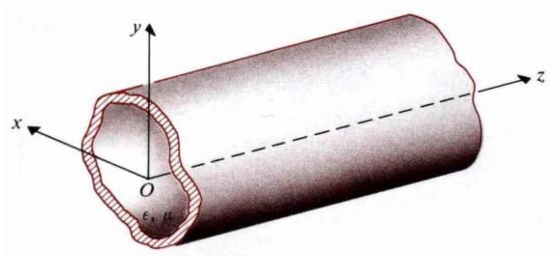
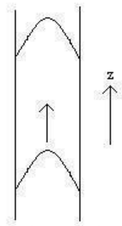
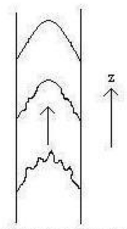
Chapter 6 Metallic Waveguide and Cavity Resonators

6-1 General Metallic Waveguides





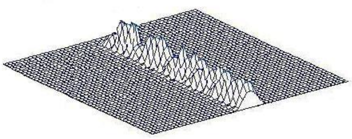
Input eigenmode

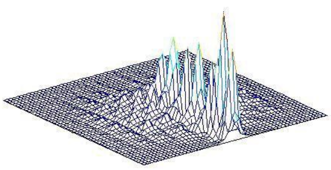


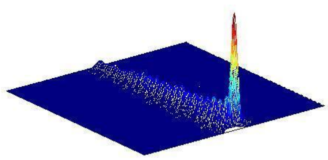
Input non-eigenmode

How to study the theory of metallic waveguides (by L. J. Chu, 朱蘭成):

1. Specify a proper coordinate system, and derive **waveguide's equations** to express the transverse components of the *E*- and *H*-fields in terms of the longitudinal components by Maxwell's equations.
2. Caculate the eigenmodes (**TM mode**, **TE mode**, **TEM mode** or **other types of modes**) of the waveguide, and obtain the eigenvalues and the longitudinal field-components of the corresponding eigenmodes by solving the wave equations. Substituting the longitudinal field-components into the longitudinal components, we can obtain the other components. If the eigenmode is injected into a waveguide, it can propagate along an infinitely-long straight waveguide without any deformation. However, in case the input EM wave is not an eigenmode, some power loss occurs and then it becomes the eigenmode gradually. All the eigenmodal functions in an infinitely-long straight metallic waveguide are orthogonal to each other. Moreover, these eigenmodes form a complete set (a basis in a vector space), such that any electromagnetic fields within the waveguide can be uniquely expressed by the eigenmodal functions.
3. Obtain the quantities of the physical characteristics for a given eigenmode, such as the **cutoff frequency (f_c)**, the **propagation constant ($\gamma=\alpha+j\beta$)**, the **phase velocity ($v_p=\omega/\beta$)**, the **group velocity** ($v_g = \partial\omega/\partial\beta$), the impedance ***Z***, etc.


Input the eigenmode of waveguide


Input non-eigenmode, and then generate radiation modes


Input non-eigenmode, and then the wave decays and gradually becomes eigenmode

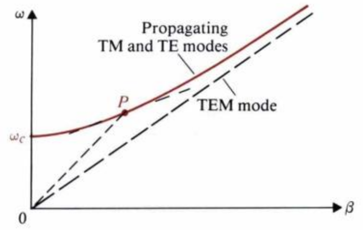
$\nabla \times \bar{E} = -j\omega\mu\bar{H}$	$\nabla \times \bar{H} = j\omega\epsilon\bar{E}$
$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x,$ $-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y,$ $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z,$	$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x,$ $-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y,$ $\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z.$

Waveguide's equations: According to Ampere's law and Faraday's law, we obtain

$$H_x = -\frac{1}{h^2}(\gamma\frac{\partial H_z}{\partial x} - j\omega\varepsilon\frac{\partial E_z}{\partial y}),\quad H_y = -\frac{1}{h^2}(\gamma\frac{\partial H_z}{\partial y} + j\omega\varepsilon\frac{\partial E_z}{\partial x})$$

$$E_x = -\frac{1}{h^2}(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}),\quad E_y = -\frac{1}{h^2}(\gamma\frac{\partial E_z}{\partial y} - j\omega\mu\frac{\partial H_z}{\partial x})$$, where $h^2=\gamma^2+k^2$,

and $\begin{cases}\nabla^2\vec{E} + k^2\vec{E} = 0 \\ \nabla^2\vec{H} + k^2\vec{H} = 0\end{cases} \Rightarrow \begin{cases}\nabla_t^2\vec{E} + (\gamma^2 + k^2)\vec{E} = 0 = [\nabla_t^2 + h^2]\vec{E} \\ \nabla_t^2\vec{H} + (\gamma^2 + k^2)\vec{H} = 0 = [\nabla_t^2 + h^2]\vec{H}\end{cases}$



***Case 1 TEM mode*:** $E_z=H_z=0$

$$h^2 = 0 = \gamma_{TEM}{}^2 + k^2 \Rightarrow \gamma_{TEM} = jk = j\omega\sqrt{\mu\varepsilon}\ ,\quad v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$Z_{TEM} = \frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{\gamma_{TEM}}{j\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \quad \text{and} \quad \vec{H} = \frac{1}{Z_{TEM}}\hat{z}\times\vec{E}$$

Note: All frequencies make γ_{TEM} is pure imaginary $\Rightarrow$ TEM wave can propagate at any frequency, no cutoff.

***Case 2 TM mode*:** $H_z=0$, $E_z\neq0$ and $\nabla_t^2E_z+h^2E_z=0$

$$H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y},\quad H_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x},\quad E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x},\quad E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y}$$

$$\Rightarrow Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\gamma}{j\omega\varepsilon}(\neq\frac{j\omega\mu}{\gamma}) \quad \text{and} \quad \vec{H} = \frac{1}{Z_{TM}}(\hat{z}\times\vec{E})$$

$$\gamma = \sqrt{h^2 - k^2} = h\sqrt{1-(\frac{f}{f_c})^2}\ ,\ \text{where}\quad f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

If $f>f_c$, $\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1-(\frac{f_c}{f})^2} \Rightarrow \beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-(\frac{f_c}{f})^2}$, $Z_{TM} = \eta\sqrt{1-(\frac{f_c}{f})^2}$, and

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}}$$

***Case 3 TE mode*:** $E_z=0$, $H_z\neq 0$ and $\nabla_t^2 H_z + h^2 H_z = 0$

$$H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x},\quad H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y},\quad E_x = -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y},\quad E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

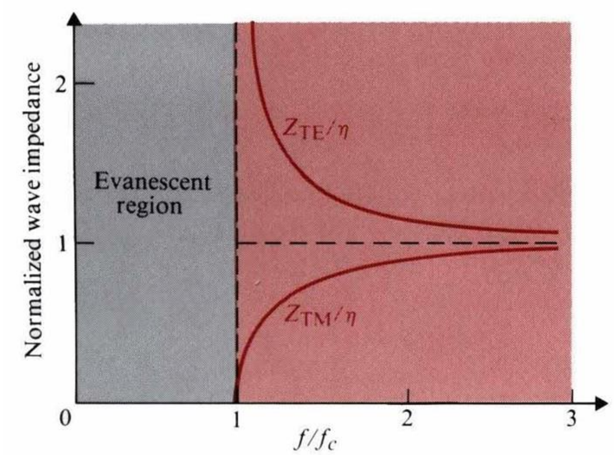
$$\Rightarrow Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma}\left(\neq \frac{\gamma}{j\omega\varepsilon}\right) \text{ and } \vec{E} = -Z_{TE}(\hat{z}\times\vec{H})$$

If $f>f_c$, $\gamma = j\beta = jk\sqrt{1-(\frac{f_c}{f})^2} = j\omega\sqrt{\mu\varepsilon}\sqrt{1-(\frac{f_c}{f})^2}$, $\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-(\frac{f_c}{f})^2}$

$$\Rightarrow Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}},\quad v_p = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2}}$$

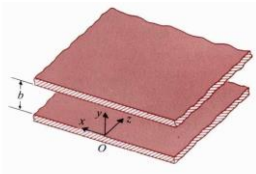
Wave Impedances and Guide Wavelengths for $f > f_c$

Mode	Wave Impedance, Z	Guide Wavelength, λ_g
TEM	$\eta = \sqrt{\frac{\mu}{\epsilon}}$	$\lambda = \frac{1}{f\sqrt{\mu\epsilon}}$
TM	$\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$	$\frac{\lambda}{\sqrt{1-(f_c/f)^2}}$
TE	$\frac{\eta}{\sqrt{1-(f_c/f)^2}}$	$\frac{\lambda}{\sqrt{1-(f_c/f)^2}}$



A case of longitudinal $v_p>0$ but longitudinal $v_g=0$ in barber's pole.

