Chapter 5 Inner Product Spaces

5-1 Inner Products and Norms

Inner product $<x,y>$: V is a vector space over F, and $<x,y>$ has the following characteristics:
(a) $<ax+bz,y>=a<x,y>+b<z,y>$, (b) $<x,y>=<y,x>^*$, (c) $<x,x>$ is positive if $x\neq 0$.

Eg. Let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$. Determine which of the following are inner products on R^3. (i) $<u,v>=u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2$, (ii) $<u,v>=2u_1v_1+u_2v_2+3u_3v_3$, (iii) $<u,v>=u_1v_1-u_2v_2+u_3v_3$. [2003交大電信所]

(Sol.) Only (ii) is the inner product. (i) does not fulfill (a), (iii) does not fulfill (c) if $u_1v_1+u_3v_3<u_2v_2$.

Eg. The following calculations fulfill the definition of the inner product:

(a) $x=(a_1,a_2,\ldots,a_n)$, $y=(b_1,b_2,\ldots,b_n)$, $<x,y>=\sum_{i=1}^{n} a_i\overline{b_i}$, (b) $A,B\in M_{nxn}(F)$, $<A,B>=tr(B^*A)$,

(c) $f,\ g\in V$, $<f,g>=\int_0^a f(x)\overline{g(x)}dx$.

In **Matlab** language, we can use the following instructions to obtain the inner product and the outer product of two vectors:

```
>>A=[4,-1,3]; B=[-2,5,2]; C=dot(A,B); D=cross(A,B)
A =
     4    -1     3
B =
    -2     5     2
C =
    -7
D=
   -17   -14    18
```

Norm $||\cdot||$: A real-valued function on V and satisfies the following conditions for all $x,\ y\in V$ and $c\in F$.
(a) $||cx||=|c|\cdot||x||$, (b) $||x||\geqq 0$, and $||x||=0$ if $x=0$, (c) $||x+y||\leqq||x||+||y||$.

Eg. Show that $||-x||=||x||$. [1993 台大電研]

(Proof) $||-x||=|-1|\cdot||x||=||x||$

Eg. The following calculations fulfill the definition of the norm:

(a) $V=M_{mxn}(F)$, $||A||=\max_{i,j}|A_{ij}|$, (b) $V=C[0,a>0]$, $||f||=\max_{t\in[0,a]}|f(t)|$, (c) $V=F^n$, $\|x\|=\left[\sum_{i=1}^{n}|a_i|^2\right]^{1/2}$

Euclidean norm of a vector: $x\in V$, $||x||=\sqrt{\langle x,x\rangle}$

Theorem (a) $|<x,y>|\leqq||x||\cdot||y||$ (Cauchy-Schwartz Inequality), (b) $||x+y||^2+||x-y||^2=2||x||^2+2||y||^2$.

(Remark) By definition of the Euclidean norm.

In **Matlab** language, we can use the following instructions to obtain the Euclidean norm of a vector:

>> x=[3 4 5]; c=norm(x)

c =

    7.0711

Orthogonal vectors x and y: $x \perp y \Leftrightarrow <x,y>=0$.
Orthonormal vectors x and y: $\|x\|=\|y\|=1$ and $<x,y>=0$ if $x \neq y$.

Eg. Show that if $\{v_1, v_2, v_3, \ldots, v_n\}$ is an orthogonal set of non-zero vectors in an inner product space, then $v_1, v_2, v_3, \ldots,$ and v_n are linearly independent. [2015 中央電研固態組、生醫電子組]
(Proof) Let $y=\sum_{i=1}^{n} a_i v_i$ be a linear combination of $v_1, v_2, v_3, \ldots,$ and v_n.
Since $\{v_1, v_2, v_3, \ldots, v_n\}$ is an orthogonal set of non-zero vectors in an inner product space, we have $<v_i, v_j>=0$ if $j \neq i$ but $<v_i, v_i> \neq 0$ in case of $j=i$.
Consider an inner product $<y,v_j>=<\sum_{i=1}^{n} a_i v_i, v_j>=\sum_{i=1}^{n} a_i <v_i, v_j>=a_i <v_i, v_i>$
Set $y=0$ and then we have $a_i=0$ because $<v_i, v_i> \neq 0$ for each i. It implies that $v_1, v_2, v_3, \ldots,$ and v_n are linearly independent.

