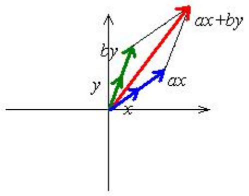
Chapter 1 Vector Spaces

1-1 Vector Spaces and Linear Combinations



Vector space V: V is a set over a field F if $\forall x, y \in V$ and $\forall a, b \in F$, $\exists !$ $ax+by \in V$.

Eg. R^2 is a vector space. For (a,b), $(c,d) \in R^2$, we can check: $-4(a,b)=(-4a,-4b) \in R^2$, $3(a,b)-7(c,d)=(3a-7c,3b-7d) \in R^2$, etc.

Eg. Show that the set of all polynomials $P(F)$ with coefficients from F is a vector space.

(Proof) $\left.\begin{matrix} x: f(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_0 \\ y: g(x)=b_n x^n + b_{n-1}x^{n-1}+\cdots+b_0 \end{matrix}\right\} \in P(F)$, $\forall a, b \in F$, $(af+bg)(x)=af(x)+bg(x) \in P(F)$

Subspace, W: A subset W of V is subspace of $V \Leftrightarrow \begin{cases} (1)\ x+y\in W, & where\ x\in W\ and\ y\in W \\ (2)\ ax\in W, & where\ a\in F\ and\ x\in W \\ (3)\ 0\ in\ V \Rightarrow 0\in W \\ (4)\ x+y=0\ for\ x\in W \Rightarrow y\in W \end{cases}$

Eg. R^2 is a subspace of R^3. For $(a,b,0)$, $(c,d,0) \in R^2$, we can check: $(a,b,0)+(c,d,0)=(a+c,b+d,0) \in R^2$, $-3(a,b,0)= (-3a,-3b,0) \in R^2$, $(0,0,0) \in R^2$, $(a,b,0)+(-a,-b,0)=(0,0,0)$ and then $(-a,-b,0) \in R^2$, etc.

Eg. V and $\{0\}$ are both subspace of V.

Theorem Any intersection of subspaces of a vector space V is a subspace of V.

Theorem W_1 and W_2 are subspaces of V, then $W_1 \cup W_2$ is a subspace $\Leftrightarrow W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Sum of two sets, S_1+S_2: $S_1+S_2=\{x+y: x\in S_1$ and $y\in S_2\}$.

Eg. Let $S_1=\{\cos(x), \cos(2x), \cos(3x), \ldots\}$ and $S_2=\{\sin(x), \sin(2x), \sin(3x), \ldots\}$, then $S_1+S_2=\{\cos(x)+\sin(2x), \cos(2x)+\sin(3x), \cos(5x)+\sin(x), \ldots\}$.

Theorem W_1 and W_2 are subspace of V, then W_1+W_2 is the smallest subspace that contains both W_1 and W_2.

Direct Sum, $W_1 \oplus W_2$: $W_1 \oplus W_2$ if $W_1 \cap W_2=\{0\}$ and $W=W_1+W_2$.

Eg. $F^n= W_1 \oplus W_2$, where $\begin{matrix} W_1=\{(a_1,\cdots,a_{n-1},0)\in F^n : a_n=0\} \\ W_2=\{(0,\cdots,0,a_n)\in F^n : a_1=a_2=\cdots=a_{n-1}=0\} \end{matrix}$

Eg. $P(F)=W_1 \oplus W_2$, where $\begin{matrix} W_1=\{f(x)=a_{2n+1}x^{2n+1}+\cdots+a_1 x : a_0=a_2=a_4=\cdots=0\} \\ W_2=\{g(x)=b_{2m}x^{2m}+\cdots+b_0 : b_1=b_3=b_5=\cdots=0\} \end{matrix}$

Even function: $f(-x)=f(x)$, **Odd function:** $f(-x)=-f(x)$

Eg. x^2, x^2-7x^{10}, $\cos(x)$ **are even functions, but** x, $3x-2x^3+5x^7$, $\sin(4x)$ **are odd functions.**

Theorem W_1 **and** W_2 **are the set of all even functions and the set of all odd functions in** $F(C,C)$**, respectively. Then** $F(C,C)=W_1\oplus W_2$**.**

(Proof) 1. W_1, W_2 are both subspaces of $F(C,C)$

2. $f(x)\in W_1\cap W_2$, $f(x)=f(-x)=-f(x)\Rightarrow f(x)=0$, $\therefore$ $W_1\cap W_2=\{0\}$

Let $\begin{cases} h(x)=\frac{1}{2}\left[g(x)+g(-x)\right] \\ i(x)=\frac{1}{2}\left[g(x)-g(-x)\right] \end{cases}$, then $h(x)\in W_1$, $i(x)\in W_2$, $\therefore$ $F(C,C)=W_1\oplus W_2$.

