Chapter 2 Linear Transformations

2-1 Linear Transformations

Linear transformation T: $V \to W$, where V and W are vector spaces over F : $T(ax+by)=aT(x)+bT(y)$.

Note : $T(0)=0$.

Eg. $T[f(t)]=\int_a^b f(t)dt$ **is a linear transformation of $f(t)$.**

(Proof) $\int_a^b [af(t)+bg(t)]dt = a\int_a^b f(t)dt + b\int_a^b g(t)dt$

Null space of T, $N(T)$: $N(T)=\{x \in V: T(x)=0\} \subseteq V$.

Range of T, $R(T)$: $R(T)=\{T(x): x \in V\} \subseteq W$.

Nullity(T)=dim(N(T)), and ***Rank(T)=dim(R(T))***

Eg. Linear operator T: $R^3 \to R^2$ is defined by $T(a_1,a_2,a_3)=(a_1-a_2,2a_3)$. Find $N(T)$ and $R(T)$.

(Sol.) $N(T)$: $T(a_1,a_2,a_3)=(a_1-a_2,2a_3)=(0,0) \Rightarrow a_1=a_2=a,\ a_3=0$

$\Rightarrow N(T)=\{(a,a,0): a \in R\}$: A straight line in R^3.

$R(T)$: $a_1,a_2,a_3 \in R \Rightarrow a_1-a_2, 2a_3 \in R$, $\therefore R(T)=R^2$

Eg. Let T: $P_3(x) \to P_3(x)$ be defined by $T(f)=[f(x)+f(-x)]/2$. Find a basis for $N(T)$. Find a basis for $R(T)$. [2001台大電研]

(Sol.) $N(T)$: $T(ax^3+bx^2+cx+d)=[(ax^3+bx^2+cx+d)+(-ax^3+bx^2-cx+d)]/2=bx^2+d=0 \Rightarrow b=d=0$

$\Rightarrow N(T)=\{ax^3+cx: a, c \in R\}$: An odd function in $P_3(x)$. A basis for $N(T)$ is $\{x^3,x\}$

$R(T)$: $T(ax^3+bx^2+cx+d)=[(ax^3+bx^2+cx+d)+(-ax^3+bx^2-cx+d)]/2=bx^2+d$: An even function in $P_3(x)$. A basis for $R(T)$ is $\{x^2,1\}$.

Theorem Linear operator T: $V \to W$, and T is one-to-one $\Leftrightarrow N(T)=\{0\}$.

(Proof) "$\Rightarrow$": $T(x)=0=T(0)$. $\because$ T is one-to-one, $\therefore$ $x=0 \Rightarrow N(T)=\{0\}$

"$\Leftarrow$": Suppose $T(x)=T(y)$, then $0=T(x)-T(y)=T(x-y)$, Hence $x-y \in N(T) \Rightarrow x=y$

Onto: One-to-one linear operator T: $V \to W$ and $Rank(T)=dim(V)=dim(W)$.

Eg. (a) T: $F^2 \to F^2$ by $T(a_1,a_2)=(a_1+a_2,a_1)$ is onto.

(b) T: $P_2(x) \to P_3(x)$ by $T(f)(x)=2f'(x)+\int_0^x 3f(t)dt$ is one-to-one but not onto.

Theorem Linear transform T: $V \to W$, where V and W are vector spaces. Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

Theorem Linear transform T: $V \to W$, where V and W are vector spaces. If the basis of V is $\beta=\{x_1,x_2,\ldots,x_n\}$, then $R(T)=Span\{T(x_1),T(x_2),\ldots,T(x_n)\}$.

Eg. Linear operator L: $R^2 \to R^3$ is defined by $L(x,y)=(x+y,x-y,x+2y)$ (a) Find a basis for the range of L. (b) Is L onto? [1990 交大工業工程所]

(Sol.) $L(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ x+2y \end{bmatrix}$

(a) $\because Rank(\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}) = 3$, $\therefore$ Choose the basis of $R(L)=(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$. (b) $R^2 \to R^3$, $\therefore$ No.

