Chapter 3 Elementary Matrix Operations and Determinants

3-1 Elementary Operations

Elementary row (or column) operations:

1. Interchanging any two adjacent rows (or columns). **Eg.** $\begin{bmatrix}1&2\\3&4\end{bmatrix}\to\begin{bmatrix}3&4\\1&2\end{bmatrix}$

2. Multiplying any row (or column) by a nonzero constant. **Eg.** $\begin{bmatrix}1&2\\3&4\end{bmatrix}\to\begin{bmatrix}1&2\\-6&-8\end{bmatrix}$

3. Adding any constant multiple of a row (or column) to another row (or column).

Eg. $\begin{bmatrix}1&2\\3&4\end{bmatrix}\to\begin{bmatrix}-5&-6\\3&4\end{bmatrix}$**: Adding minus double of the 2nd row to the 1st row.**

***Rank* of *A*, *Rank*(*A*):** $Rank(A)=dim(Range(L_A))$, where $L_A: F^n \to F^m$.

Theorem $A\in M_{m\times n}(F)$, $P\in M_{m\times m}(F)$, and $Q\in M_{n\times n}(F)$. If P and Q are both invertible, then $Rank(A)=Rank(PA)=Rank(AQ)=Rank(PAQ)$.

Theorem $A, B\in M_{m\times n}(F)$, **then** $Rank(A+B)\le Rank(A)+Rank(B)$. **[2000 台大電研]**

(Proof) $y\in Range(L_A+L_B)$. $\exists\ x$ such that $y=(L_A+L_B)(x)=L_A(x)+L_B(x)$

$\because L_A(x)\in Range(L_A)$, $L_B(x)\in Range(L_B)$, $(L_A+L_B)(x)\in Range(L_A)+Range(L_B)$

$\Rightarrow Range(L_A+L_B)\subseteq Range(L_A)+Range(L_B) \Rightarrow Rank(A+B)\le Rank(A)+Rank(B)$

Theorem $A\in M_{m\times n}(F)$, $B\in M_{n\times p}(F)$, **then** $Rank(AB)\le\begin{cases}Rank(A)\\Rank(B)\end{cases}$. **[1999 台大電研]**

(Proof) $y\in Range(L_AL_B)$. $\exists\ x$ such that $y=(L_AL_B)(x)$

$\because z=L_B(x)\in Range(L_B)$, $L_A(z)\in Range(L_A)$, $y=(L_AL_B)(x)=L_A(L_B(x))=L_A(z)\in Range(L_A)$,

$\Rightarrow R(L_AL_B)\subseteq R(L_A)\Rightarrow Rank(AB)\le Rank(A)$

Note: $Rank(AB)\le Rank(B)$ can be proved by $Rank(A)=Rank(A^t)$

Question: How to determine the rank of a matrix?

Eg. Given $A=\begin{bmatrix}1&0&1\\0&1&1\\1&0&1\end{bmatrix}$, $Rank(A)=dim(Span\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\})=2$

Eg. By elementary operation, $B=\begin{bmatrix}1&1&0&1\\2&2&0&2\\1&1&0&1\\1&1&0&1\end{bmatrix}\to\begin{bmatrix}1&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$, $Rank(B)=1$

Eg. For $A=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 4\\ 3 & 0 & a\end{bmatrix}$**, find** ***Rank***$(A)$**. [1993 台大電研]**

(Sol.) $\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 4\\ 3 & 0 & a\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 2\\ 3 & 0 & a-3\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & a-3\end{bmatrix}$, 1. $a=3\Rightarrow rank(A)=2$ 2. $a\neq 3\Rightarrow rank(A)=3$

Eg. For $A=\begin{bmatrix}1 & -3 & 2 & -1 & 3\\ 2 & -5 & 4 & -2 & 4\end{bmatrix}$**, determine the dimension of** $N(A)$ **and the basis for** $N(A)$**. What is** ***Rank***(A)**? [1993 中山應數所]**