6-2 Parallel–Plate Waveguides



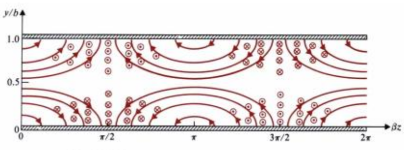
Case 1 TM$_n$ mode: $H_z=0$, $E_z(x,y)=E_z(y)e^{-\gamma z}$

$$\frac{dE_z(y)}{dy^2}+h^2E_z(y)=0,\quad E_z(y)=0 \text{ at } y=0 \text{ and } b$$

$\Rightarrow$ Eigenvalues: $h=\frac{n\pi}{b}$, $E_z(y)=A_n\sin(\frac{n\pi y}{b})$, $n=0, 1, 2, 3, \ldots$

$$\begin{cases} H_x(y)=\frac{j\omega\varepsilon}{h^2}\frac{dE_z^0(y)}{dy}=\frac{j\omega\varepsilon}{h}A_n\cos(\frac{n\pi y}{b}) \\ E_y(y)=-\frac{\gamma}{h^2}\frac{dE_z^0(y)}{dy}=-\frac{h}{\gamma}A_n\cos(\frac{n\pi y}{b}) \end{cases} \text{ and } \gamma=\sqrt{h^2-\omega^2\mu\varepsilon}=\sqrt{(\frac{n\pi}{b})^2-\omega^2\mu\varepsilon}$$

Cutoff frequency: $f_c=\frac{n}{2b\sqrt{\mu\varepsilon}}$ fulfills $\gamma=0$. (**Note:** $n=0$ is the TEM mode)



Electric field lines,
⊙ ⊗ Magnetic field lines (x-axis into the paper).

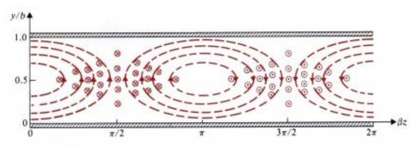
Case 2 TE$_n$ modes: $E_z=0$, $H_z(y,z)=H_z(y)e^{-\gamma z}$

$$\frac{d^2H_z(y)}{dy^2}+h^2H_z(y)=0,\quad \frac{dH_z(y)}{dy}=0 \text{ at } y=0 \text{ and } b$$

$\Rightarrow H_z(y)=B_n\cos(\frac{n\pi y}{b})$ and $f_c=\frac{n}{2b\sqrt{\mu\varepsilon}}$

$$\begin{cases} H_y(y)=-\frac{j\omega\varepsilon}{h^2}\frac{dH_z}{dy}=\frac{\gamma}{h}B_n\sin(\frac{n\pi y}{b}) \\ E_x(y)=-\frac{\gamma}{h^2}\frac{dH_z}{dy}=\frac{j\omega\mu}{h}B_n\sin(\frac{n\pi y}{b}) \end{cases} \text{ and } \gamma=\sqrt{(\frac{n\pi}{b})^2-\omega^2\mu\varepsilon},\ n=1, 2, 3, \ldots$$

Eg. (a)Write the instantaneous field expression for *TM*$_1$ mode in a parallel-plate waveguide. (b) Sketch the *E*- & *H*- field lines in the *yz*-plane.



Magnetic field lines,
⊙ ⊗ Electric field lines (x-axis into the paper).

(Sol.) (a) For $n=1$,

$$E_y(y,z,t)=\frac{\beta b}{\pi}A_1\cos(\frac{\pi y}{b})\sin(\omega t-\beta z)$$

$$H_x(y,z,t)=-\frac{\omega\varepsilon b}{\pi}A_1\cos(\frac{\pi y}{b})\sin(\omega t-\beta z),$$

$$\beta=\sqrt{\omega^2\mu\varepsilon-(\frac{\pi}{b})^2}$$

(b) $\frac{dy}{E_y}=\frac{dz}{E_z}\Rightarrow\frac{dy}{dz}=\frac{E_y(y,z,0)}{E_z(y,z,0)}=-\frac{\beta b}{\pi}\cot(\frac{\pi y}{b})\tan\beta z\Rightarrow\cos(\frac{\pi y}{b})\cos\beta z$ =constant

Eg. (a) Write the instantaneous field expression for TE_1 mode in a parallel-plate waveguide. (b) Sketch the electric and magnetic field lines in the *yz*-plane.

(Sol.) (a) For *n*=1, $H_z(y,z;t) = B_1 \cos(\frac{\pi y}{b})\cos(\omega t - \beta z)$,

$$H_y = (y,z;t) = -\frac{\beta b}{\pi} B_1 \sin(\frac{\pi y}{b})\sin(\omega t - \beta z)$$

$E_x(y,z;t) = -\frac{\omega\mu b}{\pi} B_1 \sin(\frac{\pi y}{b})\sin(\omega t - \beta z)$, where β is the same as that of the TM_1 mode.

(b) At *t*=0,

$$E_x(y,z;0) = -\frac{\omega\mu b}{\pi} B_1 \sin(\frac{\pi y}{b})\sin \beta z,$$

$$\frac{dy}{dz} = \frac{H_y(y,z;0)}{H_z(y,z;0)} = \frac{\beta b}{\pi}\tan(\frac{\pi y}{b})\tan \beta z$$

Eg. Find the electric and the magnetic fields of the propagating wave in a parallel waveguide *b*=5*cm*, filled with a dielectric ($4\varepsilon_0, \mu_0$) and excited by $\vec{H} = \hat{y}\cos 40\pi x \sin 8\pi 10^9 t$.

(Sol.) $f = 4\times10^9 Hz$, $\cos 40\pi x = \cos(\frac{2\pi x}{0.05})$: TM_2 mode

$(f_c)_{TM_2} = \frac{2}{2b\sqrt{\mu_0 4\varepsilon_0}} = 3\times10^9$, $(f_c)_{TM_2} < f$, $\therefore$ TM_2 mode can propagate!

$$\beta = \omega\sqrt{\mu_0 4\varepsilon_0}\sqrt{1-(\frac{(f_c)_{TM_2}}{f})^2} = 110.82 \Rightarrow \vec{H} = \hat{y}\cos(40\pi x)\sin(8\pi 10^9 t - 110.82z)$$

$$\vec{E} = \hat{x}124.67\cos(40\pi x)\sin(8\pi 10^9 t - 110.82z) + \hat{z}141.37\sin(40\pi x)\cos(8\pi 10^9 t - 110.82z)$$

Eg. Find the electric field of the propagating wave in an air parallel waveguide *b*=5*cm* excited by $\vec{E} = \hat{y}10(\sin 20\pi x + 0.5\sin 60\pi x)\sin 10^{10}\pi t$.

(Sol.) $f = 5\times10^9 Hz$, $\sin 20\pi x = \sin(\frac{\pi x}{0.05})$: TE_1 mode, $\sin 60\pi x = \sin(\frac{3\pi x}{0.05})$: TE_3 mode

$(f_c)_{TE_1} = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = 3\times10^9$, $(f_c)_{TE_3} = \frac{3}{2b\sqrt{\mu_0\varepsilon_0}} = 9\times10^9$, $(f_c)_{TE_1} < f < (f_c)_{TE_3}$

$\therefore$ Only TE_1 mode can propagate! $\beta = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1-(\frac{(f_c)_{TE_1}}{f})^2} = \frac{80\pi}{3}$

$$\vec{E} = \hat{y}10\sin(20\pi x)\sin(10^{10}\pi t - \frac{80\pi z}{3})$$

Energy-transport velocity and attenuation in parallel-plate waveguides:

Energy velocity: $v_{en} = \dfrac{(P_z)}{W'_{av}}$ and $(v_{en})_{TE} = (v_{en})_{TM}$

Energy velocity of TM mode:

$$(P_z)_{av} = \int_s \vec{P}_{av} \cdot d\vec{S} = \int_s \frac{1}{2}\mathrm{Re}[\vec{E}\times\vec{H}^*]\cdot\hat{z}dy = \int_s \frac{1}{2}\mathrm{Re}(-\hat{z}E_y H_x + \hat{y}E_z H_x)\cdot\hat{z}dy = \frac{\omega\varepsilon\beta b}{4h^2}A_n^{\ 2}$$

$$w'_{av} = \frac{\varepsilon}{4}\mathrm{Re}(\vec{E}\cdot\vec{E}^*) + \frac{\mu}{4}\mathrm{Re}(\vec{H}\cdot\vec{H}^*) = \frac{\varepsilon A_n^{\ 2}}{4}[\sin^2(\frac{n\pi y}{b}) + \frac{\beta^2}{h^2}\cos^2(\frac{n\pi y}{b})] + \frac{\mu}{4}(\frac{\omega^2\varepsilon^2}{h^2})A_n^{\ 2}\cos(\frac{n\pi y}{b})$$

$$W'_{av} = \int_0^b w'_{av}\,dy = \frac{\varepsilon b}{4h^2}k^2 A_n^{\ 2} \Rightarrow v_{en} = \frac{\omega\beta}{k^2} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-(\frac{f_c}{f})^2}$$

Attenuation constant: $\alpha = \alpha_d + \alpha_c$

α of TEM mode: $\alpha_d = \dfrac{G}{2}R_0 = \dfrac{\sigma}{2}\sqrt{\dfrac{\mu}{\varepsilon}} = \dfrac{\sigma}{2}\eta \approx \dfrac{\omega\varepsilon''}{2}\eta$, $\alpha_c = \dfrac{R}{2R_0} = \dfrac{1}{b}\sqrt{\dfrac{\pi f\varepsilon}{\sigma_c}}$