Theorem **V is an inner product space and $S=\{x_{1,\ldots,} x_m\}$. If $y=\sum_{i=1}^{m} a_i x_i$, and then (a) $a_j=<y,x_j>/<x_j,x_j>$ for all j if S is orthogonal, and (b) $a_j=<y,x_j>$ for all j if S is orthonormal.**
(Proof) (a) $<y,x_j>=<\sum_{i=1}^{m} a_i x_i, x_j>=\sum_{i=1}^{m} a_i <x_i,x_j>=a_j<x_j,x_j> \Rightarrow a_j=\frac{<y,x_j>}{<x_j,x_j>}$
Eg. (8,-7)=8(1,0)-7(0,1) where 8=<(8,-7),(1,0)> and -7=<(8,-7),(0,1)>

Theorem **A linear operator T: $V \rightarrow W$, let $\beta=\{x_{1,\ldots,}x_m\}$ and orthogonal set $\beta'=\{y_{1,\ldots,}y_m\}$ are the bases of V and W, respectively. Then the ij-entry $T_{ij}=\frac{<T(x_j),y_i>}{<y_i,y_i>}$. If β' is an orthonormal basis, then $T_{ij}=<T(x_j),y_i>$.**

(Proof) $\because$ $T(x_j)=\sum_{k=1}^{m} T_{kj} y_k$, $\therefore$ $<T(x_j),y_i>=<\sum_{k=1}^{m} T_{kj} y_k, y_i>=\sum_{k=1}^{m} T_{kj}<y_k,y_i>=T_{ij}<y_i,y_i>$

$$\Rightarrow T_{ij}=\frac{<T(x_j),y_i>}{<y_i,y_i>}$$

Eg. For a linear operator $T(a_1,a_2)=(a_1-a_2,a_1,2a_1+a_2)$, let $\beta=\{(1,2),(2,3)\}$ be a basis for R^2, and $\beta'=\{(1,0,0),(0,1,0),(0,0,1)\}$ be an orthonormal basis for R^3, then we have $T(1,2)=(-1,1,4)=-1(1,0,0)+1(0,1,0)+4(0,0,1)$ and $T(2,3)=(-1,2,7)=-1(1,0,0)+2(0,1,0)+7(0,0,1)$

$\Rightarrow [T]_{\beta}^{\beta'}=\begin{bmatrix} -1 & -1 \\ 1 & 2 \\ 4 & 7 \end{bmatrix}$, **where T_{11}=-1=<T(1,2),(1,0,0)>, T_{21}=1=<T(1,2),(0,1,0)>, T_{31}=4=<T(1,2),(0,0,1)>, T_{12}=-1<T(2,3),(1,0,0>, T_{22}=2=<T(2,3),(0,1,0)>, and T_{32}=7=<T(2,3),(0,0,1)>**

5-2 Gram-Schmidt Orthogonalization Process

Gram-Schmidt Orthogonalization Process: **Let $S=\{y_1,y_2,y_3,\ldots,y_n\}$ be a linearly independent subset of V and $S'=\{x_1,x_2,x_3,\ldots,x_n\}$ be an orthogonal subset of V. Set $x_1=y_1$, $x_2=y_2-\frac{<y_2,x_1>}{<x_1,x_1>}\cdot x_1$, $x_3=y_3-\frac{<y_3,x_1>}{<x_1,x_1>}\cdot x_1-\frac{<y_3,x_2>}{<x_2,x_2>}\cdot x_2$, …, and $x_k=y_k-\sum_{j=1}^{k-1}\frac{<y_k,x_j>}{<x_j,x_j>}\cdot x_j$. Then $Span(S)=Span(S')$.**

Eg. For $V=R^3$, $\beta=\{(1,1,0),(2,0,1),(2,2,1)\}$, find an orthogonal basis for V by the Gram-Schmidt orthogonalization process. [2007 台科大電研]