(Sol.) $\begin{bmatrix}1 & -3 & 2 & -1 & 3\\ 2 & -5 & 4 & -2 & 4\end{bmatrix}\to\begin{bmatrix}1 & -3 & 0 & 0 & 3\\ 2 & -5 & 0 & 0 & 4\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0 & 3\\ 2 & -1 & 0 & 0 & 4\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 0 & 0\end{bmatrix}$

Rank(A)=2, *Nullity*(A)=*dim*(V)-*Rank*(A) =5-2=3.

$$\begin{bmatrix}1 & -3 & 2 & -1 & 3\\ 2 & -5 & 4 & -2 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ v\\ w\end{bmatrix}=0\Rightarrow\begin{bmatrix}x\\ y\\ z\\ v\\ w\end{bmatrix}=r\begin{bmatrix}-2\\ 0\\ 1\\ 0\\ 0\end{bmatrix}+s\begin{bmatrix}1\\ 0\\ 0\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}3\\ 2\\ 0\\ 0\\ 1\end{bmatrix}$$

. The basis of $N(A)$ is $\{\begin{bmatrix}-2\\ 0\\ 1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 0\\ 1\\ 0\end{bmatrix},\begin{bmatrix}3\\ 2\\ 0\\ 0\\ 1\end{bmatrix}\}$

In **Matlab** language, we can use the following instructions to obtain the rank of a matrix:

```
>>A=[2 1;4 3];
>>rank(A)
A =
        2        1
        4        3
ans =
        2
```

3-2 Inverse of a Matrix

Inverse of a matrix A^{-1}**:** $A A^{-1}= A^{-1}A =I$

Eg. Show that $\frac{dA^{-1}}{dx}=-A^{-1}\frac{dA}{dx}A^{-1}$**. [交大電子所]**

(Proof) $A A^{-1}= A^{-1}A =I$, $A\frac{dA^{-1}}{dx}+\frac{dA}{dx}A^{-1}=0$, $\therefore$ $\frac{dA^{-1}}{dx}=-A^{-1}\frac{dA}{dx}A^{-1}$

Eg. Show that $(AB)^{-1}=B^{-1}A^{-1}$**.**

(Proof) $AB(AB)^{-1}=I=(AB)^{-1}AB$ and $ABB^{-1}A^{-1}=I=B^{-1}A^{-1}AB$, $\therefore$ $(AB)^{-1}=B^{-1}A^{-1}$

Augmented matrix: $[A|B]=(A^{(1)},\ldots,A^{(n)},B^{(1)},\ldots,B^{(n)})$

Method of obtaining the inverse of a matrix A: $M=E_p\cdots E_2E_1$,. If $(A|I)=(MA|M)=(I|M)$, then $M=A^{-1}$.

Eg. $A=\begin{bmatrix}1&-1&2\\3&0&1\\1&0&2\end{bmatrix}$**, find** A^{-1}**.**

(Sol.) $\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\3&0&1&0&1&0\\1&0&2&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&3&-5&-3&1&0\\0&1&0&-1&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&1&0&-1&0&1\\0&0&-5&0&1&-3\end{array}\right]$

$\to\left[\begin{array}{ccc|ccc}1&-1&2&1&0&0\\0&1&0&-1&0&1\\0&0&-5&0&1&-3\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&-1&0&1&\frac{2}{5}&-\frac{6}{5}\\0&1&0&-1&0&1\\0&0&1&0&-\frac{1}{5}&\frac{3}{5}\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&0&2/5&-1/5\\0&1&0&-1&0&1\\0&0&1&0&-1/5&3/5\end{array}\right],$

$\therefore\ A^{-1}=\begin{bmatrix}0&2/5&-1/5\\-1&0&1\\0&-1/5&3/5\end{bmatrix}$

Eg. $A=\begin{bmatrix}0&2&4\\2&4&2\\3&3&1\end{bmatrix}$**, find** A^{-1}**.**

(Sol.) $\left[\begin{array}{ccc|ccc}0&2&4&1&0&0\\2&4&2&0&1&0\\3&3&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}2&4&2&0&1&0\\0&2&4&1&0&0\\3&3&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&2&1&0&\frac{1}{2}&0\\0&2&4&1&0&0\\3&3&1&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&2&1&0&\frac{1}{2}&0\\0&2&4&1&0&0\\0&-3&-2&0&\frac{-3}{2}&1\end{array}\right]$