Theorem **Let** W_1 **and** W_2 **be two subspaces of a vector space** V **over** F**, and then** $V=W_1\oplus W_2\Leftrightarrow \forall x\in V$, $\exists!\, x_1\in W_1$, $\exists!\, x_2\in W_2$ **such that** $x=x_1+x_2$**.**

(Proof) Suppose $x=x_1+x_2=y_1+y_2$, $\begin{cases} x_1 \\ y_1 \end{cases}\in W_1$, $\begin{cases} x_2 \\ y_2 \end{cases}\in W_2$, $x_1-y_1=y_2-x_2$

$\because$ $x_1-y_1\in W_1$, $y_2-x_2\in W_2$, $\therefore$ $x_1-y_1=y_2-x_2\in W_1\cap W_2=\{0\}\Rightarrow x_1=y_1$, $x_2=y_2$.

Eg. Let W_1**,** W_2**, and** W_3 **denote the** x**-, the** y**-, and the** z**-axis, respectively. Then** $R^3=W_1\oplus W_2\oplus W_3$**,** $W_i\cap(\sum_{j\neq i} W_j)=\{0\}$**.** $\forall (a,b,c)\in R^3$**,** $(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c)$**, where** $(a,0,0)\in W_1$**,** $(0,b,0)\in W_2$**,** $(0,0,c)\in W_3$**.** $\therefore$ R^3 **is uniquely represented as a direct sum of** W_1**,** W_2**, and** W_3**.**

Eg. Let W_1 **and** W_2 **denote the** xy**- and the** yz**-planes, respectively. Then** $R^3=W_1+W_2$ **and** $W_1\cap W_2=\{(x,y,z)\mid x=z=0\}\neq\{0\}$**.** $\forall (a,b,c)\in R^3$**,** $(a,b,c)=(a,0,0)+(0,b,c)=(a,b,0)+(0,0,c)$**, where** $(a,0,0)$**,** $(a,b,0)\in W_1$ **and** $(0,b,c)$**,** $(0,0,c)\in W_2$**,** $\therefore$ R^3 **can not be uniquely represented as a direct sum of** W_1 **and** W_2**.**

Theorem W **is a subspace of** V **and** x_1**,** x_2**,** x_3**, …,** x_n **are elements of** W**, then** $\sum_{i=1}^{n} a_i x_i$ **is an element of** W **for any** a_i **over** F**.**

(Proof) $n=2$, it holds by definition. Suppose $n=k$ $(k\geq 2)$, $\sum_{i=1}^{i=k} a_i x_i$ is an element of W, and then $n=k+1$, $\sum_{i=1}^{i=k+1} a_i x_i = a_{k+1}x_{k+1}+(\sum_{i=1}^{i=k} a_i x_i)$ is also an element of W by definition. $\therefore$ the proof is complete.

Linear combination, $y=a_1x_1+a_2x_2+\ldots+a_nx_n$**:** It is called the linear combination of x_1, x_2, …, x_n in V, where $a_i\in F$, $\forall i\in n$, $x\in S$ and S is a nonempty subset of V.

***Span*(*S*)** (the subspace generated by the elements of S)**:** The subspace consists of all linear combinations of elements of S.

Eg. $S=\{x.y\}$**, then** $Span(S)=\{ax+by:\ \forall a, b\in F\}=\{3x-2y, -6x+1.5y, 4.3x+7.45y, 2x, -7y, \ldots\}$**.**

Theorem **(a)** ***S*** **is a nonempty subset of** $V \Rightarrow Span(S)$ **is a subspace of** ***V*****. (b)** $Span(S)$ **is the smallest subspace of** ***V*** **containing** ***S*** **in the sense that** $Span(S)$ **is a subset of only subspace of** ***V*** **that contains** ***S*****.**

Eg. $\because \forall\, a, b, c, d \in F$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$\therefore M_{2\times 2}(R) = Span(\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\})$.