(Sol.) Let $x_1=(1,1,0)$, $x_2=(2,0,1)-\frac{<(2,0,1),(1,1,0)>}{<(1,1,0),(1,1,0)>}\cdot(1,1,0)=(2,0,1)-\frac{2}{2}(1,1,0)=(1,-1,1)$

$$x_3=(2,2,1)-\frac{<(2,2,1),(1,1,0)>}{<(1,1,0),(1,1,0)>}\cdot(1,1,0)-\frac{<(2,2,1),(1,-1,1)>}{<(1,-1,1),(1,-1,1)>}\cdot(1,-1,1)=(-\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3}).$$

Then $\beta'=\{x_1,x_2,x_3\}$ is an orthogonal basis.

Eg. Let the vector space P_2 have the inner product $<p,q>=\int_{-1}^{1}p(x)q(x)dx$. Apply the Gram-Schmidt process to transform the standard basis $S=\{1,x,x^2\}$ into an orthonormal basis. [2005 北科大電腦通訊所]

(Sol.) $S=\{1,x,x^2\}=\{y_1,y_2,y_3\}$. Let $x_1=y_1=1$, $x_2=y_2-\frac{<y_2,x_1>}{<x_1,x_1>}\cdot x_1=x-\frac{\int_{-1}^{1}x\cdot 1dx}{\int_{-1}^{1}1\cdot 1dx}\cdot 1=x-0=x$,

$$x_3=y_3-\frac{<y_3,x_1>}{<x_1,x_1>}\cdot x_1-\frac{<y_3,x_2>}{<x_2,x_2>}\cdot x_2=x^2-\frac{\int_{-1}^{1}x^2\cdot 1dx}{\int_{-1}^{1}1\cdot 1dx}\cdot 1-\frac{\int_{-1}^{1}x^2\cdot xdx}{\int_{-1}^{1}x\cdot xdx}\cdot x=x^2-\frac{1}{3}-0=x^2-\frac{1}{3}.$$

$\therefore$ Orthogonal basis: $\{1,x,x^2-\frac{1}{3}\}\Rightarrow$ Orthonormal basis: $\{\sqrt{\frac{1}{2}},\sqrt{\frac{3}{2}}x,\sqrt{\frac{45}{8}}(x^2-\frac{1}{3})\}$

S_{perp}: $S^\perp=\{x\in V: <x,y>=0 \text{ for all } y\in S\}$ is an orthogonal set of S.

Eg. For $V=C^3$, $S=Span\{(1,0,i),(1,2,1)\}$, compute $S^\perp$.

(Sol.) Suppose $S^\perp=Span\{(a,b,c)\}$,

$$\begin{cases}<(a,b,c),(1,0,i)>=0\Rightarrow a=-ci\\ <(a,b,c),(1,2,1)>=0\Rightarrow b=-\frac{1-i}{2}c\end{cases},\quad S^\perp=Span\{(-i,-\frac{1-i}{2},1)\}$$

Eg. Show that (a) $\{x_1,x_2,\ldots,x_k\}$ is an ordered basis for W (subspace of V) and $\{x_1,x_2,\ldots,x_k,x_{k+1},\ldots,x_n\}$ is an ordered basis V, then $\{x_{k+1},\ldots,x_n\}$ is an order basis of for $W^\perp$.
(b) W is a subspace of V, then $dim(V)=dim(W)+dim(W^\perp)$. [台大電研]

(Proof) (a) $\forall x\in V \Leftrightarrow x=\sum_{i=1}^{n}<x,x_i>x_i$

If $x\in W^\perp$, $then <x,x_i>=0$ for $1\le i\le k$. Therefore, $x=\sum_{i=k+1}^{n}<x,x_i>x_i\in Span(\{x_{k+1},\cdots,x_n\})$

(b) According to (a), $dim(V)=n=k+(n-k)=dim(W)+dim(W^\perp)$

Eg. Show that (a) $\left(W_1+W_2\right)^{\perp}=W_1^{\perp}\cap W_2^{\perp}$ **and (b)** $\left(W_1\cap W_2\right)^{\perp}=W_1^{\perp}+W_2^{\perp}$**. [1990, 1999 台大電研]**