$\to\left[\begin{array}{ccc|ccc}1&2&1&0&\frac{1}{2}&0\\0&1&2&\frac{1}{2}&0&0\\0&-3&-2&0&\frac{-3}{2}&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&-3&-1&\frac{1}{2}&0\\0&1&2&\frac{1}{2}&0&0\\0&0&4&\frac{3}{2}&\frac{-3}{2}&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&-3&-1&\frac{1}{2}&0\\0&1&2&\frac{1}{2}&0&0\\0&0&1&\frac{3}{8}&-\frac{3}{8}&\frac{1}{4}\end{array}\right]$

$\to\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{8}&\frac{-5}{8}&\frac{3}{4}\\0&1&0&\frac{-1}{4}&\frac{3}{4}&\frac{-1}{2}\\0&0&1&\frac{3}{8}&\frac{-3}{8}&\frac{1}{4}\end{array}\right],\ A^{-1}=\begin{bmatrix}\frac{1}{8}&\frac{-5}{8}&\frac{3}{4}\\\frac{-1}{4}&\frac{3}{4}&\frac{-1}{2}\\\frac{3}{8}&\frac{-3}{8}&\frac{1}{4}\end{bmatrix}$

Eg. (a) Find the inverse of A, where A=$\begin{bmatrix}1 & -4 & 4\\ 0 & 1 & -1\\ 1 & -2 & 1\end{bmatrix}$. (b) Find the null space of A. [交大控制、光電研究所] (Ans.) A^{-1}=$\begin{bmatrix}1 & 4 & 0\\ 1 & 3 & -1\\ 1 & 2 & -1\end{bmatrix}$, $N(A)$={$\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$}

Eg. Compute the inverse matrix of A=$\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 10\end{bmatrix}$ by some elementary operations. [台科大電研]

Eg. Compute the inverse matrix of A=$\begin{bmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{bmatrix}$. [2003 台科大電研]

(Ans.) A^{-1}=$\begin{bmatrix}d/\Delta & -b/\Delta & 0\\ -c/\Delta & a/\Delta & 0\\ 0 & 0 & 1\end{bmatrix}$, where △=$ad$-$bc$

In **Matlab** language, we can use the following instructions to obtain the inverse of a matrix:

```
>>A=[2,1;4,3]
A =
     2     1
     4     3
>>inv(A)
ans =
    1.5000   -0.5000
   -2.0000    1.0000
```

3-3 System of Linear Equations

Matrix representation for system of linear equations:

Eg. $\begin{cases}2x_1+3x_2+x_3=1\\ x_1-x_2+2x_3=6\end{cases} \Leftrightarrow \begin{bmatrix}2 & 3 & 1\\ 1 & -1 & 2\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}1\\ 6\end{bmatrix} \Leftrightarrow Ax=B$.

Theorem **$A \in M_{n\times n}(F)$. $AX=B$ has only one solution $A^{-1}B$ if A is invertible. Conversely, if the system has exactly one solution, then A is invertible.**

Theorem ***AX*=*B* has at least one solution** $\Leftrightarrow$ ***Rank*(*A*)=*Rank*(*A*|*B*).**

Gaussian elimination method:

Eg. Solve $\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ 2x_1 + 2x_2 + x_3 = 1 \\ 3x_1 + 5x_2 - 2x_3 = -1 \end{cases}$ **.**

(Sol.) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & -2 & 3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 3 & 3 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$, $\therefore$ x_1=4, x_2=-3, x_3=-1.

In **Matlab** language, we can use the following instructions to solve a linear system of equations:

```
>>A=[1,3,5;2,4,6;1,2,1];
>>B=[2,1,3]'; % [2,1,3]' is the transpose of [2,1,3]
>>rref([A,B]) % solve
```

$\begin{cases} x + 3y + 5z = 2 \\ 2x + 4y + 6z = 1 \\ x + 2y + z = 3 \end{cases}$.

ans =

1.0000	0	0	-3.7500
0	1.0000	0	4.0000
0	0	1.0000	-1.2500

Homogeneous linear system: *AX* =0.