Eg. $\because (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = \frac{a+b-c}{2}(1,1,0) + \frac{a+c-b}{2}(1,0,1) + \frac{b+c-a}{2}(0,1,1)$ $\forall\, a, b, c \in F$, $\therefore R^3 = Span(\{(1,0,0),(0,1,0),(0,0,1)\}) = Span(\{(1,1,0),(1,0,1),(0,1,1)\})$.

Eg. Plot $Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \cup Span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$**. [2006 台科大電子所]**

(Sol.) $Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: a line of $x=y$, $Span(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: the y-axis. $Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \cup Span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$: a union of $x=y$ and the y-axis. **Note:** $Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \cup Span(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) \neq Span(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) + Span(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} a \\ a+b \end{bmatrix}$ is the xy-plane.

Eg. Let $S=\{x_1, x_2, \ldots, x_n\}$ **be linearly independent set and their coefficients be selected from {0,1}. How many elements are there in** $Span(S)$**?**

(Sol.) If $y \in Span(S)$, $y = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, $a_i = \begin{cases} 0 \\ 1 \end{cases}$, $\therefore 2^n$ elements.

Theorem $Span(\varphi) = \{0\}$**.**

Theorem **A subset** ***W*** **of a vector space** ***V*** **is a subspace of** $V \Leftrightarrow Span(W) = W$**.**

(Proof) "$\Leftarrow$" : $\because Span(W)$ is a subspace of V, $\therefore W = Span(W)$ is a subspace of V.

"$\Rightarrow$": If $Span(W) = W' \neq W$, W' is a subspace of V.

$\because W' = Span(W)$ is the smallest subspace of V containing W, $\therefore W$ is a not subspace of V.

It is contradictory to the statement. $\therefore Span(W) = W' = W$.

Theorem **(a) If** S_1 **and** S_2 **are subspace of** ***V*** **and** $S_1 \subseteq S_2$**, then** $Span(S_1) \subseteq Span(S_2)$**.**

(b) $Span(S_1 \cap S_2) \subseteq Span(S_1) \cap Span(S_2)$**. [台大電研]**

(Proof) (a) $y = a_1x_1 + a_2x_2 + \cdots + a_nx_n \in Span(S_1)$, $a_i \in F$ and $x_i \in S_1$

$\because S_1 \subseteq S_2, \therefore x_i \in S_2 \Rightarrow y \in Span(S_2), \therefore Span(S_1) \subset Span(S_2)$

$(b) \because S_1 \cap S_2 \subseteq S_1, \therefore Span(S_1 \cap S_2) \subseteq Span(S_1) \cdots (1)$

$and\ S_1 \cap S_2 \subseteq S_2, \therefore Span(S_1 \cap S_2) \subseteq Span(S_2) \cdots (2)$

$by\ (1),(2) \Rightarrow Span(S_1 \cap S_2) \subseteq Span(S_1) \cap Span(S_2)$

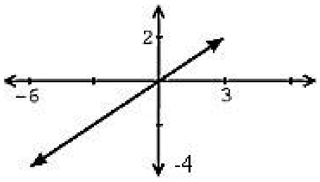
Theorem If S_1 and S_2 are subspace of V, then $Span(S_1 \cup S_2)=Span(S_1)+Span(S_2)$. [台大電研]

(Proof) $\because \begin{cases} S_1 \\ S_2 \end{cases} \subset (S_1 \cup S_2)$, $\therefore \begin{cases} Span(S_1) \\ Span(S_2) \end{cases} \subseteq Span(S_1 \cup S_2)$

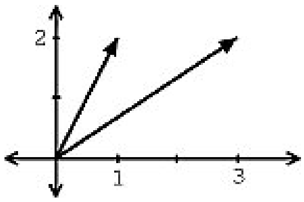
Suppose $Span(S_1 \cup S_2)=Span(S_1)+Span(S_2)+W$, where W is independent of S_1 and S_2, and $W \subseteq V$ ($\because S_1, S_2 \subseteq V$). Let $S_1=S_2 \Rightarrow W=\phi$, $\therefore Span(S_1 \cup S_2)= Span(S_1)+ Span(S_2)$

1-2 Linear Dependence and Linear Independence

Linear dependence & linear independence: For $x_1, x_2, \ldots, x_n \in S$, $\sum_{i=1}^{n} a_i x_i=0$ if $\exists\ a_1, a_2, \ldots, a_n$ are all zeros, then S is linearly independent; otherwise, S is linearly dependent.