(Proof) $\left(W_1+W_2\right)\oplus\left(W_1+W_2\right)^{\perp}=W_1\oplus W_1^{\perp}=W_2\oplus W_2^{\perp}=V$

$W_1\ and\ \ W_2\subset W_1+W_2, \therefore \left(W_1+W_2\right)^{\perp}\subset W_1^{\perp}\ and\ W_2^{\perp}\Rightarrow\left(W_1+W_2\right)^{\perp}\subseteq W_1^{\perp}\cap W_2^{\perp}$

$\forall x\in W_1^{\perp}\cap W_2^{\perp}, then\ \forall y\in W_1\ and\ \ z\in W_2\Rightarrow <y,x>=<z,x>=0$

$\because cy+dz\in W_1+W_2, <cy+dz,x>=c<y,x>+d<z,x>=0$

$\Rightarrow x\in\left(W_1+W_2\right)^{\perp}\Rightarrow W_1^{\perp}\cap W_2^{\perp}\subseteq(W_1+W_2)^{\perp},\ \therefore\ \left(W_1+W_2\right)^{\perp}=W_1^{\perp}\cap W_2^{\perp}$

On the other hand, $\left(U_1+U_2\right)^{\perp}=U_1^{\perp}\cap U_2^{\perp}\Rightarrow U_1+U_2=\left(U_1^{\perp}\cap U_2^{\perp}\right)^{\perp}$

Set $U_1=W_1^{\perp}, U_2=W_2^{\perp}\Rightarrow W_1^{\perp}+W_2^{\perp}=\left(W_1\cap W_2\right)^{\perp}$

5-3 Various Matrices

Adjoint matrix: A^* is the complex conjugate transpose of A. **Note:** $A^* \neq adj(A)$ and $(AB)^* = B^*A^*$.

Eg. For $A=\begin{bmatrix} 1-i & 2 \\ -3i & 4 \end{bmatrix}$, **then** $A^*=\begin{bmatrix} 1+i & 3i \\ 2 & 4 \end{bmatrix}$.

Eg. For T: $C^2 \to C^2$ by $T(a_1,a_2)=(2ia_1+3a_2, a_1-a_2)$, if β is the standard ordered basis, find $T^*(a_1,a_2)$. [1998 台大電研]

(Sol.) $T(1,0)=(2i,1)=2i\cdot(1,0)+1\cdot(0,1)$ and $T(0,1)=(3,-1)=3\cdot(1,0)+(-1)\cdot(0,1)$

$$\Rightarrow [T]_\beta=\begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix} \Rightarrow [T^*]_\beta=\begin{bmatrix} -2i & 1 \\ 3 & -1 \end{bmatrix}, \therefore T^*(a_1,a_2)=(-2ia_1+a_2, 3a_1-a_2).$$

Theorem For $A \in M_{m\times n}(F)$, $x \in F^n$, $y \in F^m$, then $<Ax,y>=<x,A^*y>$.

(**Note:** $<T(x),y> = <x,T^*(y)>$ for $\forall x,y \in V$, $T:V \to V$)

Eg. $$\left\langle \begin{bmatrix} 1 & 1 \\ -2i & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x+y \\ -2ix+3y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle = x\bar{u}+y\bar{u}-2ix\bar{v}+3y\bar{v} = \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 & 2i \\ 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u+2iv \\ u+3v \end{bmatrix} \right\rangle$$

Theorem (a) $A \in M_{m\times n}(F)$, $Rank(A^*A)=Rank(A)$. (b) $A \in M_{m\times n}(F)$, if $Rank(A)=n$, then A^*A is invertible.

(Proof) (a) $A^*Ax=0 \Leftrightarrow Ax=0$

1. "$\Leftarrow$": $Ax=0$ *implies* $A^*Ax=0$
2. "$\Rightarrow$": $0=\langle A^*Ax,x\rangle_n=\langle Ax,A^{**}x\rangle_m=\langle Ax,Ax\rangle_m$, $\therefore Ax=0$

(b) $A \in M_{m\times n}(F) \Rightarrow A^*A \in M_{n\times n}(F)$

$Rank(A^*A)=Rank(A)=n$, $\therefore A^*A$ is invertible

Orthogonal matrix A: $AA^t=A^tA=I$. **Unitary matrix A:**. $AA^*=A^*A=I$.

