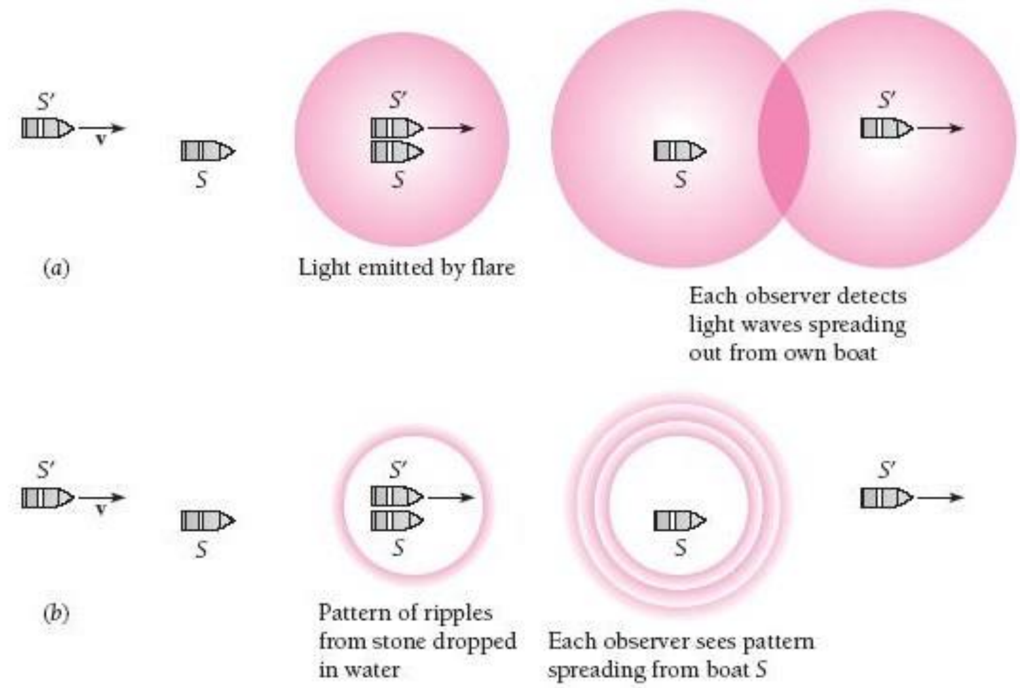
Chapter 1 Relativity

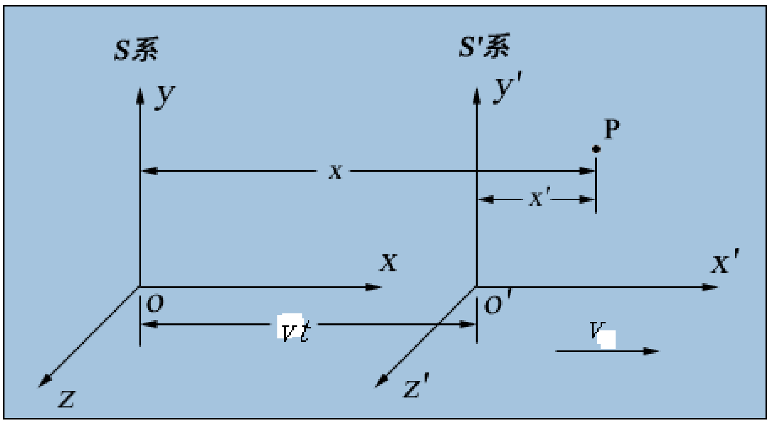
1-1 Postulates of Special Relativity and Lorentz Transformation

The 1st postulate: The laws of physics may be expressed in equations having the same form in all frames of reference moving at constant velocity with respect to one another.

The 2nd postulate: The speed of light in free spaces has the same value for all observers, regardless of their state of motion.

Eg. (a) Phenomenon of relativity. (b) Classical phenomenon.





Galilean transformation in classical physics: $\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases}$ and $\begin{cases} x = x' + vt' \\ y = y' \\ z = z' \\ t = t' \end{cases}$.