α of TM mode:

$$\gamma = j[\omega^2\mu\varepsilon(1-\frac{j\sigma}{\omega\varepsilon}) - (\frac{n\pi}{b})^2]^{1/2} \approx j\sqrt{\omega^2\mu\varepsilon - (\frac{n\pi}{b})^2}\cdot\{1-\frac{j\omega\mu\sigma}{2}[\omega^2\mu\varepsilon - (\frac{n\pi}{b})^2]^{-1}\}$$

$$= j\omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(\frac{f_c}{f})^2}\cdot\{1-\frac{j\omega\mu\sigma}{2}\cdot\frac{1}{1+(\frac{f_c}{f})^2}\} = \alpha_d + j\beta \Rightarrow \alpha_d = \frac{\sigma\eta}{2\sqrt{1-(\frac{f_c}{f})^2}},$$

$$\beta = \omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(\frac{f_c}{f})^2}$$

$$\alpha_c = \frac{P_L(z)}{2P(z)},\ \text{where}\quad P(z) = w\int_0^b -\frac{1}{2}(E_y^0)(H_x^0)^* dy = w\omega\varepsilon\beta b(\frac{bA_n}{2n\pi})^2,$$

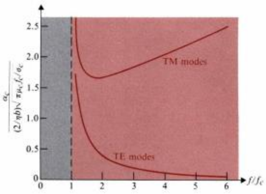
$$|J_{sz}^0| = |H_x^0(y=0)| = \frac{\omega\varepsilon bA_n}{n\pi} \Rightarrow P_L(z) = 2w(\frac{1}{2}|J_{sz}^0|^2 R_s) = w(\frac{\omega\varepsilon bA_n}{n\pi})^2 R_s,$$

$$R_s = \sqrt{\frac{\pi f\mu_c}{\sigma_c}} \Rightarrow \alpha_c = \frac{2}{\eta b}\cdot\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}}\cdot\frac{1}{\sqrt{(f_c/f)[1-(f_c/f)^2]}}$$

α of TE mode: α_d is the same as the expression in TM mode

$$P(z) = w\int_0^b \frac{1}{2}(E_x)(H_y)^* dy = \frac{w\omega\mu\beta}{2}(\frac{bB_n}{n\pi})^2 \int_0^b \sin^2(\frac{n\pi y}{b})dy = w\omega\mu\beta b(\frac{bB_n}{2n\pi})^2,$$

$$\beta = \omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(\frac{f_c}{f})^2}$$



$$P_L(z) = 2w(\frac{1}{2}\left|J_{sx}\right|^2 R_s) = w\left|H_z(y=0)\right|^2 R_s = wB^2R_s$$

$$\alpha_c = \frac{P_L(z)}{2P(z)} = \frac{2R_s}{\omega\mu\beta b}(\frac{n\pi}{b})^2 = \frac{2R_s f_c^2}{\eta b f^2\sqrt{1-(f_c/f)^2}} = \frac{2}{\eta b}\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}}\sqrt{\frac{(f_c/f)}{1-(f_c/f)^2}}$$

Note: $\because \begin{Bmatrix}\alpha_c\\ \alpha_d\end{Bmatrix}$ of the higher-order modes $> \begin{Bmatrix}\alpha_c\\ \alpha_d\end{Bmatrix}$ of the lower-order modes, $\therefore$ the lowest-order mode is often utilized in communication systems. Otherwise, the signal decays very soon.

Eg. A waveguide is formed by two parallel copper sheets, which is separated by a 5*cm* thick lossy dielectric ε_r=2.25, μ_r=1, $\sigma=10^{-10}$(*S*/m). For an operating frequency of 10*GHz*, find α_d, α_c, β, v_p, v_g and λ_g for (a) the TEM mode, (b) the *TM*$_1$ mode.

(Sol.) $\sigma_c=5.8\times10^7$(*S*/m)

(a) TEM: $\beta = \omega\sqrt{\mu\varepsilon} = 314.16 rad/m$, $\alpha_d = \frac{\sigma\eta}{2} = 1.257\times10^{-8} N_p/m$,

$\alpha_c = \frac{1}{b}\sqrt{\frac{\pi f\varepsilon}{\sigma_c}} = 2.078\times10^{-3} N_p/m$, $v_p = \frac{1}{\sqrt{\mu\varepsilon}} = 2\times10^8 m/s$, $v_g = v_p$,

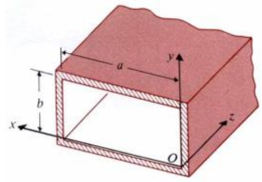
$\lambda_g = \frac{v_p}{f} = 0.02m$

(b) *TM*$_1$: $f_c = \frac{1}{2b\sqrt{\mu\varepsilon}} = 2\times10^9 Hz$, $\beta = \omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(\frac{f_c}{f})^2} = 307.88 rad/m$

$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}} = 1.282\times10^{-8} N_p/m$, $\alpha_c = \frac{1}{\eta b}\cdot\frac{1}{\sqrt{(f_c/f)[1-(f_c/f)^2]}} = 4.237\times10^{-3} N_p/m$

$v_p = \frac{\omega}{\beta} = 2.041\times10^8 m/s$, $v_g = \frac{(3\times10^8)^2}{2.25v_p} = 1.96\times10^8 m/s$, $\lambda_g = \frac{v_p}{f} = 0.0204m$

6-3 Rectangular Waveguides



Case 1 TM$_{mn}$ modes: $H_z=0$, $E_z(x,y,z)=E_z^0(x,y)e^{-\gamma z}$

$$(\nabla_t^2+h^2)E_z(x,y)=0 \Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+h^2\right)E_z(x,y)=0,$$

$E_z(x,y)=0$ at $x=0$, a, and $y=0$, b

Eigenvalues: $h^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$, $E_z(x,y)=E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

Waveguide's equations $\Rightarrow$

$$\begin{cases} E_x(x,y)=-\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), & E_y(x,y)=-\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ H_x(x,y)=\frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), & H_y(x,y)=-\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \end{cases}$$

$\gamma=j\beta=j\sqrt{\omega^2\mu\varepsilon-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$. **Note:** ***TM$_{mn}$*** mode, neither m nor n can be zero.

Cutoff frequency: $(f_c)_{TM_{mn}}=\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$

In case of ***f*>*f*$_c$: waves can propagate**, else if ***f*<*f*$_c$: evanescent waves** (**cutoff**).

Case 2 TE$_{mn}$ modes: $E_z=0$, $H_z(x,y,z)=H_z(x,y)e^{-\gamma z}$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+h^2\right)H_z(x,y)=0, \quad \begin{cases} \frac{\partial H_z}{\partial x}=0 \ (or\ E_y=0) \\ \frac{\partial H_z}{\partial y}=0 \ (or\ E_x=0) \end{cases} \text{ at } \begin{matrix} x=0,\ a \\ y=0,\ b \end{matrix}$$

$$\Rightarrow H_z(x,y)=H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Waveguide's equations $\Rightarrow$

$$\begin{cases} E_x(x,y)=\frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), & E_y(x,y)=-\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \\ H_x(x,y)=\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), & H_y(x,y)=\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \end{cases}$$

where $\gamma=j\beta=j\sqrt{\omega^2\mu\varepsilon-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$ and $(f_c)_{TE_{mn}}=\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$

Note: We prefer the **single-mode waveguide** because the **fundamental mode** has **the lowest attenuation** and **avoid modal dispersion**. The **bandwidth** of the single-mode waveguide is $(f_c)_1 < f < (f_c)_2$.

Eg. Calculate in ascending order the cutoff frequencies of an $a \times b$ rectangular waveguide for the following modes: TE_{01}, TE_{10}, TE_{11}, and TE_{02} if $a=2b$.

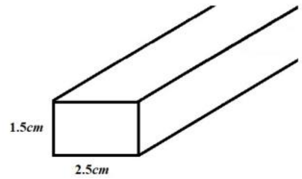
(Sol.) $f_c = \dfrac{1}{2\sqrt{\mu\varepsilon}}\sqrt{(\dfrac{m}{a})^2 + (\dfrac{n}{b})^2}$, $a=2b \Rightarrow (f_c)_{TE_{10}} = \dfrac{1}{4b\sqrt{\mu\varepsilon}}, (f_c)_{TE_{01}} = \dfrac{1}{2b\sqrt{\mu\varepsilon}}$

$(f_c)_{TE_{11}} = (f_c)_{TM_{11}} = \dfrac{\sqrt{5}}{4b\sqrt{\mu\varepsilon}}$, $(f_c)_{TE_{02}} = \dfrac{1}{b\sqrt{\mu\varepsilon}}$

Note: In case of $2b>a>b$, the fundamental mode of the rectangular waveguide is TE_{10} mode. It has the lowest cutoff frequency $(f_c)_{TE_{10}} = \dfrac{1}{2a\sqrt{\mu\varepsilon}}$ and the corresponding cutoff wavelength $(\lambda_c)_{TE_{10}} = 2a$. It means that the TE_{10} mode will be cut off in case of $f < (f_c)_{TE_{10}} = \dfrac{1}{2a\sqrt{\mu\varepsilon}}$ or $\lambda > (\lambda_c)_{TE_{10}} = 2a$

Dispersion: Waves have different velocities
Modal Dispersion: Different modes have distinct phase velocities.