Eg. Solve *AX*=0 for *A*=$\begin{bmatrix} 1 & -5 & 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$**. [2003 台科大電研]**

(Sol.) $\begin{bmatrix} 1 & -5 & 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Theorem $A \in M_{m\times n}(F)$. $AX=0$ has a nontrivial solution if $m<n$ and $Rank(A)=m$.

Theorem Let K be the set of all solutions to $AX=B$, and let K_H be the set of the solutions of $AX=0$. Then for any solution S to $AX=B$, $K=\{S\}+K_H=\{S+h: h\in K_H\}$.

Eg. Solve $\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}7\\ -4\end{bmatrix}$.

(Sol.) $\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow K_H=\{t\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}, t\in F\}$

$S=\begin{bmatrix}1\\ 1\\ 4\end{bmatrix}$ fulfills $\begin{bmatrix}1 & 2 & 1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}7\\ -4\end{bmatrix} \Rightarrow K=\{\begin{bmatrix}1\\ 1\\ 4\end{bmatrix}+t\begin{bmatrix}1\\ -2\\ 3\end{bmatrix},\ t\in F\}$

Eg. Solve the system of linear equations: $\begin{cases} x_1-4x_2-x_3+x_4=3\\ 2x_1-8x_2+x_3-4x_4=9\\ -x_1+4x_2-2x_3+5x_4=-6\end{cases}$. [中正電研]

(Sol.) $\left[\begin{array}{cccc|c}1 & -4 & -1 & 1 & 3\\ 2 & -8 & 1 & -4 & 9\\ -1 & 4 & -2 & 5 & -6\end{array}\right] \to \left[\begin{array}{cccc|c}1 & -4 & -1 & 1 & 3\\ 0 & 0 & 3 & -6 & 3\\ 0 & 0 & -3 & 6 & -3\end{array}\right]$

$\to \left[\begin{array}{cccc|c}1 & -4 & -1 & 1 & 3\\ 0 & 0 & 1 & -2 & 1\\ 0 & 0 & -1 & 2 & -1\end{array}\right] \to \left[\begin{array}{cccc|c}1 & -4 & -1 & 1 & 3\\ 0 & 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$

$\Leftrightarrow \begin{cases} x_1-4x_2-x_3+x_4=3\\ \qquad x_3-2x_4=1\\ \qquad\qquad 0=0\end{cases} \Rightarrow x_3=1+2x_4 \Rightarrow x_1=4x_2+x_4+4$

Let $x_2=r$, $x_4=s \Rightarrow$ $\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}4\\ 0\\ 1\\ 0\end{bmatrix}+r\begin{bmatrix}4\\ 1\\ 0\\ 0\end{bmatrix}+s\begin{bmatrix}1\\ 0\\ 2\\ 1\end{bmatrix}$

3-4 Determinants of Matrices

Determinant: $det(A)=\sum_{i=1}^{n}(-1)^{i+j}A_{ij}\cdot\det(\tilde{A}_{ij})$, where $\tilde{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j.

Eg. $det\left(\begin{bmatrix}1&1&0&1\\1&0&1&1\\0&0&1&1\\1&1&1&1\end{bmatrix}\right)=\sum_{i=1}^{4}(-1)^{i+2}A_{i2}\cdot\det(\tilde{A}_{i2})$

$$=(-1)^{1+2}\cdot1\cdot\det\begin{bmatrix}1&1&1\\0&1&1\\1&1&1\end{bmatrix}+(-1)^{2+2}\cdot0\cdot\det\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\end{bmatrix}+(-1)^{3+2}\cdot0\cdot\det\begin{bmatrix}1&0&1\\1&1&1\\1&1&1\end{bmatrix}+(-1)^{4+2}\cdot1\cdot\det\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}$$

Theorem $A\in M_{n\times n}(F)$ **and** $Rank(A)<n$, **then** $det(A)=0$.