Lorentz transformation in relativity: $\begin{cases} x' = \dfrac{x - vt}{\sqrt{1-v^2/c^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - vx/c^2}{\sqrt{1-v^2/c^2}} \end{cases}$ and $\begin{cases} x = \dfrac{x' + v't'}{\sqrt{1-v^2/c^2}} \\ y = y' \\ z = z' \\ t = \dfrac{t' + v'x'/c^2}{\sqrt{1-v^2/c^2}} \end{cases}$.

Proof of Lorentz transformation: Suppose that a flare is set off at a common origin of S and S' at $t=t'=0$, and the observers in each system proceed to measure the speed light c at x and x', respectively. We have $x=ct$ and $x'=ct'$.

Let $\begin{cases} x' = k(x - vt) \\ y' = y \\ z' = z \\ t' = f(t, x, v) \end{cases}$ and $\begin{cases} x = k(x' + vt') \\ y = y' \\ z = z' \\ t = f^{-1}(t', x', v') \end{cases}$ $\Rightarrow x=k[k(x-vt)+vt']=k^2(x-vt)+kvt'$

$\Rightarrow t'=kt+\dfrac{(1-k^2)x}{kv}$ $\Rightarrow x'=k(x-vt)=ct'=c[kt+\dfrac{(1-k^2)x}{kv}]\Rightarrow x=ct[\dfrac{1-v/c}{1+(1-1/k^2)c/v}]=ct$

$\Rightarrow \dfrac{1-v/c}{1+(1-1/k^2)c/v}=1\Rightarrow k=\dfrac{1}{\sqrt{1-v^2/c^2}}$. ∴ Lorentz transformation holds.

Lorentz contraction: An observer in S' measures the length of a moving rod, of which ends to x_1' and x_2' in S'. The length in S' is $L_0=x_1'-x_2'=\dfrac{x_2-x_1}{\sqrt{1-v^2/c^2}}=\dfrac{L}{\sqrt{1-v^2/c^2}}$, where $L=x_1-x_2$ in $S\Rightarrow L=L_0\cdot(\sqrt{1-v^2/c^2})<L_0$.

∴ The length of an object in motion with respect to an observer is shorter than it is at rest with respect to him.

Eg. A meter stick appear only 50cm long to an observer. What is its relative velocity? How long does it take to pass the observer?

(Sol,) $0.5=1\cdot(\sqrt{1-v^2/c^2})\Rightarrow v=\dfrac{\sqrt{3}}{2}c$, $t=0.5/v=1.92\times10^{-9}$sec

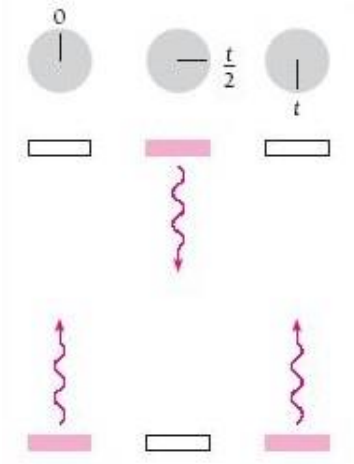
Time dilation: A clock is at x'in S'. When an observer in S' find the time is t_1', after a time interval t_0, and the time is up to t_2' (all in S'). The observer in S find the time interval is $t=t_2-t_1=\dfrac{t_2'-t_1'}{\sqrt{1-v^2/c^2}}=\dfrac{t_0}{\sqrt{1-v^2/c^2}}>t_0$. ∴ A stationary clock measure a longer time interval between events occurring in a moving frame of reference than a clock does in the moving frame.

Eg. (Twin paradox) A man leaves the earth in a spacecraft that makes a round trip to a star, 4 light-year distant, at a speed of 0.9*c*. How many years younger is she upon her return than his twin brother who remained behind?