Eg. Calculate and compare the values of β, v_p, v_g, λ_g and $Z_{TE_{10}}$ for a 2.5cm×1.5cm rectangular waveguide operating at 7.5GHz. if the waveguide is hollow.

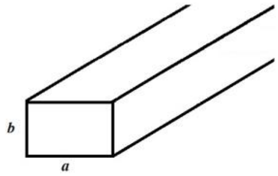


(Sol.) $\mu=\mu_0, \varepsilon=\varepsilon_0$, $(f_c)_{TE_{10}} = \dfrac{1}{2a}\dfrac{1}{\sqrt{\mu\varepsilon}} = 6\times10^9\,Hz$, $\sqrt{1-(\dfrac{f_c}{f})^2}$ =0.6

$\beta = \omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(\dfrac{f_c}{f})^2}$ =94.25rad/m, $v_p = \dfrac{1}{\sqrt{\mu\varepsilon}}\cdot\dfrac{1}{\sqrt{1-(f_c/f)^2}} = 5\times10^8\ m/s$

$\lambda_g = v_p/f$=0.067m, $Z_{TE_{10}} = \dfrac{120\pi}{\sqrt{1-(f_c/f)^2}}$ =628.3Ω, $v_g = \dfrac{1}{\sqrt{\mu\varepsilon}}\cdot\sqrt{1-(\dfrac{f_c}{f})^2}$ =1.8×10^8 m/s

Eg. An air-filled $a \times b$ ($b<a<2b$) rectangular waveguide is to be constructed to operate at 3*GHz* in the dominant mode. We desire the operating frequency to be at least 20% higher than the cutoff frequency of the dominant mode and also at least 20% below the cutoff frequency of the next higher-order mode. (a) Give a typical design for the dimensions *a* and *b*. (b) Calculate for your design β, v_p, λ_g and the wave impedance at the operating frequency. [台大電研]



(Sol.) (a) $f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \cdot \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$. $b<a<2b$, the dominant mode: TE_{10}, the next mode: TE_{01}

$$(f_c)_{TE_{10}} = \frac{1}{2a\sqrt{\mu\varepsilon}},\ (f_c)_{TE_{01}} = \frac{1}{2b\sqrt{\mu\varepsilon}},\ \frac{3\times 10^9 - (1/2a\sqrt{\mu\varepsilon})}{(1/2a\sqrt{\mu\varepsilon})} > 20\%,$$

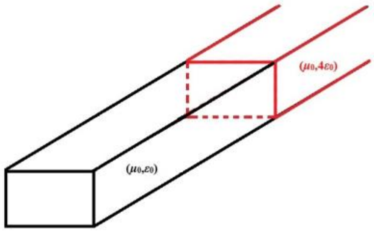
$$\frac{(1/2b\sqrt{\mu\varepsilon}) - 3\times 10^9}{(1/2b\sqrt{\mu\varepsilon})} > 20\% \Rightarrow a \ge 0.06m,\ b \le 0.04m, \text{ and } a<2b$$

(b) Choose a=0.065*m*, b=0.035*m*, $(f_c)_{TE_{10}} = 2.3\times 10^9 (Hz)$, $\sqrt{1-(\frac{f_c}{f})^2} = 0.679$,

$$\beta = \omega\sqrt{\mu\varepsilon} \cdot \sqrt{1-(\frac{f_c}{f})^2} = 40.15 rad/m,\ v_p = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \frac{1}{\sqrt{1-(f_c/f)^2}} = 4.7\times 10^8 m/s,$$

$$\lambda_g = \frac{v_p}{f} = 0.157m,\ Z_{TE_{10}} = \eta_0 \Big/ \sqrt{1-(f_c/f)^2} = 120\pi / 0.639 = 590\ \Omega$$

Eg. A 3*cm*×1.5*cm* rectangular waveguide operating at 6*GHz* has a dielectric discontinuity between medium 1 (μ_0,ε_0) and medium 2 ($\mu_0,4\varepsilon_0$). (a) Find the *SWR* in the free-space region. (b) Find the length and the permittivity of a quarter-wave section to achieve a match between two media.

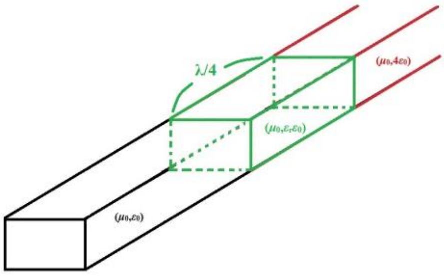


(Sol.) (a) For TE_{10} mode, $f_{c1}=\frac{1}{2a}\frac{1}{\sqrt{\mu_0\varepsilon_0}}=5\times10^9$,

$$f_{c2}=\frac{1}{2a}\frac{1}{\sqrt{\mu_0 4\varepsilon_0}}=2.5\times10^9$$

$$Z_1=\sqrt{\mu_0/\varepsilon_0}\Big/\sqrt{1-(f_{c1}/f)^2}=682\Omega,\quad Z_2=\sqrt{\mu_0/4\varepsilon_0}\Big/\sqrt{1-(f_{c2}/f)^2}=207\ \Omega$$

$$\Rightarrow \Gamma=\frac{Z_2-Z_1}{Z_2+Z_1}=-0.5337\ \Rightarrow SWR=\frac{1+|\Gamma|}{1-|\Gamma|}=3.289$$

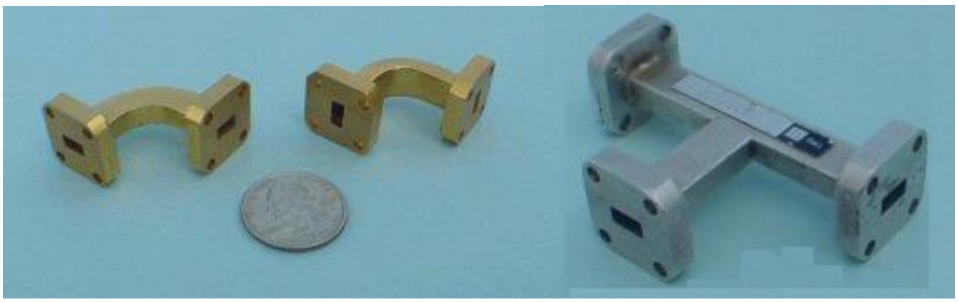


(b) $f_{c3}=\frac{1}{2a}\frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}}$ and

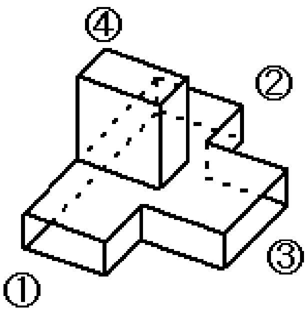
$$Z_3=\sqrt{\mu_0/\varepsilon_r\varepsilon_0}\Big/\sqrt{1-(f_{c3}/f)^2}=\sqrt{Z_1Z_2}\Rightarrow \varepsilon_r=1.6995$$

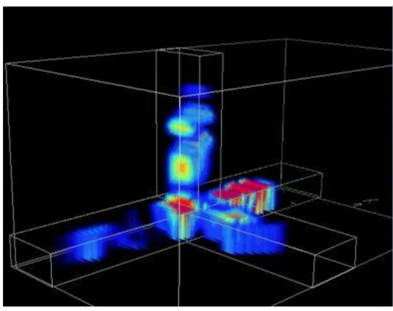
$d=\lambda_3/4=v_{p3}/4f=1.24685\times10^{-2}m$

Bending and *T*-Branches of Waveguides:



Magic-*T*-junction: (made of rectangular metallic waveguide)





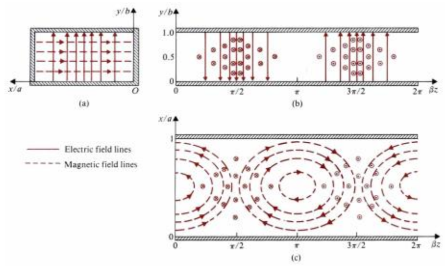
Eg. (a) Write the instantaneous field expression for the TE_{10} mode in a rectangular waveguide having sides *a* and *b*. (b) Sketch the electric and magnetic field lines in typical *xy*-, *yz*-, and *xz*-planes. (c) Sketch the surface currents on the guide walls.