Eg. (3,2) and (-6,-4) are linearly dependent because of 2(3,2)+1(-6,-4)=0, but (1,2) and (3,2) are linearly independent because of only 0(1,2)+0(3,2)=0



Theorem V is a vector space, $S_1 \subseteq S_2 \subseteq V$.

(a) If S_1 is linearly dependent, then S_2 is also linearly dependent.

(b) If S_2 is linearly independent, then S_1 is also linearly independent.

Basis: A basis β for a vector space V is a linearly independent subset of V that generates V.

Dimension, *dim*(V): The unique number of elements in each basis for V.

Theorem If $V=W_1 \oplus W_2$, then $dim(V)=dim(W_1)+dim(W_2)$.

Eg. $\forall \begin{bmatrix} a \\ b \end{bmatrix} \in R^2$, **we have** $\begin{bmatrix} a \\ b \end{bmatrix}=a\begin{bmatrix} 1 \\ 0 \end{bmatrix}+b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. **Thus** $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ **is the basis of R^2 and $dim(R^2)=2$.**

Eg. $\forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$, **we have** $\begin{bmatrix} a \\ b \\ c \end{bmatrix}=a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **Thus** $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ **is the basis of R^3 and $dim(R^3)=3$.**

Eg. For $W=\{(a_1,a_2,a_3,a_4,a_5) \in R^5: a_1+a_3+a_5=0,\ a_2=a_4\}$, find a basis of W and $dim(W)$.

(Sol.) $a_1+a_3+a_5=0$. Set $a_1=r$, $a_3=s$, $a_5=-r-s$, and set $a_2=a_4=t$.

$(a_1,a_2,a_3,a_4,a_5)=(r,t,s,t,-r-s)=r(1,0,0,0,-1)+t(0,1,0,1,0)+s(0,0,1,0,-1)$

Basis of W: $\{(1,0,0,0,-1),(0,1,0,1,0),(0,0,1,0,-1)\}$ and $dim(W)=3$.

Eg. Let V=$Span\{A_1,A_2,A_3,A_4\}$, where $A_1=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$, $A_2=\begin{bmatrix}1&0\\0&1\\-1&1\end{bmatrix}$, $A_3=\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}$, and $A_4=\begin{bmatrix}1&0\\0&-1\\1&0\end{bmatrix}$. Find a basis for V. [2005台大電研]

(Sol.) $aA_1+bA_2+cA_3+dA_4=\begin{bmatrix}a+b+c+d&0\\0&a+b+c-d\\a-b+d&a+b+c\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}\Rightarrow\begin{cases}a=r\\b=r\\c=-2r\\d=0\end{cases}$

$\therefore$ A_1, A_2, A_3, and A_4 are linearly dependent. In fact, $0.5A_1+0.5A_2=A_3$.

We can drop A_3,

$a'A_1+b'A_2+c'A_4=\begin{bmatrix}a'+b'+c'&0\\0&a'+b'-c'\\a'-b'+c'&a'+b'\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}\Rightarrow\begin{cases}a'=0\\b'=0\\c'=0\end{cases}$, $\therefore$ A_1, A_2, and A_4 are linearly independent.

$\therefore$ $\{A_1,A_2,A_4\}$ is the basis of V.

Theorem $\beta=\{x_1,x_2,\ldots,x_n\}$ is a basis for $V\Leftrightarrow y\in V$ can be uniquely expressed as a linear combination of vectors in β.

(Proof) If $y=a_1x_1+\cdots+a_nx_n=b_1x_1+\cdots+b_nx_n$, $0=(a_1-b_1)x_1+\cdots+(a_n-b_n)x_n$

$\because$ β is linearly independent, $\therefore$ $a_1-b_1=\cdots=a_n-b_n=0\Rightarrow a_1=b_1,\cdots,a_n=b_n$

Theorem S is a linearly independent subset of V, and let $x\in V$ but $x\notin S$. Then $S\cup\{x\}$ is linearly dependent $\Leftrightarrow x\in Span(S)$.