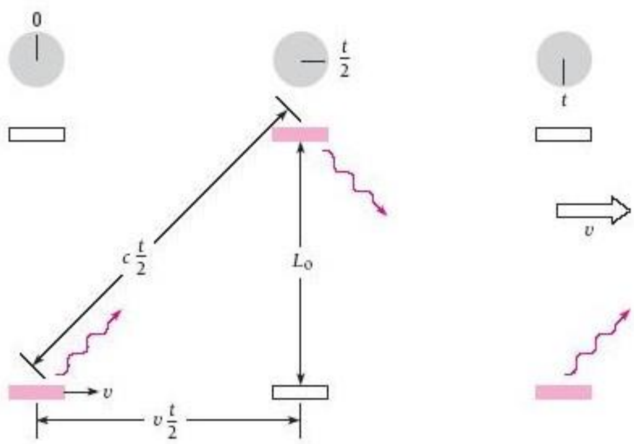
(Sol.) $2\times4c/0.9c-2\times4c(\sqrt{1-v^2/c^2})/0.9c=5.01$years

Another simple formulation of time dilation:

(a):



(b):



(a) A light-pulse clock at rest as seen by observer in S, the time interval t_0 is the light pulse travel between mirrors. Therefore we have $t_0=2L_0/c$.

(b) A light-pulse clock in S' as seen by observer in S, and hence we have

$(ct/2)^2 = L_0^2+(vt/2)^2$, $L_0=ct_0/2 \Rightarrow t=\dfrac{t_0}{\sqrt{1-v^2/c^2}}$

Simultaneity: Consider that a pair of time-bombs explodes at the same time t_0 at x_1 and x_2 in S, respectively. But in S', $t_1'=\dfrac{t_0-vx_1/c^2}{\sqrt{1-v^2/c^2}}$, $t_2'=\dfrac{t_0-vx_2/c^2}{\sqrt{1-v^2/c^2}}$

$\because$ $t_2'-t_1'=\dfrac{v(x_1-x_2)/c^2}{\sqrt{1-v^2/c^2}}\neq 0$ if $x_2\neq x_2$, $\therefore$ The explosions do not simultaneously occur in S' frame.

Relativistic velocities: Define $V_x=dx/dt$, $V_y=dy/dt$, $V_z=dz/dt$ in S, and $V_x'=dx'/dt'$, $V_y'=dy'/dt'$, $V_z'=dz'/dt'$ in S'.

$\because$ $dx=\dfrac{dx'+vdt'}{\sqrt{1-v^2/c^2}}$, $dy=dy'$, $dz=dz'$, $dt=\dfrac{dt'+\dfrac{vdz'}{c^2}}{\sqrt{1-v^2/c^2}}$, $\therefore$ we have

$$\begin{cases} V_x=\dfrac{dx}{dt}=\dfrac{dx+vdt'}{dt'+\dfrac{vdx'}{c^2}}=\dfrac{V_x'+v}{1+\dfrac{vV_x'}{c^2}} \\ V_y=\dfrac{V_y'\sqrt{1-v^2/c^2}}{1+\dfrac{vV_x'}{c^2}} \\ V_z=\dfrac{V_z'\sqrt{1-v^2/c^2}}{1+\dfrac{vV_x'}{c^2}} \end{cases} \quad \text{and} \quad \begin{cases} V_x'=\dfrac{V_x-v}{1-\dfrac{vV_x}{c^2}} \\ V_y'=\dfrac{V_y\sqrt{1-v^2/c^2}}{1-\dfrac{vV_x}{c^2}} \\ V_z'=\dfrac{V_z\sqrt{1-v^2/c^2}}{1-\dfrac{vV_x}{c^2}} \end{cases}.$$

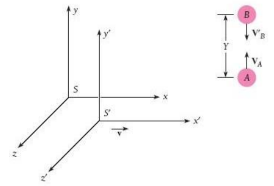
Eg. Spacecraft *A* has a velocity with respect to the earth of 0.9*c*. If spacecraft *B* is to pass spacecraft *A* at a relative velocity 0.5*c*, what velocity must spacecraft *B* have with respect to the earth?

