Chapter 6 Jordan Forms of Matrices

6-1 Jordan Forms and Dot Diagrams

Generalized eigenvector: $X \in V$, $X \neq 0$. If $\exists \lambda$ fulfills $(A-\lambda I)^P X=0$ for some positive integer P.

Generalized eigenspace: $K_\lambda=\{x \in V: (A-\lambda I)^P(x)=0 \text{ for some } P\}$.

Eg. For $A=\begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$, find the generalized eigenspace of A.

(Sol.) $f(\lambda)=\det(A-\lambda I)=-(\lambda-3)(\lambda-2)^2=0$, $\lambda_1=3$, $\lambda_2=2$

For λ_1, $K_{\lambda_1}=E_{\lambda_1}=N(A-3I)$,

$$(A-3I)\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Rightarrow \quad K_{\lambda 1}=E_{\lambda 1}=\left\{t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}: t \in C\right\}$$

For λ_2, $$(A-2I)\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\mu\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \Rightarrow E_{\lambda_2}=\left\{\mu\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \ \mu \in C\right\},$$

$$(A-2I)^2\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 & 1 & -1 \\ -4 & -2 & 2 \\ -2 & -1 & 1 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad K_{\lambda_2}=\left\{\mu\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}+v\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}: \mu, v \in C\right\}$$

$$\Rightarrow \quad K_{\lambda_2} \neq E_{\lambda_2} \quad \text{in this case,}$$

$$\beta=\left\{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\} \quad \text{and} \quad J=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{is a Jordan form.}$$

Eg. For $A=\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, find the generalized eigenspace of A.

(Sol.) $\det(A-\lambda I)=-(\lambda-1)^2(\lambda-2)=0$, $\lambda_1=1,\ \lambda_2=2$

For λ_2, $K_{\lambda_2}=E_{\lambda_2}=N(A-2I)=\left\{ u\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : u\in C \right\}$

For λ_1, $K_{\lambda_1}=N((A-I)^2)$, $E_{\lambda_1}=N((A-I))$

$$(A-I)\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$(A-I)^2\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix},\ \therefore\ E_{\lambda_1}=K_{\lambda_1}$$ in this case,

$\beta=\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $J=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}=D$ is a Diagonal form

Jordan Form and Block: $J=\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}$, in which $J_j=\begin{bmatrix} \lambda_j & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_j & 1 & & & \vdots \\ 0 & 0 & \lambda_j & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \lambda_j & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_j \end{bmatrix}$ is called the Jordan block corresponding to λ_j.

Eg. Let $J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, where $J_1=\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ and $J_2=\begin{bmatrix} 3.5 & 1 \\ 0 & 3.5 \end{bmatrix}$. And then we have

$$J=\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3.5 & 1 \\ 0 & 0 & 0 & 3.5 \end{bmatrix}.$$

Eg. For $A=\begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}$, find Q such that $Q^{-1}AQ=J$.

(Sol.) $\det(A-\lambda I)=(\lambda-2)^2(\lambda-4)^2$, $\lambda_1=2$, $\lambda_2=4$

1. λ_1: r_1=4-$Rank(A\text{-}2I)$=4–2=2=m_{λ_1}, $\therefore$ Dot diagram: • • $\Rightarrow J_{\lambda_1}=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

2. λ_2: r_1=4–$Rank(A\text{-}4I)$=4–3=1 and r_2=$Rank(A\text{-}4I)$–$Rank((A-4I)^2)$=3-2=1

$\therefore$ Dot diagram:
•
•
$\Rightarrow J_{\lambda_2}=\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \Rightarrow J=J_{\lambda_1}\oplus J_{\lambda_2}=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Bases for K_{λ_1} and K_{λ_2}:

For λ_1, $(A-2I)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=0 \Rightarrow \beta_1=\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}\right\}$

For λ_2,

$$(A-4I)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=0 \Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(A-4I)\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \beta_2=\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}\right\}$$

$\therefore\ Q=\begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ fulfills $Q^{-1}AQ=J$.

Eg. For $A=\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, find its Jordan canonical form (or diagonal form).

(Sol.) $det(A-\lambda I) = -(\lambda-1)^2(\lambda-2)=0$, $\lambda_1 = 1(double)$, $\lambda_2 = 2$.

For λ_1, $\because$ $r_1=n\text{-}Rank(A\text{-}1I)=3\text{-}1=2$, dot diagram: • • $\Rightarrow$ $J_{\lambda_1} = \left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \end{array}\right]$

$$\Rightarrow D_\lambda = J_{\lambda_1} \oplus J_{\lambda_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = J_\lambda .$$

Eg. Let T be a linear operator on V defined by $T(f)=\dfrac{\partial f}{\partial x}$, and a basis $\alpha=\{1,x,y,x^2,y^2,xy\}$. Find a Jordan canonical basis for T.