Eg. Linear operator T: $R^2 \to R^2$ is defined by $T(1,0)=(1,4)$, $T(1,1)=(2,5)$. (a) $T(3,5)=$? (b) What is the null space of T? (c) What is $R(T)$? (d) Is T one-to-one? (e) Is T onto? [1990 交大電子所]

(Sol.) $T(x,y)=(ax+by,cx+dy)=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

$T(1,0)=(1,4) \Rightarrow a=1, c=4$, $T(1,1)=(2,5) \Rightarrow a+b=2, c+d=5 \Rightarrow b=1, d=1$

(a) $T(3,5) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$, or $T(3,5)=(8,17)$

(b) For the standard basis of R^2, $[T]_\beta = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 4x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+y=0 \\ 4x+y=0 \end{matrix} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \Rightarrow N(T)=\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$$

(c) $\because x+y, 4x+y \in R$, $\therefore R(T)=R^2$.

(d) $\because N(T)=\{0\}$, $\therefore$ Yes. (e) $\because$ one-to-one and $R^2 \to R^2$, $\therefore$ Yes.

Theorem Linear operator T: $V \to W$, if V is finite-dimensional, then $Nullity(T)+Rank(T)=dim(V)$. [1999 台大電研]

(Proof) Suppose $\dim(V)=n$ and $\{x_1,\cdots,x_k\}$: basis of $N(T)$ extend $\{x_1,\cdots,x_k\}$ to $\{x_1,\cdots,x_k,x_{k+1},\cdots,x_n\}$: basis of V

It is desired to show that $S=\{T(x_{k+1}),\cdots,T(x_n)\}$ is a basis for $R(T)$.

1. For $1 \le i \le k$, $T(x_i)=0$
2. $R(T)=Span\{T(x_1),\cdots,T(x_n)\}=Span(S)=Span\{T(x_{k+1}),\cdots,T(x_n)\}$
3. If $T(x_i)$ is linearly independent, $\forall\ k+1 \le i \le n$, then the proof is complete.

$$\sum_{i=k+1}^{n} b_i T(x_i) = 0 = \sum_{i=k+1}^{n} T(b_i x_i) = T(\sum_{i=k+1}^{n} b_i x_i),\ \forall k+1 \le i \le n \Rightarrow \sum_{i=k+1}^{n} b_i x_i \in N(T) = Span(x_1,\cdots,x_k)$$

$$\therefore \exists c_i \in F\ \forall i \le k\ \ s.t. \sum_{i=k+1}^{n} b_i x_i = \sum_{i=1}^{k} c_i x_i \ or\ \sum_{i=1}^{k}(-c_i)x_i + \sum_{i=k+1}^{n} b_i x_i = 0$$

$\because \beta=\{x_1,\cdots,x_n\}$ is a basis (linear independent), $\therefore b_i=0\ \forall i \Rightarrow T(x_i)$ is linearly independent

Eg. Let T: $V \to W$ be a one-to-one linear transformation and V be a finite-dimensional vector space. Show that $Rank(T)=dim(V)$. [台大電研類似題]

(Proof) T: $V \to W$ is one-to-one $\Leftrightarrow N(T)=\{0\}$, $\therefore Nullity(T)=0 \Rightarrow Nullity(T)+Rank(T)=Rank(T)=dim(V)$.