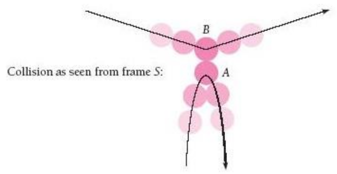
(Sol.) $V_x=\dfrac{V_x'+v}{1+\dfrac{vVx'}{c^2}}=\dfrac{0.5c+0.9c}{1+\dfrac{0.5c\times 0.9c}{c^2}}=0.9655c$

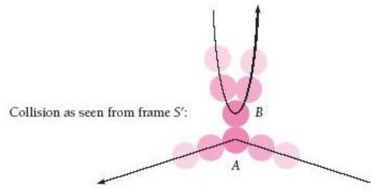
Eg. Show that c is a universal constant.

(Proof) $V_x'=\dfrac{V_x'-v}{1-\dfrac{vVx'}{c^2}}=\dfrac{c-v}{1-\dfrac{c\times v}{c^2}}=c$

1-2 Relativistic Mechanics







Suppose A and B collide at y=0.5Y, and $V_A=V_B'$. The round-trip time T_0 for A measured in S is $T_0=Y/V_A$, which is identical $T_0=Y/V_B'$ in S'.

$m_AV_A=m_BV_B'=m_BY/T=\dfrac{m_BY}{\dfrac{T_0}{\sqrt{1-v^2/c^2}}}=\dfrac{m_BY}{\dfrac{Y}{V_A}\cdot\dfrac{1}{\sqrt{1-v^2/c^2}}}\Rightarrow m_A=m_B\sqrt{1-v^2/c^2}$.

Let $m_A=m_0$, $m_B=m\Rightarrow m=\dfrac{m_0}{\sqrt{1-v^2/c^2}}$

Momentum: $p=mv=\dfrac{m_0v}{\sqrt{1-v^2/c^2}}$

Force: $F=dp/dt=mdv/dt+vdm/dt$

Kinetic energy: $T=\int_0^s Fds=\int_0^s\dfrac{d(mv)}{dt}ds=\int_0^v vd(mv)=\int_0^v vd(\dfrac{m_0v}{\sqrt{1-v^2/c^2}})$

$=\dfrac{m_0v^2}{\sqrt{1-v^2/c^2}}-m_0\int_0^v\dfrac{vdv}{\sqrt{1-v^2/c^2}}=\dfrac{m_0v^2}{\sqrt{1-v^2/c^2}}+m_0c^2\cdot(\sqrt{1-v^2/c^2}\,\Big|_0^v)=mc^2-m_0c^2$.

Eg. An electron has a kinetic energy of 0.1MeV. Find its velocity according to classical and relativistic mechanics.

(Sol.) Classical mechanics:

$0.1MeV=0.1\times10^6\times1.602\times10^{-19}=1.602\times10^{-14}J=9.109\times10^{-31}v^2/2\Rightarrow v=1.87\times10^8m/\text{sec}$

Relativistic mechanics:

$T=1.602\times10^{-14}=mc^2-m_0c^2=9.109\times10^{-31}c^2[\dfrac{1}{\sqrt{1-v^2/c^2}}-1]=1.64\times10^8m/\text{sec}$

Total energy: $E=mc^2=T+m_0c^2$

Eg. Dynamite liberates 5.6×10⁶J/Kg when it explodes. What fraction of its total energy contents in this?

(Sol.) Suppose that the dynamite is 1Kg.

$E=mc^2=9\times10^{16}$, $5.6\times10^6/9\times10^{16}=6\times10^{-11}$

1-3 Doppler Effect

Classical Doppler effect: $f=f_0 \cdot \left(\frac{1+v/w}{1-V/w}\right)$, where w is the velocity of wave, v is the velocity of observer, and V is the velocity of source.

1. $v>0$ if the observer moves toward the source; $v<0$ if the observer moves away from the source.
2. $V>0$ if the source moves toward the observer; $V<0$ if the source moves away from the observer.

Eg. The velocity of sound is 340*m*/sec. A train generates some noise as high as 3000*Hz* when its velocity is 50*m*/sec away from an observer who is standing at a fixed position. What frequency does he hear?