Theorem

(a) For $A, B\in M_{n\times n}(F)$, **then** $det(AB)=det(A)det(B)$.

(b) If A **is invertible,** $det(A^{-1})=1/det(A)$.

(c) $det(A^t)=det(A)$. **Eg.** $det\left(\begin{bmatrix}1&2\\3&4\end{bmatrix}\right)=-2=det\left(\begin{bmatrix}1&3\\2&4\end{bmatrix}\right)$

(d) $det(A)=\prod_{i=1}^{n}A_{ii}$ **if** A **is a triangular matrix. Eg.** $det\left(\begin{bmatrix}2&0&0\\11.4&-3&0\\-6.2&5.3&-4\end{bmatrix}\right)=2\times(-3)\times(-4)=24$.

Theorem Q **is invertible** $\Leftrightarrow$ $det(Q)\neq0$.

Orthogonal matrix B**:** $B^tB=BB^t=I$.

Eg. $B=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}$ **is orthogonal because** $\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

Unitary matrix B**:** $BB^*=I$, where $(B^*)_{ij}=B_{ji}^*$.

Theorem (a) If B **is orthogonal, then** $det(B)=\pm1$.

(b) If B **is unitary, then** $|det(B)|=1$.

(Proof) (a) $det(I)=1=det(BB^t)=det(B)\,det(B^t)=[det(B)]^2\Rightarrow det(B)=\pm1$.

(b) $det(I)=1=det(BB^*)=det(B)det(B^*)=det(B)det(B)^*=|det(B)|^2\Rightarrow|det(B)|=1$.

Summary of properties of the determinants:

1. If B is obtained by exchanging two adjacent rows (or two adjacent columns) of A, then $det(B)$=-$det(A)$. **Eg.** $det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix})$=-2 and $det(\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix})$=2.

2. If B is obtained by multiplying each entry of some row (or column) of A by c, then $det(B)=c \cdot det(A)$.
Eg. $det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix})$=-2 and $det(\begin{bmatrix} 1 & 2 \\ -9 & -12 \end{bmatrix})$=(-12)-(-18)=6=(-3)×(-2).

3. If B is obtained from A by adding a multiple of row i (or column i) to row j (or column j), where $i \neq j$, then $det(A)=det(B)$. **Eg.** $det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix})$=-2 and $det(\begin{bmatrix} -5 & -6 \\ 3 & 4 \end{bmatrix})$=-2.

4. $det(I)$=1.

5. $det(A)=\prod_{i=1}^{n} A_{ii}$ if A is a triangular matrix. **Eg.** $det(\begin{bmatrix} 1 & 0 & 0 \\ 11.4 & -3 & 0 \\ -6.2 & 5.3 & 2 \end{bmatrix})$=(1)×(-3)×(2)=-6.

Eg. Find the determinants of $A=\begin{bmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{bmatrix}$ **and** A^{-1}.

(Sol.) Triangularize A, we obtain

$$det(A)=\begin{vmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{vmatrix}=\begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & -12 & 5 & -7 \\ 0 & -2 & 4 & -6 \end{vmatrix}=\begin{vmatrix} 1 & 4 & -2 & -3 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 8/3 & -16/3 \end{vmatrix}=\begin{vmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -8 \end{vmatrix}=-144,$$

$$\therefore\ det(A^{-1}) = -1/144$$

Eg. $A=\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$, $det(A)$=?

(Sol.) $$A \rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 3 & 3 & 2 \\ 0 & -4 & -1 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & -13 \\ 0 & 0 & 3 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}$$

$$det(A)=-[1 \cdot (-1) \cdot 3 \cdot (-13)]=-39$$

Eg. Digits 1 to 9 can be arranged into 3×3 matrices in 9! ways. Find the sum of the determinants of these matrices. [台大電研]

(Sol.) $[det(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix})+det(\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix})]+[det(\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix})+det(\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix})]+\ldots=0.$