Eg. $A=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $A^t=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, **$AA^t=A^tA=I$, $\therefore$ A is orthogonal.**

Eg. $A=\begin{bmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$, $A^*=\begin{bmatrix} -i/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$, **$AA^*=A^*A=I$, $\therefore$ A is unitary.**

Normal Matrix A: $AA^*=A^*A$.

Eg. $A=\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$, $A^*=\begin{bmatrix} -i & -i \\ i & -i \end{bmatrix}$, **$AA^*=A^*A$, $\therefore$ A is normal.**

Eg. T is a normal operator on V. Show that (a) $\|T(x)\| = \|T^*(x)\|$, (b) $T(x)=\lambda x \Rightarrow T^*(x)=\bar{\lambda}x$, (c) λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvector x_1 and x_2. Then x_1 and x_2 are orthogonal. [2012 台大電研]

(Proof) (a) $\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle = \langle TT^*(x), x\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2$

(b) Suppose $T(x)=\lambda x$ for some $x \in V$. Let $U = T - \lambda I$,

$0 = \|U(x)\| = \|U^*(x)\| = \|(T^* - \bar{\lambda}I)(x)\|$, $\therefore T^*(x) = \bar{\lambda}x$.

(c) Let $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$,

$\lambda_1\langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle = \langle T(x_1), x_2\rangle = \langle x_1, T^*(x_2)\rangle = \langle x_1, \bar{\lambda}_2 x_2\rangle = \lambda_2\langle x_1, x_2\rangle \quad \because \lambda_1 \neq \lambda_2, \quad \therefore \langle x_1, x_2\rangle = 0$.

Self-adjoint (Hermitian) matrix A: $A=A^*$.

Eg. $\begin{bmatrix} 1 & 2-5i \\ 2+5i & -3 \end{bmatrix}$ **is a self-adjoint matrix.**

(**Note:** A self-adjoint matrix is also a normal matrix, but a normal matrix may not be self-adjoint)

Theorem If A is a self-adjoint matrix, all eigenvalues of A are real. [2000 台大電研]

(Proof) $\lambda x = Ax = A^* x = \bar{\lambda}x$ for some $x \neq 0$, $\lambda = \bar{\lambda} \Rightarrow \lambda$ is real.

Gramian matrix A: $\exists B \in M_{m\times n}(F)$ such that $A=BB^t$ then A is called the Gramian matrix.

Theorem A is a Gramian matrix $\Leftrightarrow \begin{cases} A \text{ is symmetric.} \\ \text{all eigenvalues of } A \text{ are nonnegative.} \end{cases}$

(Proof) (1) $A^t=(BB^t)^t=BB^t=A$, $\therefore$ A is symmetric. (2) For some $x \neq 0$, and $Ax=\lambda x$,

then $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle = \langle BB^t x, x\rangle = \langle B^t x, B^t x\rangle \geq 0, \ \because \langle x, x\rangle \geq 0, \ \therefore \lambda \geq 0$

Eg. Which of the following matrices is Hermitian? Which is normal?

$A = \begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix}$ $B = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix}$ $C = \begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix}$ [交大電子所]

(Sol.) $A=A^*$ and $D=D^*$, $\therefore$ A and D are both Hermitian matrices,

$CC^*=C^*C$, $BB^*=\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$, $B^*B=\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$, $\therefore$ C is normal but B is not normal.

5-4 Special Characteristics of Matrices

Positive definite: If $\langle Ax, x\rangle>0$ for all $x\neq 0$, then A is positive definite.

Eg. If A is a positive-definite matrix, then all the eigenvalues of A are positive.

(Proof) For some $x\neq 0$ and $Ax=\lambda x$, positive definite: $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle > 0 \Rightarrow \lambda > 0$.