(Sol.) (a) m=1, n=0,

$$E_x(x,y,z;t)=0,\quad E_z(x,y,z;t)=0,\quad E_y(x,y,z;t)=\frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_0\sin\left(\frac{\pi}{a}x\right)\sin(\omega t-\beta z),$$

$$H_x(x,y,z;t)=-\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_0\sin\left(\frac{\pi}{a}x\right)\cos(\omega t-\beta z),\; H_y(x,y,z;t)=0$$

where $\beta=\sqrt{k^2-h^2}=\sqrt{\omega^2\mu\varepsilon-\left(\frac{\pi}{a}\right)^2}$



(b) $\left(\frac{dx}{dz}\right)_H=\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)\tan\left(\frac{\pi}{a}x\right)\tan\beta z$

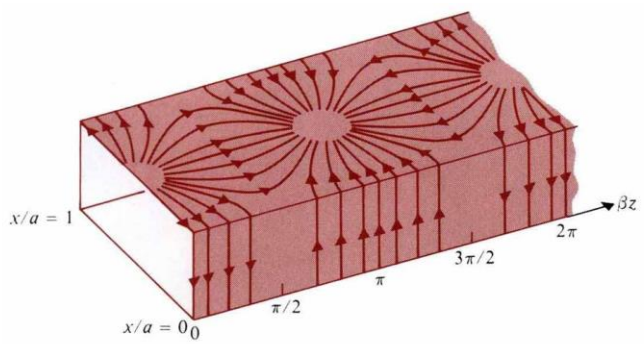
(c) $\vec{J}_s=\hat{a}_n\times\vec{H}$. At t=0,

$$\vec{J}_s(x=0)=-\hat{y}H_z(0,y,z;0)=-\hat{y}H_0\cos\beta z$$

$$\vec{J}_s(x=a)=\hat{y}H_z(a,y,z;0)=\vec{J}_s(x=a)$$

$$\vec{J}_s(y=0)=\hat{x}H_z(x,0,z;0)-\hat{z}H_x(x,0,z;0)=\hat{x}H_0\cos\left(\frac{\pi}{a}x\right)\cos(\beta z)-\hat{z}\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_0\sin\left(\frac{\pi}{a}x\right)\sin\beta z$$

$$\vec{J}_s(y=b)=-\vec{J}_s(y=0)$$



Attenuation in the rectangular waveguide:

$\alpha=\alpha_d+\alpha_c$, where $\alpha_d = \dfrac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}}$ and $\alpha_c = \dfrac{P_L(z)}{2P(z)}$.

Consider TE_{10} mode:

$$P(z)=\int_0^b\int_0^a -\frac{1}{2}(E_y)(H_x)^* dxdy = \frac{1}{2}\omega\mu\beta(\frac{a}{\pi})^2 H_0^2\int_0^b\int_0^a \sin^2(\frac{\pi}{a}x)dxdy = \omega\mu\beta ab(\frac{aH_0}{2\pi})^2$$

and

$$\begin{cases}\vec{J}_s(x=0)=\vec{J}_s(x=a)=-\hat{y}_y H_z(x=0)=-\hat{y}H_0\\ \vec{J}_s(y=0)=-\vec{J}_s(y=b)=\hat{x}H_z(y=0)-\hat{z}H_x(y=0)=\hat{x}H_0\cos(\frac{\pi}{a}x)-\hat{z}\frac{\beta a}{\pi}H_0\sin(\frac{\pi}{a}x)\end{cases}$$

$$P_L(z)=2[P_L(z)]_{x=0}+2[P_L(z)]_{y=0},\quad [P_L(z)]_{x=0}=\int_0^b \frac{1}{2}\left|J_s(x=0)\right|^2 R_s dy = \frac{b}{2}H_0^2 R_s$$

and $[P_L(z)]_{y=0}=\int_0^a \frac{1}{2}[\left|J_{sx}(y=0)\right|^2+\left|J_{sx}(y=0)\right|^2]R_s dx = \frac{a}{4}[1+(\frac{\beta a}{\pi})^2]H_0^2 R_s$

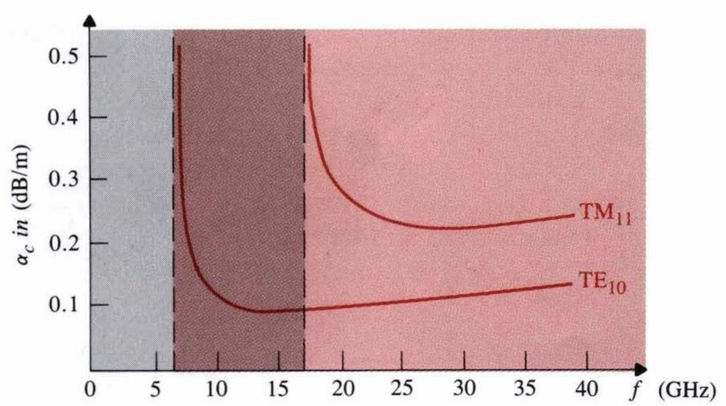
$$\Rightarrow P_L(z)=\left\{b+\frac{a}{2}[1+(\frac{\beta a}{\pi})^2]\right\}H_0^2 R_s = [b+\frac{a}{2}(\frac{f_c}{f})^2]H_0^2 R_s$$

$$(\alpha_c)_{TE_{10}} = \frac{R_s[1+(2b/a)(f_c/f)^2]}{\eta b\sqrt{1-(f_c/f)^2}} = \frac{1}{\eta b}\sqrt{\frac{\pi f\mu_c}{\sigma_c[1-(f_c/f)^2]}}[1+\frac{2b}{a}(\frac{f_c}{f})^2],\quad R_s=\sqrt{\frac{\pi f\mu_c}{\sigma_c}}$$

Similar approach $\Rightarrow (\alpha_c)_{TM_{11}} = \dfrac{2R_s[(b/a^2)+(a/b^2)]}{\eta ab\sqrt{1-(f_c/f)^2}\,[(1/a^2)+(1/b^2)]}$

General Cases:

$$\begin{cases}(\alpha_c)_{TE_{mn}} = \dfrac{2R_s}{\eta b\sqrt{1-(f_c/f)^2}}\cdot\left\{1+(\dfrac{b}{a})(\dfrac{f_c}{f})^2+[\dfrac{1}{2}-(\dfrac{f_c}{f})^2]\cdot\dfrac{\frac{b}{a}(\frac{b}{a}m^2+n^2)}{\frac{b^2}{a^2}m^2+n^2}\right\}\\ (\alpha_c)_{TM_{mn}} = \dfrac{2R_s}{\eta b\sqrt{1-(f_c/f)^2}}\cdot\dfrac{m^2(\frac{b}{a})^3+n^2}{m^2(\frac{b}{a})^2+n^2}\end{cases}$$



Eg. A TE_{10} wave at 10GHz propagates in a brass σ_c=1.57×10^7(S/m) rectangular waveguide with inner dimensions a=1.5cm and b=0.6cm, which is filled with ε_r=2.25, μ_r=1, loss tangent=4×10^{-4}. Determine (a) the phase constant, (b) the guide wavelength, (c) the phase velocity, (d) the wave impedance, (e) the attenuation constant due to loss in the dielectric, and (f) the attenuation constant due to loss in the guide walls.

(Sol.) f=10^{10}Hz, $\lambda = \frac{v}{f} = \frac{3\times10^8}{\sqrt{2.25}\times10^{10}} = \frac{2\times10^8}{10^{10}} = 0.02\,(m)$

For TE_{10} mode, $f_c = \frac{v}{2a} = \frac{2\times10^8}{2\times(1.5\times10^{-2})} = 0.667\times10^{10}\,Hz$

$$\beta = \frac{\omega}{v}\sqrt{1-(\frac{f_c}{f})^2} = 234\ (rad/m)\,,.\ \lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} = 0.0268\,(m)$$

$$v_p = \frac{v}{\sqrt{1-(f_c/f)^2}} = 2.68\times10^8\,m/s\,,\ \ Z_{TE_{10}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-(f_c/f)^2}} = 337.4(\Omega)$$

$$\sigma = 4\times10^{-4}\,\omega\varepsilon = 5\times10^{-4}\,S/m,\ \ \alpha_d = \frac{\sigma}{2}Z_{TE_{10}} = 0.084\,Np/m = 0.73dB/m$$

$$R_s = \sqrt{\frac{\pi f\mu_c}{\sigma_c}} = 0.05101\,(\Omega)\,,$$

$$\alpha_c = \frac{R_s[1+(2b/a)(f_c/f)^2]}{\eta b\sqrt{1-(f_c/f)^2}} = 0.0526\,Np/m = 0.457\,dB/m\,.$$

Eg. (a) Determine the value of f_c/f at which the attenuation constant due to conductor losses in an a×b rectangular waveguide for the TE_{10} mode is a minimum. What is the minimum obtainable α_c in a 2cm×1cm guide? At what frequency? (b) Determine the value of $(f/f_c)_{TM_{11}}$ at which this attenuation constant is a minimum.

(Sol.) (a) $(\alpha_c)_{TE_{10}} = \frac{1}{\eta b}\cdot\sqrt{\frac{\pi f\mu_c}{\sigma_c[1-(f_c/f)^2]}}\cdot[1+\frac{2b}{a}(\frac{f_c}{f})^2],\ \ \frac{d(\alpha_c)_{TE_{10}}}{df} = 0$

$$\Rightarrow f = f_c\cdot[\frac{(6b+3a)-\sqrt{(6b+3a)^2-8ab}}{2a}]^{-1/2}$$

(b) $(\alpha_c)_{TM_{11}} = \frac{2R_s(b/a^2+a/b^2)}{\eta ab\sqrt{1-(f_c/f)^2}\cdot(1/a^2+1/b^2)},\ \ \frac{d(\alpha_c)_{TM_{11}}}{df} = 0 \Rightarrow f = \sqrt{3}f_c$

Eg. An air-filled rectangular waveguide made of copper and having transverse dimensions a=7.20cm and b=3.40cm operates at a frequency 3GHz in the dominant mode. Find (a) f_c, (b) λ_g, (c) α_c, and (d) the distance over which the field intensities of the propagating wave will be attenuated by 50%.