(Sol.) $f_0=3000$, $w=340$, $v=0$, $V=-50$, $f=f_0 \cdot \left(\frac{1+v/w}{1-V/w}\right)=2615.3Hz$

Doppler effect in relativity: $f=f_0 \cdot \left(\frac{\sqrt{1-v^2/c^2}}{1-v\cos\theta/c}\right)$, where θ is the angle between the moving direction of the light source and the line from the source to an observer. And v is the relative velocity of the source ($v>0$ if the source and the observer approaching each other; $v<0$ if they move away from each other).

Special case 1: Longitudinal Doppler effect, $\theta=0 \Rightarrow f=f_0 \cdot \sqrt{\frac{1+v/c}{1-v/c}}$

Special case 2: Transverse Doppler effect, $\theta=\pi/2 \Rightarrow f=f_0 \cdot \left(\sqrt{1-v^2/c^2}\right)$

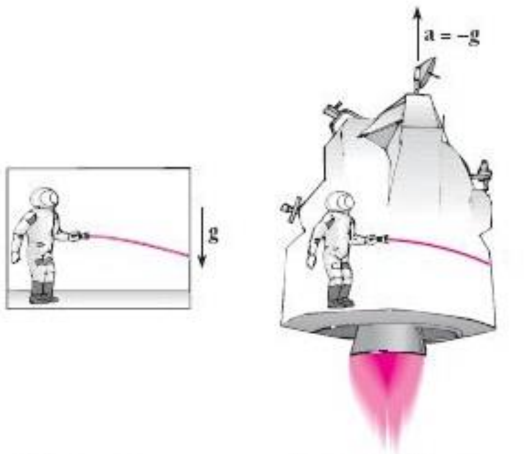
Eg. A spacecraft receding from the earth at 0.97*c* transmits data at the rate of 10^4pulse/sec. At what rate are they received?

(Sol.) $v=-0.97c$, $f_0=10^4$, $f=f_0 \cdot \sqrt{\frac{1+v/c}{1-v/c}}=1.234\times10^3$pulse/sec

Eg. (Red shift) A distant galaxy in the constellation Hydra is receding from the earth at 6.12×10^7m/sec. By how much is a green spectral line of wavelength 500nm emitted by this galaxy shifted toward the red end of the spectrum?

(Sol.) $v=-6.12\times10^7$, $\lambda_0=500$, $f=f_0 \cdot \sqrt{\frac{1+v/c}{1-v/c}} \Rightarrow \lambda=\lambda_0 \cdot \sqrt{\frac{1-v/c}{1+v/c}}=615nm$

1-4 Some Phenomena of General Relativity

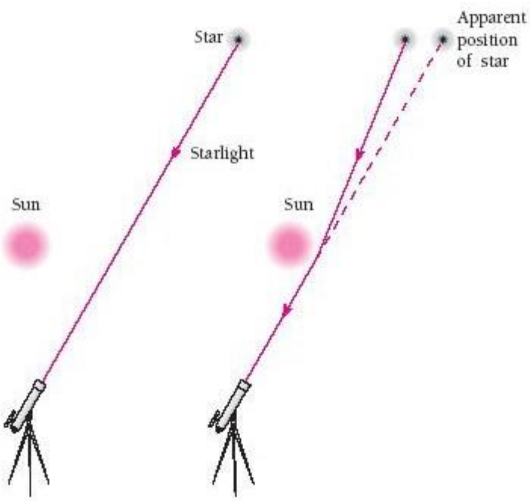


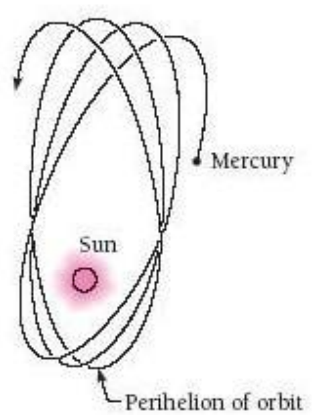
Laboratory in gravitational field

Accelerated laboratory

The principle of equivalence: An observer in a closed laboratory can not distinguish between the effects produced by a gravitational field and those produced by an acceleration of the laboratory.

Starlight passing near the sun is deflected by sun's strong gravitational field: This phenomenon has been proved by observing and measuring the precession of the perihelion of Mercury's orbit.





The precession of the perihelion of Mercury's orbit.