Eg. Let A be a complex normal (or real symmetric) $n\times n$ matrix with eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$, and then show that (a) $tr(A)=\sum_{i=1}^{n}\lambda_i$ and $tr(A^*A)=\sum_{i=1}^{n}|\lambda_i|^2$, (b) $det(A)=\prod_{i=1}^{n}\lambda_i$. [1998 台大電研]

(Proof) Let $A=QDQ^{-1}$ and D be diagonal, $tr(A)=tr(QDQ^{-1})=tr(DQQ^{-1})=tr(D)=\sum_{i=1}^{n}\lambda_i$

$$\det(A)=\det(QDQ^{-1})=\det(Q)\det(D)\det(Q^{-1})=\det(Q)\det(D)[\det(Q)]^{-1}=\det(D)=\prod_{i=1}^{n}\lambda_i .$$

Eg. Let λ_i be the eigenvalues of $A=\begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 4 & 2 \\ -1 & 4 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}$, $1 \leq 4$, (a) $\sum_{i=1}^{4}\lambda_i=?$ (b) $\prod_{i=1}^{4}\lambda_i=?$

(c) Is A positive definite? [台大機研]

(So1.) ∵ A is a real symmetric matrix, ∴ (a) $\sum_{i=1}^{4}\lambda_i=tr(A)=2+3+1+1=7$, (b) $\prod_{i=1}^{4}\lambda_i=det(A)=10$,

(c) Method 1: $\exists\lambda<0 \Rightarrow A$ is not positive definite.

Method 2: $det([2])=2>0$ and $det(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix})=5>0$, but $det(\begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 4 & 1 \end{bmatrix})=-38<0 \Rightarrow$ Not positive definite.

***A* is unitarily equivalent to *B*:** $\exists P$ fulfills $B=P^*AP$ and $P^{-1}=P^*$.

Theorem (a) A is complex normal $\Leftrightarrow$ A is unitary equivalent to a diagonal matrix. (b) A is real symmetric $\Leftrightarrow$ A is orthogonally equivalent to a real diagonal matrix.

(Proof) $AA^*=(PDP^*)(PDP^*)^*=PDP^*PD^*P^*=PDD^*P^*$

$$A=PDP^*,\ A^*A=(PDP^*)^*(PDP^*)=PD^*DP^*.\quad \because DD^*=D^*D,\quad \therefore A^*A=AA^*$$

Projection: $V=W_1\oplus W_2$, $x_1\in W_1$, and $x_2\in W_2$, $x=x_1+x_2$. If $T(x)=x_1$, then T is projection on W_1. That is, $R(T)=W_1=\{x: T(x)=x\}$ and $N(T)=W_2$. (**Note:** $T^2=T$ if T is a projection.)

Orthogonal Projection: If $R(T)^{\perp}=N(T)$ and $N(T)^{\perp}=R(T)$ for T: $V\to V$ be a projection.

Theorem T is a linear operator on V. Then T is an orthogonal projection. $\Leftrightarrow$ $T^2=T=T^*$.

Eg. Let T: $R^2 \rightarrow R^2$ be a linear operator. For any $z \in R^2$, $T(z)=p$, where p is the projection of z on the line $x=y$. Find $[T]$. [交大電信所]

(Sol.) The orthonormal basis of line $x=y$ is $\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\}$ and the standard basis of R^2 is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

Inner product: $<\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}>=\frac{1}{\sqrt{2}}$, so the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto the line $x=y$ is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}=\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Inner product: $<\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}>=\frac{1}{\sqrt{2}}$, so the orthogonal projection of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto the line $x=y$ is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}=\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \therefore [T]=\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$, and we have $[T]^2=[T]=[T]^*$

Eg. Find the orthogonal projection of the vector $v=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ onto the subspace $S=Span(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})$.

[交大電子所]

(Sol.) Transform the orthogonal basis $\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\}$ into the orthonormal one: $\{\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\}$.

$<\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}>=-\sqrt{2}$, so the orthogonal projection of the vector $v=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is

$$-\sqrt{2}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$< \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} >=-\sqrt{2}$, so the orthogonal projection of the vector $v=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is

$-\sqrt{2}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$.