(Sol.) σ_c=5.8×10^7S/m (a) $(f_c)_{TE_{10}} = \dfrac{1}{2a\sqrt{\mu\varepsilon}} = 2.083GHz < 3GHz$

(b) $\lambda_g = \dfrac{\lambda}{\sqrt{1-(f_c/f)^2}} = 0.109m$

(c) $(\alpha_c)_{TE_{10}} = \dfrac{1}{\eta b}\sqrt{\dfrac{\pi f\mu_c}{\sigma_c[1-(f_c/f)^2}}\cdot[1+\dfrac{2b}{a}(\dfrac{f_c}{f})^2] = 2.26\times10^{-3}N_p/m$

(d) $0.5 = e^{-\alpha_c d} \Rightarrow d = 307.25m$

Eg. An average power of 1kW at 10GHz is to be delivered to an antenna at the TE_{10} mode by an air-filled rectangular copper waveguide 1m long and having sides a=2.25cm and b=1.00cm. Find (a) the attenuation constant due to conductor losses, (b) the maximum values of the electric and magnetic field intensities within the waveguide, (c) the maximum value of the surface current density on the conducting walls, (d) the total amount of average power dissipated in the waveguide. [台大電研]

(Sol.) (a) $(f_c)_{TE_{10}} = \dfrac{1}{2a\sqrt{\mu\varepsilon}} = 6\times10^9(Hz)$, $f = 10\times10^9 Hz$, $\sqrt{1-(\dfrac{f_c}{f})^2} = 0.7454$

$(\alpha_c)_{TE_{10}} = \dfrac{1}{\eta b}\cdot\sqrt{\dfrac{\pi f\mu_c}{\sigma_c[1-(f_c/f)^2]}}\cdot[1+\dfrac{2b}{a}(\dfrac{f_c}{f})^2] = 0.13Np/m$

(b) $P = \omega\mu\beta ab(\dfrac{aH_0}{2\pi})^2$, $\beta = \omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-(f_c/f)^2} = 156.1rad/m$, $h^2 = (\dfrac{\pi}{a})^2$

$\Rightarrow 1000 = 3.56\times10^{-2}H_0^2 \Rightarrow H_0 = 167A/m \Rightarrow |E_{\max}| = |E_y^0(x,y)|_{\max} = \dfrac{\omega\mu}{h^2}\cdot\dfrac{\pi}{a}H_0 = 94800V/m$

$|H_x^0(x,y)|_{\max} = \dfrac{\beta a}{\pi}H_0 = 187.4A/m$, $|H_z^0(x,y)|_{\max} = H_0 = 167A/m$.

If at input end, all factor×$e^{0.13}$ (=1.138)

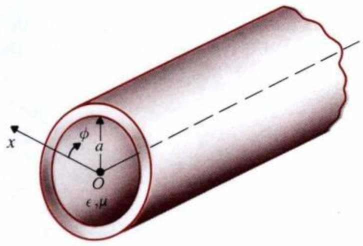
(c) $\vec{J}_s(x=0) = -\hat{y}H_0\cos\beta z \Rightarrow |\vec{J}_s(x=0)| = H_0$, $\vec{J}_s(y=0) = \hat{y}\times(\hat{x}H_x^0+\hat{z}H_z^0)_{y=0}$

$\Rightarrow |J_s(y=0)| = \dfrac{E_0}{\eta_0}\cdot\left\{(\dfrac{f_c}{f})^2 + [1-2(\dfrac{f_c}{f})^2\cdot\sin(\dfrac{\pi x}{\sigma})]\right\}^{1/2}$, ∴ Max of $|J_s(y=0)| = \dfrac{E_0}{\eta_0}\cdot\sqrt{1-(\dfrac{f_c}{f})^2}$

At the input end, this factor×1.138

(d) $\Delta P = 1000\times[e^{2\alpha_c\cdot 1}-1] = 26.2W$ ($\because$ $P \propto |E|^2 \propto e^{-2\alpha\cdot\ell}$)

6-4 Circular Waveguides

Circular waveguide's equations:

$$\begin{cases} E_r = -\frac{j}{h^2}[\beta\frac{\partial E_z}{\partial r} + \frac{\omega\mu}{r}\frac{\partial H_z}{\partial \phi}], & H_r = \frac{j}{h^2}[\frac{\omega\varepsilon}{r}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial r}] \\ E_\phi = \frac{j}{h^2}[-\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} + \omega\mu\frac{\partial H_z}{\partial r}], & H_\phi = -\frac{j}{h^2}[\omega\varepsilon\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \phi}] \end{cases}$$

Case 1 TM$_{np}$ modes: $H_z=0$, and $\nabla_t^2 E_z + h^2 E_z = 0$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial E_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + h^2 E_z = 0$$

$E_z(r,\varphi) = C_n J_n(hr)\cos n\varphi$ and $E_z(r,\varphi,z) = E_z(r,\varphi)e^{-\gamma z}$

$$\Rightarrow \begin{cases} E_r = -\frac{j\beta}{h}C_n J'_n(hr)\cos n\varphi, & E_\varphi = \frac{j\beta n}{h^2 r}C_n J_n(hr)\sin n\varphi \\ H_r = -\frac{j\omega\varepsilon n}{h^2 r}C_n J_n(hr)\sin n\varphi, & H_\varphi = -\frac{j\omega\varepsilon}{h}C_n J'_n(hr)\cos n\varphi \end{cases}$$

$E_z = H_z = 0$ at $r=a$ and $h^2=\gamma^2+k^2 \Rightarrow$ h fulfills $J_n(ha)=0$ (the first root of $J_0(x)$ is 2.405)

Case 2 TE$_{np}$ modes: $E_z=0$, $\nabla_t^2 H_z + h^2 H_z = 0 \Rightarrow H_z^0(r,\phi) = C'_n J_n(hr)\cos n\phi$

Waveguide's equations

$$\Rightarrow \begin{cases} H_r = -\frac{j\beta}{h}C'_n J'_n(hr)\cos n\varphi, & H_\varphi = \frac{j\beta n}{h^2 r}C'_n J_n(hr)\sin n\varphi \\ E_r = \frac{j\omega\mu n}{h^2 r}C'_n J_n(hr)\sin n\varphi, & E_\varphi = -\frac{j\omega\mu}{h}C'_n J'_n(hr)\cos n\varphi \end{cases}$$

$E_z = 0,\ \frac{\partial H_z}{\partial r} = 0$ at $r=a \Rightarrow h$ fulfills $J'_n(ha)=0$ (the first root of $J'_1(x)$ is 1.841)

Note: The cutoff frequency of ***TE$_{11}$*** mode: $(f_c)_{TE_{11}} = \frac{h_{TE_{11}}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.293}{a\sqrt{\mu\varepsilon}}$ and the corresponding cutoff wavelength is $(\lambda_c)_{TE_{11}} = \frac{a}{0.293}$ where ($h_{TE_{11}} = \frac{1.841}{a}$).

Note: ***TE*₁₁** mode is the fundament (dominant) mode of a circular waveguide.

Zeros of $J_n(x)$, x_{np}

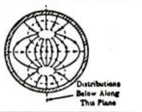
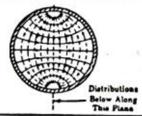
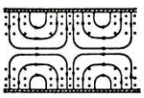
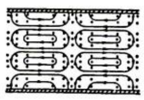
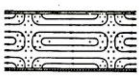
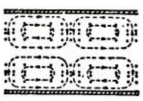
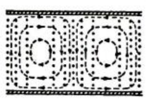
$p \backslash n$	$n=0$	$n=1$	$n=2$
1	2.405	3.832	5.136
2	5.520	7.016	8.417

Zeros of $J'_n(x)$, x'_{np}

$p \backslash n$	$n=0$	$n=1$	$n=2$
1	3.832	1.841	3.054
2	7.016	5.331	6.706

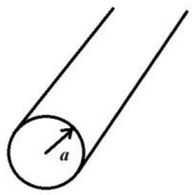
Note: For TM_{np} mode, h fulfills the p^{th} root of $J_n(ha)=0$, but for TE_{np} mode, h fulfills the p^{th} root of $J'_n(ha)=0$.

Summary of Wave Types for Circular Guides*

Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$
p_{nl} or p'_{nl}	2.405	5.52	3.83	3.83	1.84
$(k_c)_{nl}$	$\frac{2.405}{a}$	$\frac{5.52}{a}$	$\frac{3.83}{a}$	$\frac{3.83}{a}$	$\frac{1.84}{a}$
$(\lambda_c)_{nl}$	$2.61a$	$1.14a$	$1.64a$	$1.64a$	$3.41a$
$(f_c)_{nl}$	$\frac{0.383}{a\sqrt{\mu\varepsilon}}$	$\frac{0.877}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.293}{a\sqrt{\mu\varepsilon}}$
Attenuation due to imperfect conductors	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}\left[\left(\frac{f_c}{f}\right)^2+0.420\right]$

* Electric field lines are shown solid and magnetic field lines are dashed.

Eg. (a) A 10*GHz* signal is to be transmitted inside a hollow circular conducting pipe. Determine the inside diameter of the pipe such that its lowest cutoff frequency is 20% below this signal frequency. (b) If the pipe is to operate at 15*GHz*, what waveguide modes can propagate in the pipe?