2-2 Matrix Representations and Coordinate (Basis) Transformations

In **Matlab** language, we can use the following instructions to find out the ij^{th}-entry, the m^{th}-row, and the n^{th}-column of a matrix:

```
>>A=[0,1,2;3,4,5]
A =
      0     1     2
      3     4     5
>>A(2,1)
ans =
      3
>>A(2,3)
ans =
      5
>>ROW1=A(1,:)
ROW1 =
      0     1     2
>>COL1=A(:,2)
COL1 =
      1
      4
```

Matrix representation of a linear transformation: Linear transformation T: $V \rightarrow W$, $\beta=\{x_1,x_2,\ldots,x_n\}$ is a basis for V, $\gamma=\{y_1,y_2,\ldots,y_m\}$ is a basis for W, then $\exists!\, a_{ij} \in F$ for i=1, 2, …, m and j=1, 2, …, n such that $T(x_j)=\sum_{i=1}^{m} a_{ij} y_i$ for j=1, 2, …, n. That is, $\begin{pmatrix} T(x_1) = a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \cdots \\ T(x_2) = a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + \cdots \\ T(x_3) = a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + \cdots, etc. \end{pmatrix}$, and then

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

Eg. Linear operator T: $P_3(R) \rightarrow P_2(R)$ by $T(f)=f'$, find $[T]_\beta^\gamma$, where β, γ are the standard ordered bases for $P_3(R)$ and $P_2(R)$, respectively.

(Sol.) $\beta = \{1, x, x^2, x^3\}$, $\gamma = \{1, x, x^2\}$

$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$, $T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$, $T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$, and

$$T(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 \Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Eg. Let W be the subspace $sp(\sin 2x, \cos 2x)$ of the vector space of all real-valued functions with domain R, and let $B=(\sin 2x, \cos 2x)$. Find the matrix representation A relative to B for the linear transformation T: $W\to W$ defined by $T(f)=D^2(f)+3D(f)+2f$, where D presents the derivative operator. (where $sp(X)$ denotes the set of all linear combinations of vectors in X) [2005 成大電腦通訊所]

(Sol.) $T[\sin(2x)]=-4\sin(2x)+6\cos(2x)+2\sin(2x)=-2\sin(2x)+6\cos(2x)$,

$T[\cos(2x)]=-4\cos(2x)-6\sin(2x)+2\cos(2x)=-6\sin(2x)-2\cos(2x)$, $\therefore A=\begin{bmatrix}-2 & -6\\ 6 & -2\end{bmatrix}$

Eg. Find the 4 by 4 matrix A represents a cyclic permutation: each vector (x_1,x_2,x_3,x_4) is transformed to (x_2,x_3,x_4,x_1). What is the effect of A^2? Show that $A^3=A^{-1}$. [2005 北科大電腦通訊所]

(Sol.) $\beta=\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$$A(1,0,0,0)=(0,0,0,1)=0(1,0,0,0)+0(0,1,0,0)+0(0,0,1,0)+1(0,0,0,1)$$
$$A(0,1,0,0)=(1,0,0,0)=1(1,0,0,0)+0(0,1,0,0)+0(0,0,1,0)+0(0,0,0,1)$$
$$A(0,0,1,0)=(0,1,0,0)=0(1,0,0,0)+1(0,1,0,0)+0(0,0,1,0)+0(0,0,0,1)$$
$$A(0,0,0,1)=(0,0,1,0)=0(1,0,0,0)+0(0,1,0,0)+1(0,0,1,0)+0(0,0,0,1)$$

$$\therefore A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}, A^2=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}, A^3=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}=A^{-1}$$

Eg. $V=R^{2\times 2}$, T: $V\to V$ is a linear transformation given by $T(A)=\dfrac{3A^T-A}{2}$. Find $[T]_\beta$, where β is a basis of V: $\beta=\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&2\\-1&0\end{bmatrix},\begin{bmatrix}0&-1\\2&0\end{bmatrix},\begin{bmatrix}0&0\\0&2\end{bmatrix}\}$. [2006 台科大電研]

(Sol.)