$\because \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$-$(-\sqrt{2}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix})$-$(-\sqrt{2}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix})$=$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$-$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$-$\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\therefore v=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}+\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

and $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\} \perp S=Span(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})$

Rotation: $\exists$ orthogonal basis $\beta=\{x_1,x_2\}$ for W, and $\exists\theta$ is real and fulfills $T(x_1)=x_1\cos\theta+x_2\sin\theta$, $T(x_2)=-x_1\sin\theta+x_2\cos\theta$, and $T(y)=y$, $\forall y\in W^\perp$, where W is a 2-dimensional subspace. And then T is a rotation of W about $W^\perp$, where $W^\perp$ is the axis of rotation.

Reflection: $T(x)=-x$, $\forall x\in W$ and $T(y)=y$, $\forall y\in W^\perp$. And then T is a reflection of W about $W^\perp$, where W is a 1-dimensional subspace.
Eg. T: $R^2\to R^2$ by $T(a,b) = (-a,b)$. Let $W=Span(\{e_1\})$ and $W^\perp=Span(\{e_2\})$. It is reflection of R^2 about $W^\perp$ (the y-axis).

Theorem T is an orthogonal operator on a 2-dimensional real inner product space. If $det(T)$=1, then T is a rotation. If $det(T)$=-1, then T is a reflection.

(Proof) By definition, $[T]_\beta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is a rotation, $\therefore det(T) = 1$.

Choose $\gamma = \{z_1, z_2, \ldots\}$ be an orthogonal basis for V and $z_1\in W$,

$$[T]_\gamma = \begin{bmatrix} -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \Rightarrow det(T) = -1.$$

Theorem The composition of a reflection and a rotation is a reflection.

(Proof) $det(T_1T_2)=det(T_1)det(T_2)=-1$

Spectral Theorem If T is $\begin{cases}\text{normal on V over F = C}\\ \text{self - adjoint on V over F = R}\end{cases}$, and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T. Let W_i be the eigenspace of T corresponding to λ_i, and T_i be an orthogonal projection on W_i. ($T_i(x)=x_i \in W_i$). Then (a) $V=W_1\oplus W_2\oplus W_3\cdots\oplus W_k$, (b) $W_i^{\perp}=\sum_{j\neq i} W_j$ (direct sum), (c) $T_iT_j=\delta_{ij}T_i$, (d) $I=T_1+T_2+\cdots+T_k$, (e) $T=\lambda_1T_1+\lambda_2T_2+\cdots+\lambda_kT_k$.

(Proof) (c) Let $x \neq 0$ and $T_j(x)=x_j$, then $\mathrm{T_iT_j}(x)=\mathrm{T_i}(x_j)=\begin{cases}0, i\neq j\\ x_i, i=j\end{cases}=\delta_{ij}T_i(x) \Rightarrow T_iT_j=\delta_{ij}T_i$

(d) $x=Ix=x_1+\ldots+x_k=T_1(x)+T_2(x)+\ldots+T_k(x)=(T_1+T_2+\ldots+T_k)(x)$, $\therefore\ I=T_1+T_2+\ldots+T_k$.

(e) For $x\in V$, then $x=x_1+\ldots+x_k$, where $x_i\in W_i$,

$T(x)=T(x_1)+\ldots+T(x_k)=\lambda_1x_1+\ldots+\lambda_kx_k=\lambda_1T_1(x)+\ldots+\lambda_kT_k(x)=(\lambda_1T_1+\ldots+\lambda_kT_k)(x)$,

$\therefore\ T=\lambda_1T_1+\ldots+\lambda_kT_k$

Theorem (a) $F=C$, T is a unitary operator $\Leftrightarrow$ T^* is normal and $|\lambda|=1$ for all eigenvalues. (b) $T=-T^*$ $\Leftrightarrow$ each λ is pure imaginary. (c) T is a projection $\Leftrightarrow$ each λ is either 0 or 1.

5-5 Bilinear Forms

Bilinear form, H: If (a) $H(ax_1+bx_2,y)=aH(x_1,y)+bH(x_2,y)$ and (b) $H(x,ay_1+by_2)=aH(x,y_1)+bH(x,y_2)$ for $x_1, y_1, x_2, y_2, x, y\in V$, $a, b\in F$.