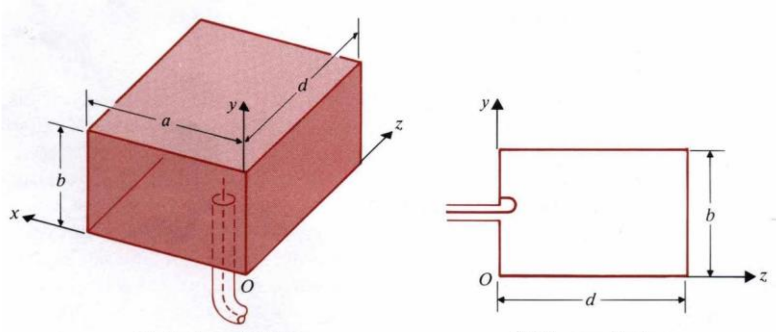


(Sol.) (a) $(f_c)_{TE_{11}} = \dfrac{0.293}{a\sqrt{\mu_0\varepsilon_0}} = \dfrac{0.879}{a}\times10^8 = 10^{10}\times(1\text{-}20\%) = 8\times10^9$, $2a=0.022m$.

(b) f_c of waveguide with $a=0.011$(m) is $(f_c)_{TE_{11}}=8GHz<15GHz$.

$(f_c)_{TM_{01}} = \dfrac{2.405}{2\pi a\sqrt{\mu_0\varepsilon_0}} = 10.45(GHz)$, $(f_c)_{TE_{21}} = \dfrac{3.054}{2\pi a\sqrt{\mu_0\varepsilon_0}} = 13.27(GHz)$

6-5 Rectangular Cavity Resonators



(a) Probe excitation. (b) Loop excitation.

Case 1 TM$_{mnp}$ mode: $H_z=0$, neither m nor $n=0$, p can be 0.

$$\begin{cases} E_z(x,y,z)=E_0\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\cos(\frac{p\pi}{d}z) \\ E_x(x,y,z)=-\frac{1}{h^2}(\frac{m\pi}{a})(\frac{p\pi}{d})E_0\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\sin(\frac{p\pi}{d}z) \\ E_y(x,y,z)=-\frac{1}{h^2}(\frac{n\pi}{b})(\frac{p\pi}{d})E_0\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\sin(\frac{p\pi}{d}z) \\ H_x(x,y,z)=\frac{j\omega\varepsilon}{h^2}(\frac{n\pi}{b})E_0\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\cos(\frac{p\pi}{d}z) \\ H_y(x,y,z)=-\frac{j\omega\varepsilon}{h^2}(\frac{m\pi}{a})E_0\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\cos(\frac{p\pi}{d}z) \end{cases},$$

where $h^2=(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2+(\frac{p\pi}{d})^2$

Case 2 TE$_{mnp}$ mode: $E_z=0$, $p\neq0$. Either m or $n=0$, but not both.

$$\begin{cases} H_z(x,y,z)=H_0\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\sin(\frac{p\pi}{d}z) \\ E_x(x,y,z)=\frac{j\omega\mu}{h^2}(\frac{n\pi}{b})H_0\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\sin(\frac{p\pi}{d}z) \\ E_y(x,y,z)=-\frac{j\omega\mu}{h^2}(\frac{m\pi}{a})H_0\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\sin(\frac{p\pi}{d}z) \\ H_x(x,y,z)=-\frac{1}{h^2}(\frac{m\pi}{a})(\frac{p\pi}{d})H_0\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)\cos(\frac{p\pi}{d}z) \\ H_y(x,y,z)=-\frac{1}{h^2}(\frac{n\pi}{b})(\frac{p\pi}{d})H_0\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)\cos(\frac{p\pi}{d}z) \end{cases}$$

Both have the same resonant frequency (degenerate modes): $f_{mnp}=\frac{1}{2\sqrt{\mu\varepsilon}}\cdot\sqrt{(\frac{m}{a})^2+(\frac{n}{b})^2+(\frac{p}{d})^2}$

Note: ***TE$_{101}$*** mode is the dominant mode of the rectangular resonator in case of $a>b<d$.

Eg. Given an air-filled lossless rectangular cavity resonator with dimensions $8cm \times 6cm \times 5cm$, find the first twelve lowest-order modes and their resonant frequencies.

(Sol.) $f_r = \frac{c}{2}\sqrt{(\frac{m}{0.08})^2 + (\frac{n}{0.06})^2 + (\frac{p}{0.05})^2} = 1.5\times 10^{10}\cdot\sqrt{(\frac{m}{8})^2 + (\frac{n}{6})^2 + (\frac{p}{5})^2}$

$TM_{110} \Rightarrow f_r = 3.125GHz$, $TE_{101} \Rightarrow f_r = 3.54GHz$,

$TE_{111}, TM_{111} \Rightarrow f_r = 4.33GHz$, ……

Quality Factor of Rectangular Cavity Resonators: $Q=\frac{\omega W}{P_L}$, $W=W_e+W_m$

Q=2π×(Time-average energy stored at a resonant frequency)/(Energy dissipated in one period)

Consider TE_{101} mode:

$$W_e = \frac{\varepsilon_0}{4}\int |E_y|^2 dv = \frac{\varepsilon_0\omega^2\mu_0^2\pi^2}{4h^4a^2}H_0^2\int_0^d\int_0^b\int_0^a \sin^2(\frac{\pi}{a}x)\sin^2(\frac{\pi}{d}z)dxdydz = \frac{\varepsilon_0\omega_{101}^2\mu_0^2\pi^2}{4\pi^2}H_0^2(\frac{a}{2})b(\frac{d}{2})$$
$$= \frac{1}{4}\varepsilon_0\mu_0^2a^3bdf_{101}^2H_0^2$$

$$W_m = \frac{\mu_0}{4}\int\{|H_x|^2 + |H_y|^2\}dv = \frac{\mu_0}{4}H_0^2\int_0^d\int_0^b\int_0^a\{\frac{\pi^4}{h^4a^2d^2}\sin^2(\frac{\pi}{a}x)\cos^2(\frac{\pi}{d}z) + \cos^2(\frac{\pi}{a}x)\sin^2(\frac{\pi}{d}z)\}dxdydz$$
$$= \frac{\mu}{4}H_0^2\{\frac{a^2}{d^2}(\frac{a}{2})b(\frac{d}{2}) + (\frac{a}{2})b(\frac{d}{2})\} = \frac{\mu_0}{16}abd(\frac{a^2}{d^2}+1)H_0^2$$

where $h^2=(\frac{\pi}{a})^2 + (\frac{\pi}{d})^2$ and the resonant frequency: $(f_r)_{TE_{101}} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\cdot\sqrt{\frac{1}{a^2}+\frac{1}{d^2}}$

At resonance, $W=2W_e=2W_m=\frac{\mu_0}{8}\cdot H_0^2abd(\frac{a^2}{d^2}+1)$

Power loss per unit area: $P_{av}=\frac{1}{2}|J_s|^2R_s = \frac{1}{2}|H|^2R_s$

$$P_L = \oint P_{av}ds = R_s\left\{\int_0^b\int_0^a |H_x(z=0)|^2dxdy + \int_0^d\int_0^b |H_z(x=0)|^2dydz + \int_0^d\int_0^a |H_x|^2dxdz + \int_0^d\int_0^a |H_z|^2dxdz\right\}$$

$$= \frac{R_sH_0^2}{2}\left\{\frac{a^2}{d}(\frac{b}{d}+\frac{1}{2}) + d(\frac{b}{a}+\frac{1}{2})\right\} \Rightarrow Q_{TE_{101}} = \frac{\pi f_{101}\mu_0abd(a^2+d^2)}{R_s[2b(a^3+d^3)+ad(a^2+d^2)]}$$

Similarly, the expression for the Q of an air-filled $a\times b\times d$ rectangular resonator for the TM_{110} mode is $(Q)_{TM_{110}} = \frac{\pi(f_r)_{TM_{110}}\mu_0abd(a^2+b^2)}{R_s[2d(a^3+d^3)+ab(a^2+b^2)]}$.

Eg. An air-filled rectangular cavity with brass walls σ=1.57×10^7(S/m) has the following dimensions: a=4cm, b=3cm, and d=5cm. (a) Determine the dominant mode and its resonant frequency for this cavity. (b) Find the Q and the time-average stored electric and magnetic energies at the resonant frequency, assuming H_0 to be 0.1A/m.

(Sol.) (a) $f_r = \frac{c}{2}\sqrt{(\frac{m}{0.04})^2 + (\frac{n}{0.03})^2 + (\frac{p}{0.05})^2}$, dominant mode: TE_{101}, $(f_r)_{TE_{101}} = 4.8GHz$

(b) $(Q)_{TE_{101}} = \frac{\pi (f_r)_{TE_{101}} \mu_0 abd(a^2+d^2)}{R_s[2b(a^3+b^3)+ad(a^2+d^2)]} = 6869$, $R_s = \sqrt{\frac{\pi (f_r)_{TE_{101}} \mu_0}{\sigma_c}}$

At the resonant frequency, $W_e = W_m = \frac{\varepsilon_0}{4}\mu^2 a^3 bd (f_r)^2_{TE_{101}} H_0^2 = 7.73\times 10^{-14}(J)$

Eg. For an air-filled rectangular copper cavity resonator, determine how much b should be increased in order to make Q 20% higher.