$$T(\begin{bmatrix}1&0\\0&0\end{bmatrix})=(3\begin{bmatrix}1&0\\0&0\end{bmatrix}-\begin{bmatrix}1&0\\0&0\end{bmatrix})/2=\begin{bmatrix}1&0\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&2\\-1&0\end{bmatrix}+0\begin{bmatrix}0&-1\\2&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

$$T(\begin{bmatrix}0&2\\-1&0\end{bmatrix})=(3\begin{bmatrix}0&-1\\2&0\end{bmatrix}-\begin{bmatrix}0&2\\-1&0\end{bmatrix})/2=\begin{bmatrix}0&-2.5\\3.5&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+(-\frac{1}{2})\begin{bmatrix}0&2\\-1&0\end{bmatrix}+\frac{3}{2}\begin{bmatrix}0&-1\\2&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

$$T(\begin{bmatrix}0&-1\\2&0\end{bmatrix})=(3\begin{bmatrix}0&2\\-1&0\end{bmatrix}-\begin{bmatrix}0&-1\\2&0\end{bmatrix})/2=\begin{bmatrix}0&3.5\\-2.5&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+\frac{3}{2}\begin{bmatrix}0&2\\-1&0\end{bmatrix}+(-\frac{1}{2})\begin{bmatrix}0&-1\\2&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

$$T(\begin{bmatrix}0&0\\0&2\end{bmatrix})=(3\begin{bmatrix}0&0\\0&2\end{bmatrix}-\begin{bmatrix}0&0\\0&2\end{bmatrix})/2=\begin{bmatrix}0&0\\0&2\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&2\\-1&0\end{bmatrix}+0\begin{bmatrix}0&-1\\2&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

$$\therefore [T]_\beta=\begin{bmatrix}1&0&0&0\\0&-\frac{1}{2}&\frac{3}{2}&0\\0&\frac{3}{2}&-\frac{1}{2}&0\\0&0&0&1\end{bmatrix}$$

Eg. Let T: $M_{2\times2}(F)\to M_{2\times2}(F)$ be the linear operator defined by $T(x)=\begin{bmatrix}1&1\\0&0\end{bmatrix}x+x\begin{bmatrix}0&0\\1&1\end{bmatrix}$. Find the matrix representation for T. [交大電信所]

(Sol.) Choose the standard basis β for $M_{2\times2}(F)$ as $\beta=\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\}$

$$T(\begin{bmatrix}1&0\\0&0\end{bmatrix})=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T(\begin{bmatrix}0&1\\0&0\end{bmatrix})=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}1&2\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+2\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T(\begin{bmatrix}0&0\\1&0\end{bmatrix})=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}+\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T(\begin{bmatrix}0&0\\0&1\end{bmatrix})=\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}0&0\\1&1\end{bmatrix}=\begin{bmatrix}0&1\\1&1\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\0&0\end{bmatrix}+1\begin{bmatrix}0&0\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$\therefore\ [T]_\beta=\begin{bmatrix}1&1&1&0\\0&2&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}$. $Rank(T)$=3.

Eg. T: $R^2\to R^3$. $T(a_1,a_2)=(a_1-a_2,a_1,2a_1+a_2)$. Let α={(1,2),(2,3)}, β={(1,0),(0,1)} for R^2 and γ={(1,1,0),(0,1,1),(2,2,3)} for R^3, then find $[T]_\alpha^\gamma$ and $[T]_\beta^\gamma$.

(Sol.)

1. $T(1,2)=(-1,1,4)=a(1,1,0)+b(0,1,1)+c(2,2,3),\ T(2,3)=(-1,2,7)=d(1,1,0)+e(0,1,1)+f(2,2,3)$

$$\Rightarrow a=-\frac{7}{3},\ b=2,\ c=\frac{2}{3},\ d=-\frac{11}{3},\ e=3,\ f=\frac{4}{3},\ \therefore [T]_\alpha^\gamma=\begin{bmatrix}-\frac{7}{3}&-\frac{11}{3}\\2&3\\\frac{2}{3}&\frac{4}{3}\end{bmatrix}$$