Eg. Find $det(A^{-1})$ for $A=\begin{bmatrix} 1+x_1 & x_2 & x_3 & \ldots\ldots & x_n \\ x_1 & 1+x_2 & x_3 & \ldots\ldots & x_n \\ x_1 & x_2 & 1+x_3 & \ldots\ldots & x_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_1 & x_2 & x_3 & \ldots\ldots & 1+x_n \end{bmatrix}$. [1990 中央土木所]

(Sol.) $det(\begin{bmatrix} 1+x_1 & x_2 \\ x_1 & 1+x_2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & -1 \\ x_1 & 1+x_2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & -1 \\ 0 & 1+x_1+x_2 \end{bmatrix}) = 1+x_1+x_2$,

$$\det(\begin{bmatrix} 1+x_1 & x_2 & x_3 \\ x_1 & 1+x_2 & x_3 \\ x_1 & x_2 & 1+x_3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & -1 & 0 \\ x_1 & 1+x_2 & x_3 \\ x_1 & x_2 & 1+x_3 \end{bmatrix}) = \det(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ x_1 & x_2 & 1+x_3 \end{bmatrix})$$

$$=\det(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & x_1+x_2 & 1+x_3 \end{bmatrix})=\det(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1+x_1+x_2+x_3 \end{bmatrix})=1+x_1+x_2+x_3,\ldots, \ \therefore \ \det(A^{-1})=\frac{1}{\det(A)}=\frac{1}{1+\sum_{i=1}^{n}x_i}$$

Eg. Triangularize the matrix: $\begin{bmatrix} -2 & 1 & 0 & 3 \\ 1 & -3 & 2 & 4 \\ 3 & 0 & 2 & -1 \\ 2 & -2 & 4 & 6 \end{bmatrix}$. [台大電研]

In **Matlab** language, we can use the following instructions to obtain the determinant of a matrix:

```
>>A=[1,3,0;-1,5,2;1,2,1];

>>det(A)

ans =

10
```

Eg. Let $f_1(t)$, $f_2(t)$, …, $f_n(t)$ be the polynomials of degrees $\le n$-2. Prove that $det(A)=0$ for

$$A=\begin{bmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) & \cdots & f_1(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n(a_1) & f_n(a_2) & f_n(a_3) & \cdots & f_n(a_n) \end{bmatrix}$$

, where $a_1, a_2, \ldots, a_n$ are arbitrary constants. [1991 台大電研]

(Sol.) $\forall i \le n,\ f_i(t) = a_{i0} + a_{i1}t + \ldots + a_{i,n-2}t^{n-2} \in Span\{1, t, \ldots, t^{n-2}\}$

$\because\ \dim\left(Span\langle 1, t, \ldots, t^{n-2}\rangle\right) = n-1 < n$, $\therefore$ $\{f_1(t), f_2(t), \ldots, f_n(t)\}$ is linearly dependent.

That is, $\exists\, j \quad s.t \quad f_j(t) = c_1 f_1(t) + c_2 f_2(t) + \ldots + c_{j-1} f_{j-1}(t) + c_{j+1} f_{j+1}(t) + \ldots + c_n f_n(t),\quad 1 \le j \le n$

$\Rightarrow$ *for arbitrary* a_i, $f_j(a_i) = c_1 f_1(a_i) + c_2 f_2(a_i) + \ldots + c_{j-1} f_{j-1}(a_i) + c_{j+1} f_{j+1}(a_i) + \ldots + c_n f_n(a_i)$

$$\Rightarrow A = \begin{bmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) & \cdots & f_1(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_j(a_1) & f_j(a_2) & \cdots & \cdots & f_j(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n(a_1) & f_n(a_2) & f_n(a_3) & \cdots & f_n(a_n) \end{bmatrix}$$

$$= \begin{bmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) & \cdots & f_1(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i\ne j} c_i f_i(a_1) & \sum_{i\ne j} c_i f_i(a_2) & \cdots & \cdots & \sum_{i\ne j} c_i f_i(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n(a_1) & f_n(a_2) & f_n(a_3) & \cdots & f_n(a_n) \end{bmatrix} \Rightarrow rank(A) < n \Rightarrow \det(A) = 0$$

3-5 Classical Adjoint Matrix

Cofactor: For $det(A)=\sum_{1}^{n}(-1)^{i+j} A_{ij}\det(\tilde{A}_{ij})$, the scalar factor $(-1)^{i+j}\det(\tilde{A}_{ij})$ is the cofactor of A_{ij}.

