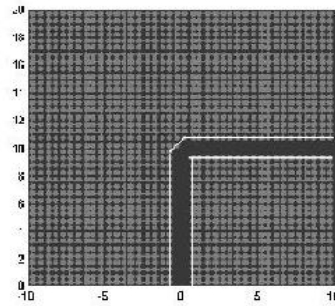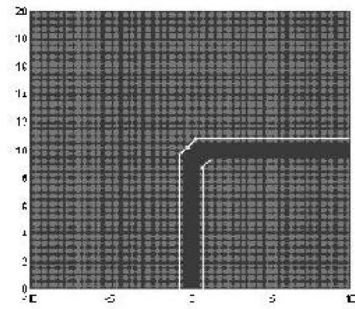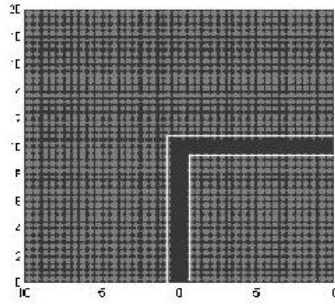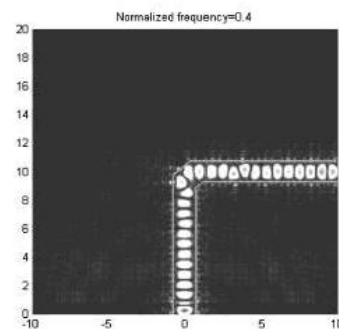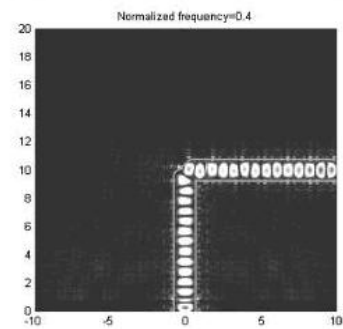& Shawn Lin (Sandia)

Eg. Comparison between 3 types of 90° bent photonic crystal waveguides. (by K. –Y. Lee, C. –C. Tsai 蔡佳辰, T. –C. Weng 翁宗誠, Y. –L. Kuo 郭奕麟, C. –W. Kao 高智偉, K. –Y. Chen 陳奎元, and Y. –J. Lin 林宇仁)

$a/\lambda = 0.4, \ w_0 = 0.5 \mu m$

February 6, 2008

Dr. Keh-Yi Lee
Department of Electrical Engineering
Chinese Culture University
Taipei
Taiwan
Republic of China

Dear Dr. Lee,

Congratulations! Our research indicates that your article "Transmission Characteristics of 90-degree Bent Photonic Crystal Waveguides" published in *Fiber and Integrated Optics* volume 25, number 1 is the 3rd most-downloaded article published in 2006.

I would like to personally invite you to publish more of your work in *Fiber and Integrated Optics*. The journal is covered by all the appropriate indexing and abstracting services, including ISI/Thomson's *Science Citation Index* and *Current Contents*. As reported by Thomson, the 2006 impact factor for the journal is 0.600 (© 2007 Thomson Scientific, *Journal Citation Reports* ®). The journal has over 140 institutional and personal subscribers from six different regions and 33 different countries, and is also available online at an additional 560 institutions worldwide.

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If you have any questions regarding the relevance of a manuscript or any other questions about the journal, please feel free to contact me at henri.hodara@L-3com.com.

I wish you the best of luck in all your future research and publishing endeavors.

Sincerely,

Henri Hodara
Editor-in-Chief, *Fiber and Integrated Optics*