2. $T(1,0)=(1,1,2)=a(1,1,0)+b(0,1,1)+c(2,2,3),\ T(0,1)=(-1,0,1)=d(1,1,0)+e(0,1,1)+f(2,2,3)$

$$\Rightarrow a=-\frac{1}{3},\ b=0,\ c=\frac{2}{3},\ d=-1,\ e=1,\ f=0,\ \therefore [T]_\beta^\gamma=\begin{bmatrix}-\frac{1}{3}&-1\\0&1\\\frac{2}{3}&0\end{bmatrix}$$

Eg. Let β'={$2x^2-x,3x^2+1,x^2$} and β={$1,x,x^2$} be both the ordered bases for $P_2(x)$. And let T: $P_2(x)\to P_2(x)$ be defined by $T(ax^2+bx+c)=cx^2+bx+a$, then find $[T]_\beta^\beta$, $[T]_{\beta'}^{\beta'}$, $[T]_\beta^{\beta'}$, $[T]_{\beta'}^\beta$. [文化電機轉學考]

(Ans.) $[T]_\beta^\beta=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, $[T]_{\beta'}^{\beta'}=\begin{bmatrix}1&0&0\\2&3&1\\-8&-8&-3\end{bmatrix}$, $[T]_\beta^{\beta'}=\begin{bmatrix}0&-1&0\\0&0&1\\1&2&-3\end{bmatrix}$, $[T]_{\beta'}^\beta=\begin{bmatrix}2&3&1\\-1&0&0\\0&1&0\end{bmatrix}$.

Annihilator of S, S^0: $S^0=\{T(x): T(x)=0 \text{ for } x\in S\}$, where V and W are subspaces, S is a subspace of V.

Theorem (a) $S_1\subseteq S_2$, then $S_2^0 \subseteq S_1^0$. (b) V_1 and V_2 are subspaces of V, then $(V_1+V_2)^0=V_1^0\cap V_2^0$.

(Proof) (a) $S_1 \subseteq S_2 \Leftrightarrow x\in S_1$, then $x\in S_2 \Rightarrow$ If $T(x)=0\ \forall\ x\in S_2$, then $T(x)=0$ for all $x\in S_1\subseteq S_2$

$\Leftrightarrow S_2^0\subseteq S_1^0$

$$(b)\begin{cases}V_1\subseteq V_1+V_2\\ V_2\subseteq V_1+V_2\end{cases}\Rightarrow\begin{cases}(V_1+V_2)^o\subseteq V_1^0\\ (V_1+V_2)^o\subseteq V_2^0\end{cases}\Rightarrow (V_1+V_2)^0\subseteq V_1^0\cap V_2^0$$

For $T\in V_1^0\cap V_2^0 \Leftrightarrow T\in V_1^0$ and $T\in V_2^0$, $\forall a\in V_1+V_2 \Leftrightarrow a=\alpha a_1+\beta a_2$

for $a_1\in V_1$ and $a_2\in V_2$

$\Rightarrow T(a)=\alpha T(a_1)+\beta T(a_2)=0$, $T\in (V_1+V_2)^0 \Rightarrow V_1^0\cap V_2^0\subseteq (V_1+V_2)^0$, $\therefore$ $(V_1+V_2)^0=V_1^0\cap V_2^0$

Multiplication of matrices: If $C=AB$, then $C_{ij}=\sum_{k=1}^{m}A_{ik}B_{kj}$, where A is a $m\times n$ matrix, B is an $n\times p$ matrix, and C is an $m\times p$ matrix.