Theorem For each $H\in B(F^n)$, $\exists!\, A\in M_{n\times n}(F)$ fulfills $H(x,y)=x^tAy$ for all $x, y\in F^n$.

Eg. Define $H:\ R^2\times R^2\to R$ **by** $H(\begin{bmatrix}a_1\\a_2\end{bmatrix},\begin{bmatrix}b_1\\b_2\end{bmatrix})=2a_1b_1+3a_1b_2+4a_2b_1-a_2b_2$ **for all** $\begin{bmatrix}a_1\\a_2\end{bmatrix},\begin{bmatrix}b_1\\b_2\end{bmatrix}\in R^2$,

then let $x=\begin{bmatrix}a_1\\a_2\end{bmatrix}$, $y=\begin{bmatrix}b_1\\b_2\end{bmatrix}$, $A=\begin{bmatrix}2&3\\4&-1\end{bmatrix}\Rightarrow H(x,y)=x^tAy=[a_1\ a_2]\begin{bmatrix}2&3\\4&-1\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

Symmetric bilinear form: $H(x,y)=H(y,x)$ for all $x, y\in V$.

Quadratic form: $K: V\to F$ if $\exists$ $H\in B(V)$ fulfills $K(x)=H(x,x)=x^tAx$.

Eg. Let $f(t_1,t_2,t_3)$= $2t_1^2-t_2^2+6t_1t_2-4t_2t_3=K(\begin{bmatrix}t_1\\t_2\\t_3\end{bmatrix})$**, and then A=**$\begin{bmatrix}2&3&0\\3&-1&-2\\0&-2&0\end{bmatrix}$

$\Rightarrow f(t_1,t_2,t_3)=[t_1\quad t_2\quad t_3]\begin{bmatrix}2&3&0\\3&-1&-2\\0&-2&0\end{bmatrix}\begin{bmatrix}t_1\\t_2\\t_3\end{bmatrix}$ **is a quadratic form.**

Eg. There are two bases (x_1,y_1) and (x_2,y_2) in R^2. If the equations of the same ellipse represented by two distinct bases are described as follows, respectively: $2x_1^2-4x_1y_1+5y_1^2-36=0$ and $x_2^2+6y_2^2-36=0$. Please find the transformation matrix between these two bases. [2004 台大電研]

(Sol.) Let $K\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 2x_1^2 + 5y_1^2 - 4x_1y_1$, $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, $\det\left(\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\right) = (\lambda-1)(\lambda-6)$, $\lambda = 1, 6$

$$\lambda_1 = 1 \Rightarrow [A - \lambda_1 I]\cdot\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\cdot\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \quad \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad a \neq 0,$$

$$\lambda_2 = 6 \Rightarrow [A - \lambda_2 I]\cdot\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \quad \Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad b \neq 0$$

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \text{ fulfills } Q^{-1}AQ = Q^tAQ = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \text{ for } \beta = \left\{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}.$$

$$\therefore \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}\cdot\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow K\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_2^2 + 6y_2^2 \Rightarrow 2x_1^2 - 4x_1y_1 + 5y_1^2 - 36 = 0 \Rightarrow x_2^2 + 6y_2^2 - 36 = 0$$

Eg. $[t_1,t_2,t_3]^t \in R^3$, $3t_1^2+3t_2^2+3t_3^2-2t_1t_3+2\sqrt{2}\,(t_1+t_3)+1=0$. Find a basis β for R^3 such that the above equation is simplified.

(Sol.) Let $K\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = 3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1t_3$, then $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. It is found that an orthogonal matrix $Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ fulfills $Q^tAQ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ for $\beta = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

$$\therefore \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\cdot\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \Rightarrow K(x) = 3s_1^2 + 4s_2^2 + 2s_3^2$$

$$\Rightarrow 3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1t_3 + 2\sqrt{2}(t_1 + t_3) + 1 = 0 \Rightarrow 3s_1^2 + 4s_2^2 + 2s_3^2 - 4s_3 + 1 = 0.$$