Eg. For $A=\begin{bmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{bmatrix}$**, find the cofactor of** A_{41}**.**

(Sol.) The cofactor of $A_{41}=(-1)^{4+1}\det\begin{bmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 0 & -3 & 1 \end{bmatrix}=23$

Classical adjoint matrix: $adj(A)= det(A)A^{-1}\equiv C$ is called the classical adjoint matrix of A if $det(A)\neq 0$, and C_{ij}=cofactor of A_{ji}.

Note: $adj(A)\cdot A = det(A)\cdot I$ and $A^{-1}=adj(A)/det(A)$

Eg. For $A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$**, find** $adj(A)$ **and** A^{-1}**.**

(Sol.) $det(A)$=-2, $adj(A)=\begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix}=\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

$$A^{-1}=\frac{adj(A)}{\det(A)}=-\frac{1}{2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}=\begin{bmatrix} \frac{7}{2} & -3 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

Theorem A^{-1} **does not exist** $\Leftrightarrow$ $det(A)=0$**.**

Eg. For $A=\begin{bmatrix} \beta & 0 & 2 \\ 0 & 3 & \beta \\ 0 & \beta & 3 \end{bmatrix}$**, find** β **such that** A^{-1} **does not exist. [1990 清大物理所]**

(Sol.) $det(A)=9\beta-\beta^3=0$, $\beta=0, \pm 3$.

Theorem **If $C=adj(A)\in M_{n\times n}(F)$, then $det(C)=[det(A)]^{n-1}$.**

(Proof) $C = adj(A) = \det(A)A^{-1} \Rightarrow CA = \det(A)I_n = \begin{bmatrix} \det(A) & 0\cdots & 0 \\ 0 & & 0 \\ \vdots & \cdots\cdots & \vdots \\ 0 & 0\cdots & \det(A) \end{bmatrix}$

$\det(CA) = \det(C)\det(A) = [\det(A)]^n \Rightarrow \det(C) = [\det(A)]^{n-1}$.

Eg. For $adj(A)=\begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, find $det(A)$ and A. [1990 台大機研]

(Sol.) $n=3$, $det(adj(A))=[det(A)]^{n-1}=4 \Rightarrow det(A)=\pm 2$

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A) \Rightarrow A^{-1} = \pm \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \Rightarrow A = \pm \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Vander monde matrix: $\begin{bmatrix} 1 & c_0{}^1 & c_0{}^2 & \ldots & c_0{}^n \\ 1 & c_1{}^1 & c_1{}^2 & \ldots & c_1{}^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & c_n{}^1 & c_n{}^2 & \ldots & c_n{}^n \end{bmatrix}$

Theorem **$det(M)=\prod_{j=1}^{n}\prod_{i=0}^{j-1}(c_j - c_i)$ if M is the Vander monde matrix.**

(Proof) If some $j=i$, then $c_j=c_i \Rightarrow det(M)=0 \Rightarrow \exists (c_j-c_i)$ factor

Let $det(M)=k \cdot \prod_{j=1}^{n}\prod_{i=0}^{j-1}(c_j - c_i)$. For $n=1$, $det(M)= (c_1-c_0)=k(c_1-c_0)$, $k=1$

Eg. Let $M=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \end{bmatrix}$, determine $det(M)$.

(Sol.) $n=3$, $c_0=1$, $c_1=2$, $c_2=-3$, $c_3=-1$,

$det(M)=\prod_{j=1}^{n}\prod_{i=0}^{j-1}(c_j - c_i)=(c_1-c_0)(c_2-c_0)(c_2-c_1)(c_3-c_0)(c_3-c_1)(c_3-c_2)$

$=(2-1)(-3-1)(-3-2)(-1-1)(-1-2)(-1+3)=240$