In **Matlab** language, we can use the following instructions to obtain the product of two matrices:

```
>>A=[2,5,1;0,3,-1];
>>B=[1,0,2;-1,4,-2;5,2,1];
>>C=A*B
A =
       2     5     1
       0     3    -1
B =
       1     0     2
      -1     4    -2
       5     2     1
C =
       2    22    -5
      -8    10    -7
```

Eg. $A=\begin{bmatrix} \pi & \sqrt{e} & \frac{1}{3} & \sqrt{2} \\ 3.7 & 10^5 & 7 & 0 \\ \ell n2 & i & \sin(2) & -1 \end{bmatrix}\cdot\begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 0.2 & -0.3 & 0.1 \\ 2 & -1 & -1 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\cdot\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\cdot\begin{bmatrix} -1 & 2.3 & 1 & -3 \\ \sqrt{5} & \frac{1}{2} & 1 & 2 \\ 3 & -2 & 1 & \sqrt{2} \end{bmatrix}$**, find the third column of *A*. [1993 中山應數所]**

(Sol.) *A* is a (3×4)・(4×3)・(3×3)・(3×4)=(3×4) matrix. $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$, $A\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$,

$$A\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} -1 & 2.3 & 1 & -3 \\ \sqrt{5} & \frac{1}{2} & 1 & 2 \\ 3 & -2 & 1 & \sqrt{2} \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}=\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$=\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 0.2 & -0.3 & 0.1 \\ 2 & -1 & -1 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\cdot\begin{bmatrix} a+b+c \\ a+b+c \\ a+b+c \end{bmatrix}=(a+b+c)\begin{bmatrix} ' & ' & ' \\ ' & ' & ' \\ ' & ' & ' \end{bmatrix}\cdot\begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 0.2 & -0.3 & 0.1 \\ 2 & -1 & -1 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$=(a+b+c)\begin{bmatrix} \pi & \sqrt{e} & \frac{1}{3} & \sqrt{2} \\ 3.7 & 10^5 & 7 & 0 \\ \ell n2 & i & \sin(2) & -1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}=(a+b+c)\begin{bmatrix} \sqrt{2} \\ 0 \\ -1 \end{bmatrix}.$$ ∴ The third column is $\begin{bmatrix} \sqrt{2}(a+b+c) \\ 0 \\ -(a+b+c) \end{bmatrix}$.

Theorem ***T*: *V*→*W*, let *β*, *β'* be the ordered bases of *V*, and *γ*, *γ'* be the ordered bases of *W*. If [*β*]=*Q*[*β'*] and [*γ*]=*P*[*γ'*], we have** $[T]_{\beta'}^{\gamma'}=P^{-1}[T]_{\beta}^{\gamma}Q$.

(Proof) $[\gamma]=[T]_{\beta}^{\gamma}[\beta]$, $[\gamma]=P[\gamma']$, $[\beta]=Q[\beta']\Rightarrow[\gamma]=[T]_{\beta}^{\gamma}Q[\beta']$

$$[\gamma']=P^{-1}[T]_{\beta}^{\gamma}Q[\beta']=[T]_{\beta'}^{\gamma'}[\beta']\Rightarrow[T]_{\beta'}^{\gamma'}=P^{-1}[T]_{\beta}^{\gamma}Q$$

Eg. Assume $T(\begin{bmatrix} x \\ y \end{bmatrix})=\begin{bmatrix} y \\ -x+y \end{bmatrix}$ **in the standard basis. Find the matrix representation for *T* with respect to the new basis {(1,1),(1,2)}. [1993 中山應數研]**

(Sol.) $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})=\begin{bmatrix} 0 \\ -1 \end{bmatrix}=0\begin{bmatrix} 1 \\ 0 \end{bmatrix}-1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})=\begin{bmatrix} 1 \\ 1 \end{bmatrix}=1\begin{bmatrix} 1 \\ 0 \end{bmatrix}+1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Rightarrow T(\begin{bmatrix} x \\ y \end{bmatrix})=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\Rightarrow[T]_{\beta}=\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}=1\begin{bmatrix} 1 \\ 0 \end{bmatrix}+1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}=1\begin{bmatrix} 1 \\ 0 \end{bmatrix}+2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $P=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $P^{-1}=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$,

$$[T]_{\beta'}=\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$$

Eg. A linear transform $T(a_1,a_2,a_3)=(2a_1+a_2,a_1+a_2-a_3)$: $R^3\to R^2$ and $\beta=\{(1,0,0),(0,1,0),(0,0,1)\}$, $\beta'=\{(2,0,0),(0,-1,0),(0,0,-2)\}$, $\gamma=\{(1,0),(0,1)\}$, $\gamma'=\{(1,1),(1,-1)\}$, then find $[T]_\beta^\gamma$ and $[T]_{\beta'}^{\gamma'}$.