Eg. Show that in case $\beta=\{x_1,x_2,x_3\}$ be a basis in R^3, then $\beta'=\{x_1,x_1+x_2,x_1+x_2+x_3\}$ is also a basis in R^3. [文化電機轉學考]

(Proof) Set $a_1x_1+a_2(x_1+x_2)+a_3(x_1+x_2+x_3)=0\cdots(1)$. If $a_1=a_2=a_3=0$, then x_1, x_1+x_2, $x_1+x_2+x_3$ are linearly independent

$(1)\Rightarrow(a_1+a_2+a_3)x_1+(a_2+a_3)x_2+a_3x_3=0\cdots(2)$

$\Rightarrow\begin{cases}a_1+a_2+a_3=0\\a_2+a_3=0\\a_3=0\end{cases}\Rightarrow\begin{cases}a_1=0\\a_2=0\\a_3=0\end{cases}$, $\therefore$ x_1, x_1+x_2, $x_1+x_2+x_3$ are linearly independent.

$\because$ $dim(R^3)=3$, $\therefore$ β' is a basis of R^3.

Another method: $\begin{bmatrix}x_1\\x_1+x_2\\x_1+x_2+x_3\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $\det\left(\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\right)=1\neq0$, $\therefore$ β' is also a basis of R^3.

Eg. Determine whether the given set of vectors is linearly independent? [交大電信所]

(a) {(1,0,0),(1,1,0),(1,1,1)} in R^3.

(b) {(1,-2,1),(3,-5,2),(2,-3,6),(1,2,1)} in R^3.

(c) {(1,-3,2),(2,-5,3),(4,0,1)} in R^3.

(Sol.) (a) $\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)=1\neq 0$, ∴ Linearly independent. (b) 4 vectors in R^3, ∴ Linearly dependent.

(c) $\det\left(\begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix}\right)=-5-36+40+6=5\neq 0$, ∴ Linearly independent.

Eg. Are $(x-1)(x-2)$ and $|x-1|\cdot(x-2)$ linearly independent? [1990 中央土木所]

(Sol.) 1. If $1<x$, $c_1(x-1)(x-2)+c_2|x-1|(x-2)=c_1(x-1)(x-2)+c_2(x-1)(x-2)=0\Rightarrow c_1=-c_2$

2. If $x<1$, $c_1(x-1)(x-2)+c_2|x-1|(x-2)=c_1(x-1)(x-2)-c_2(x-1)(x-2)=0\Rightarrow c_1=c_2$

(1), (2) hold $\Rightarrow c_1=c_2=0$, ∴ Linearly independent.

Eg. Given a matrix $A=\begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}$ and a set of matrices $S=\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$. Determine if S is a linearly independent subset of $M_{2\times 2}$, the vector space of all 2×2 matrices? Represent the matrix A as a linear combination of the vectors in the set S. What are the corresponding coefficients? [台大電研]

(Sol.) Let $a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}+c\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}+d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Rightarrow\begin{cases} a=0 \\ 2c+d=0 \\ b+d=0 \\ b-c=0 \end{cases}\Rightarrow a=b=c=d=0$

∴ $S=\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$ is a linearly independent subset of $M_{2\times 2}$.

Let $\begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}=e\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}+f\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}+g\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}+h\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\Rightarrow\begin{cases} e=5 \\ e+2g+h=3 \\ e+f+h=1 \\ f-g=-2 \end{cases}\Rightarrow\begin{cases} e=5 \\ f=-2 \\ g=0 \\ h=-2 \end{cases}$

Eg. W_1 and W_2 are finite-dimensional subspace of V, and $dim(W_1)=m$, $dim(W_2)=n$ ($m\geq n$), then $dim(W_1\cap W_2)\leqq n$ and $dim(W_1+W_2)\leq m+n$. [2010 台大電研]

(Proof) 1. $W_1\cap W_2\subseteq W_2$, ∴ $dim(W_1\cap W_2)\leqq dim(W_2)=n$

2. $dim(W_1+W_2)=dim(W_1)+dim(W_2)-dim(W_1\cap W_2)$, ∵ $dim(W_1\cap W_2)\geqq 0$, ∴ $dim(W_1+W_2)\leq m+n$

Eg. Let v be the span of the set of vectors S={(1,-1,3),(0,2,1),(1,3,5)}.(a) What is the dimension of v? (b) Can we use S as a basis of v? [2006台科大電研]

(Sol.) (a) Let $a(1,-1,3)+b(0,2,1)+c(1,3,5)=(0,0,0) \Rightarrow a=-1,\ b=-2,\ c=1$, $\therefore$ S={(1,-1,3),(0,2,1),(1,3,5)} is linearly dependent. $\because$ S'={(0,2,1),(1,3,5)} is linearly independent, $\therefore$ $dim(v)=2$.