(Sol.) $\frac{Q_2}{Q_1} = 1.2 = \sqrt{\frac{b_2}{b_1}} \Rightarrow b_2 = 1.44 b_1$

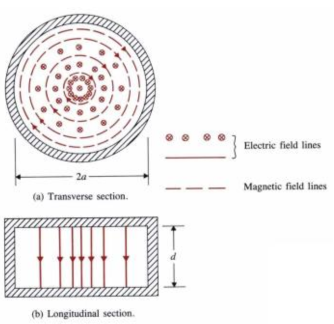
Eg. (a) What should be the size of a hollow cubic cavity made of copper in order for it to have a dominant resonant frequency of 10GHz? (b) Find the Q at that frequency.

(Sol.) (a) For a cubic cavity, a=b=d, TM_{110}, TE_{011}, and TE_{101} are degenerate dominant modes. $f_{101} = \frac{3\times 10^8}{\sqrt{2}a} = 10^{10}(Hz)$, $a = \frac{3\times 10^8}{\sqrt{2}\times 10^{10}} = 2.12\times 10^{-2} m$.

(b) $Q_{101} = \frac{\pi f_{101}\mu_0 a}{3R_s} = \frac{a}{3}\sqrt{\pi f_{101}\mu_0 \sigma}$

For copper, $\sigma = 5.80\times 10^7 (S/m)$, $Q_{101} = (\frac{2.12}{3}\times 10^{-2})\sqrt{\pi 10^{10}(4\pi 10^{-7})(5.80\times 10^7)} = 10700$.

6-6 Circular Cavity Resonators



For an air-filled circular cylindrical cavity resonator of radius a and length d. The resonant frequencies are

$$(f_r)_{TM_{mnp}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{X_{mn}}{a})^2 + (\frac{p\pi}{d})^2}$$, where $J_m(X_{mn})=0$

$$(f_r)_{TE_{mnp}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\frac{X'_{mn}}{a})^2 + (\frac{p\pi}{d})^2}$$, where $J'_m(X'_{mn})=0$

Note: In case of $2d>2a>d$, the dominant mode of the circular cylindrical cavity is ***TM***TM_{010} mode:

$$E_z = C_0 J_0(hr) = C_0 J_0(\frac{2.405}{a}r), \quad H_\phi = -\frac{jC_0}{\eta_0} J'_0(hr) = \frac{jC_0}{\eta_0} J_1(\frac{2.405}{a}r)$$

Quality factor of the TM_{010} mode: $Q_{TM_{010}} = \frac{\omega W}{P_L}$.

$$W = 2W_e = \frac{\varepsilon_0}{2}\int_v |E_z|^2 dv = \frac{\varepsilon_0 C_0}{2}(2\pi d)\int_0^a J_0^2(\frac{2.405}{a}r)rdr = (\pi\varepsilon_0 d)C_0^2[\frac{a}{2}J_1^2(2.405)]$$

$$P_L = \frac{R_s}{2}\left\{2\int_0^a |J_r|^2 2\pi r dr + (2\pi a d)|J_z|^2\right\} = \pi R_s\left\{2\int_0^a |H_\phi|^2 rdr + (ad)|H_\phi(r=a)|^2\right\}$$

$$= \frac{\pi R_s C_0^2}{\eta_0^2}\left\{2\int_0^a J_1^2(\frac{2.405}{a}r)dr + (ad)J_1^2(2.405)\right\} = \frac{\pi a R_s C_0^2}{\eta_0^2}(a+d)J_1^2(2.405)$$

$$\Rightarrow Q_{TM_{010}} = (\frac{\eta_0}{R_s})\frac{2.405}{2(1+a/d)}, \quad (f_r)_{TM_{010}} = \frac{2.405}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{0.115}{a}\times 10^9\, Hz$$

Eg. A hollow circular cylindrical cavity resonator is to be constructed of copper such that its length *d* equals its diameter 2*a*. (a) Determine *a* and *d* for a resonant frequency of 10*GHz* at the TM_{010} mode. (b) Find the *Q* of the cavity at resonance.

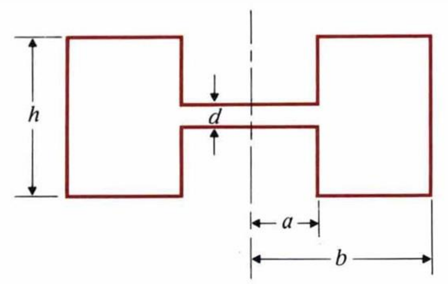
(Sol.) (a) $(f_r)_{TM_{010}} = \frac{0.115}{a}\times 10^9 = 10\times 10^9$, $a = 1.15\times 10^{-2} m$, $d=2a=2.30cm$.

(b) $R_s = \sqrt{\frac{\pi f \mu_0}{\sigma}} = \sqrt{\frac{\pi\times 10^{10}\times(4\pi 10^{-7})}{5.80\times 10^7}} = 2.61\times 10^2(\Omega)$, $Q = (\frac{120\pi}{2.61\times 10^{-2}})\frac{2.405}{2(1+1/2)} = 11{,}580.$

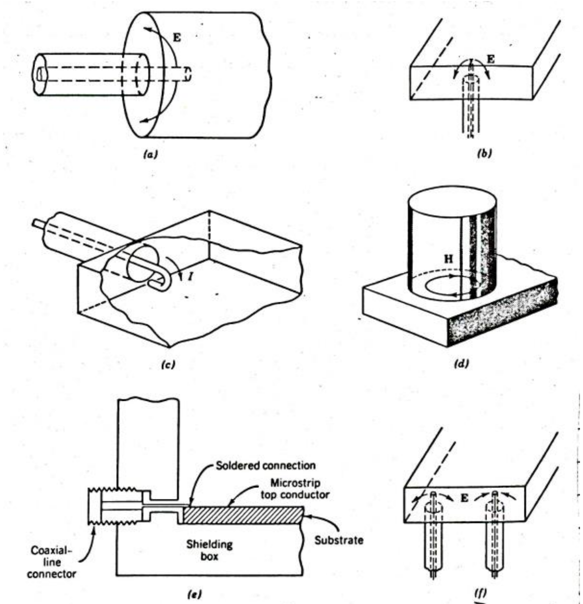
Eg. In some microwave applications, ring-shaped cavity resonators with a very narrow center part are used. A cross section of such a resonator is shown in the figure, in which *d* is very small in comparison with the resonant wavelength. Assuming that this the narrow center part and the inductance of the rest of the structure, find (a) the approximate resonant frequency. (b) the approximate resonant wavelength.

(Sol.) $C = \frac{\varepsilon\pi a^2}{d}$, $L = \frac{\mu h}{2\pi}\ln(\frac{b}{a})$

(a) $f_r = \frac{1}{2\pi\sqrt{LC}}$ (b) $\lambda_r = \frac{1}{f_r\sqrt{\mu\varepsilon}}$

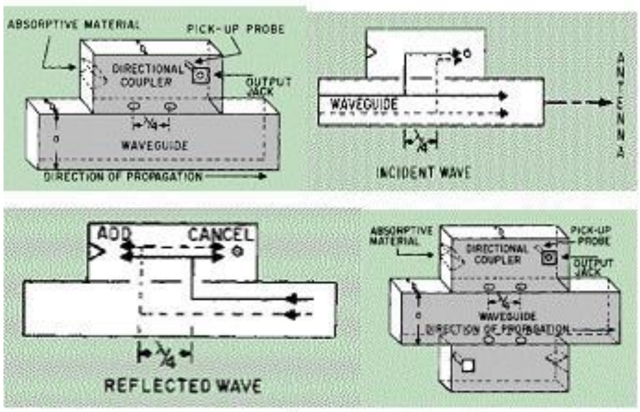


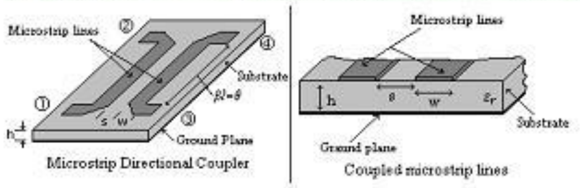
6-7 Excitations of Waveguides



(a) Antenna in end of circular guide for excitation of TM_{01} wave. (b) Antenna in bottom of rectangular guide for excitation of the TE_{10} wave. (c) Loop in end of rectangular guide for excitation of TE_{10} wave. (d) Junction between circular guide (TM_{01} wave) and rectangular guide (TE_{10} wave); large-aperture coupling. (e) Coaxial line coupling to microstrip. (f) Excitation of the TE_{20} wave in rectangular guide by two oppositely phased antennas.

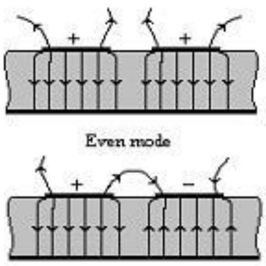
6-8 Directional couplers





Microstrip Directional Coupler

Coupled microstrip lines



Even mode

Odd mode

The electric fields for parallel-coupled microstrip lines