(Sol.) $T(1,0,0)=(2,1)=2(1,0)+1(0,1)$, $T(0,1,0)=(1,1)=1(1,0)+1(0,1)$, $T(0,0,1)=(0,-1)=0(1,0)-1(0,1)$

$\Rightarrow [T]_\beta^\gamma=\begin{bmatrix}2&1&0\\1&1&-1\end{bmatrix}$. $(1,1)=1(1,0)+1(0,1)$, $(1,-1)=1(1,0)-1(0,1)$, $\therefore P=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$, $P^{-1}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$

$(2,0,0)=2(1,0,0)+0(0,1,0)+0(0,0,1)$, $(0,-1,0)=0(1,0,0)-1(0,1,0)+0(0,0,1)$

$(0,0,-2)=0(1,0,0)+0(0,1,0)-2(0,0,1)\Rightarrow Q=\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-2\end{bmatrix}\Rightarrow [T]_{\beta'}^{\gamma'}=P^{-1}[T]_\beta^\gamma Q=\begin{bmatrix}3&-1&1\\1&0&-1\end{bmatrix}$

Another method:

$T(2,0,0)=(4,2)=3(1,1)+1(1,-1),\ \ T(0,-1,0)=(-1,-1)=-(1,1)+0(1,-1)$

$T(0,0,-2)=(0,2)=1(1,1)-1(1,-1)\Rightarrow Q=\begin{bmatrix}3&-1&1\\1&0&-1\end{bmatrix}$

Trace of an $n\times n$ matrix, $tr(M)$: $tr(M)=\sum_{i=1}^{n}M_{ii}\Rightarrow tr(aA+bB)=atr(A)+btr(B)$

Eg. $A=\begin{bmatrix}1&2\\3&-4\end{bmatrix}$, **$tr(A)=1+(-4)=-3$.**

Eg. Show that $tr(AB)=tr(BA)$ and $tr(A)=tr(A^t)$. [2005 台大電研]

(Proof) $tr(AB)=\sum_i(AB)_{ii}=\sum_i\sum_k A_{ik}B_{ki}=\sum_i\sum_k B_{ki}A_{ik}=\sum_k\sum_i B_{ki}A_{ik}=\sum_k(BA)_{kk}=tr(BA)$

Eg. Show that no matrices A and $B\in M_{n\times n}(F)$ such that $AB-BA=I$, where I is an $n\times n$ unit matrix. [台大電研]

(Proof) $\exists A$ and B such that $AB-BA=I$, then $tr(AB-BA)=tr(AB)-tr(BA)=0\neq n=tr(I)$. It is contradictory to the assumption; hence no such matrices A and B exist.

Similar matrices: $\begin{cases}A\\B\end{cases}\in M_{n\times n}(F)$, B is similar to A if $\exists$ invertible $Q\in M_{n\times n}(F)$ fulfills $B=Q^{-1}AQ$.

Eg. If A and B are similar to each other, then $tr(A)=tr(B)$. [台大電研]

(Proof) $B=Q^{-1}AQ\ \ tr(B)=tr(Q^{-1}AQ)=tr(AQQ^{-1})=tr(A)$

Eg. Show that $A=\begin{bmatrix}1&4\\-2&3\end{bmatrix}$ and $B=\begin{bmatrix}2&9\\3&-7\end{bmatrix}$ are not similar to each other.

(Proof) $tr(A)=1+3=4\neq -5=2+(-7)=tr(B)$, $\therefore$ A and B are not similar to each other.