(b) No!

Transpose of a $m\times n$ matrix M, M^t: An $n\times m$ matrix M^t, in which $(M^t)_{ij}=M_{ji}$.

Eg. $M=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$, **then** $M^t=\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$.

In **Matlab** language, we can use the following instructions to obtain the transpose of a matrix:

```
>>A=[2,5;0,3]
A =
        2        5
        0        3

>>C=A'
C =
        2        0
        5        3
```

Symmetric matrix: $M=M^t$; that is, $M_{ij}=M_{ji}$.

Skew symmetric matrix: $M=-M^t$; that is, $M_{ij}=-M_{ji}$, $\forall\ i\neq j$, and $M_{ij}=0$, $\forall\ i=j$.

Eg. $A=\begin{bmatrix}1 & 3 & -0.6\\ 3 & 2 & 7\\ -0.6 & 7 & -8\end{bmatrix}$ **is a symmetric matrix.** $B=\begin{bmatrix}0 & -3.5\\ 3.5 & 0\end{bmatrix}$ **is a skew symmetric matrix.**

Eg. Show that the set of all square matrices can be decomposed into the direct sum of the set of the symmetric matrices and that of the skew-symmetric ones. [文化電機轉學考]

(Proof) 1. The set of the symmetric matrices W_1 and the set of the skew-symmetric matrices W_2 are both subspaces of $M_{n\times n}(F)$

2. $A\in W_1\cap W_2$, $A=A^t=-A^t \Rightarrow A=0$, $\therefore$ $W_1\cap W_2=\{0\}$

Let $\begin{cases} B=\dfrac{1}{2}\left[A+A^t\right] \\ C=\dfrac{1}{2}\left[A-A^t\right]\end{cases}$, then $B\in W_1$, $C\in W_2$, $\therefore$ $M_{n\times n}(F)=W_1\oplus W_2$.

Eg. The set of symmetric $n\times n$ matrices $M_{n\times n}(F)$ is a subspace W. Find a basis for W and $dim(W)$. [文化電機轉學考]

(Sol.) $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$, where $a_{ij}=a_{ji}$.

$$[\cdots] = a_{11}\begin{bmatrix} 1 & & & \\ & 0 & 0 & \\ & & 0 & \\ & 0 & \ddots & \\ & & & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ & 1 & & \\ & & 0 & \\ & 0 & & \ddots \\ & & & 0 \end{bmatrix} + \cdots + a_{nn}\begin{bmatrix} 0 & & & \\ & 0 & 0 & \\ & & \ddots & \\ & 0 & 0 & \\ & & & 1 \end{bmatrix}$$

$$+ a_{12}\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 0 & \\ & & 0 & \\ & 0 & \ddots & \\ & & & 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix} + \cdots + a_{1n}\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & 0 & & 0 & \\ & & \ddots & & \\ 0 & 0 & & 0 & \\ 1 & & & & 0 \end{bmatrix}$$

$$+ a_{23}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & & \\ 0 & 0 & & \ddots & \\ & & & & 0 \end{bmatrix} + a_{24}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & & \\ 0 & 1 & & \ddots & \\ & & & & 0 \end{bmatrix} + \cdots + a_{2n}\begin{bmatrix} 0 & & & \\ & 0 & 0 & 1 \\ & & \ddots & \\ & 0 & 0 & \\ & 1 & & 0 \end{bmatrix}$$

$$+ \cdots$$

$$\vdots$$

$$\dim(W) = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}.$$

Note: The dimension of set of skew-symmetric $n\times n$ matrices $M_{n\times n}(F)$ is $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Eg. What are the dimensions of the set of all the 5×5 symmetric matrices and that of all the 5×5 skew-symmetric ones, respectively?

(Sol.) Dimensions of the set of all the 5×5 symmetric matrices$=\frac{5(5+1)}{2}=15$

Dimensions of the set of all the 5×5 skew-symmetric matrices$=\frac{5(5-1)}{2}=10$