(Sol.)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \det(A-\lambda I) = \lambda^6 = 0, \ \lambda=0, 0, 0, 0, 0, 0.$$

$r_1=6-Rank(A\text{-}0I)=6\text{-}3=3$

$r_2=Rank((A\text{-}0I))-Rank((A-0I)^2)=3-1=2$

$r_3=Rank((A-0I)^2)-Rank((A-0I)^3)=1$

$\therefore$ dot diagram:

1 • • •
2 • •
3 •

$$\Rightarrow J = \left[\begin{array}{ccc:cc:c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\because$ The 1st column of dot diagram consists of 3 dots, $\therefore$ $\dfrac{\partial^2}{\partial x^2}(x_1) \neq 0$

$\therefore$ $x_1 = x^2$, $(T-\lambda I)(x_1) = 2x$, $(T-\lambda I)^2(x_1) = 2$.

$\because$ The 2nd column of dot diagram consists of 2 dots, $\therefore$ $\dfrac{\partial}{\partial x}(x_2) \neq 0$

$x_2 = xy$, $(T-\lambda I)(x_2) = y$. Finally, $x_3 = y^2$

$\Rightarrow \beta=\{2,2x,x^2,y,xy,y^2\}$ is a Jordan canonical basis for T.

Eg. Suppose that the dot diagrams of *A* are as follows: $\begin{array}{c}\lambda_1=2\\ \begin{array}{ccc}\bullet&\bullet&\bullet\\ \bullet&\bullet&\\ \bullet&&\end{array}\end{array}$, $\begin{array}{c}\lambda_2=4\\ \begin{array}{cc}\bullet&\bullet\\ \bullet&\\ \bullet&\end{array}\end{array}$, $\begin{array}{c}\lambda_3=-3\\ \begin{array}{cc}\bullet&\bullet\end{array}\end{array}$ **. Find the Jordan canonical form.**

(Sol.)

$$\left[\begin{array}{ccc:cc:c:ccc:c:c:c}
2 & 1 & 0 & 0 & 0 & & & & & & & \\
0 & 2 & 1 & 0 & 0 & & & & & & & \\
0 & 0 & 2 & 0 & 0 & & & & & & & \\
\hdashline
0 & 0 & 0 & 2 & 1 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 2 & 0 & & & & & & \\
\hdashline
 & & & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
 & & & & & & 4 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 4 & 1 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 4 & 0 & 0 & 0 \\
 & & & & & & & & & 4 & 0 & 0 \\
 & & & & & & & & & 0 & -3 & 0 \\
 & & & & & & & & & & & -3 \\
 & & & & & & & & & & &
\end{array}\right]$$

Nipotent: If $A^P=0$ for some positive integer P.

Theorem (a) *A* is nipotent$\Rightarrow$*A* has only zero eigenvalue. (b) Any diagonalizable nipotent matrix is always equal to the zero matrix.

Eg. For *B*=$\begin{bmatrix}-1 & -1\\ 4 & 3\end{bmatrix}$**, calculate B^5.**

(Sol.) Method1: $J=\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\2&1\end{bmatrix}\cdot\begin{bmatrix}-1&-1\\4&3\end{bmatrix}\cdot\begin{bmatrix}-1&1\\2&-1\end{bmatrix}=Q^{-1}BQ$,

$J=\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}=D+M$, $M^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\Rightarrow M^3=0\Rightarrow M^4=0\Rightarrow\ldots$

$\therefore\ J^5=(D+M)^5=\begin{bmatrix}1&0\\0&1\end{bmatrix}^5+5\begin{bmatrix}1&0\\0&1\end{bmatrix}^4\begin{bmatrix}0&1\\0&0\end{bmatrix}+0+0+0+0=\begin{bmatrix}1&5\\0&1\end{bmatrix}$

$\Rightarrow B^5=QJ^5Q^{-1}=\begin{bmatrix}-1&1\\2&-1\end{bmatrix}\cdot\begin{bmatrix}1&5\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&1\\2&1\end{bmatrix}=\begin{bmatrix}-9&-5\\20&11\end{bmatrix}$.

Method2: $f(\lambda)=\lambda^2-2\lambda+1=0\Rightarrow f(B)=B^2-2B+I=0$, $g(B)=B^5=h(B)(B^2-2B+I)+aB+bI=aB+bI$, $g'(B)=5B^4=aB$

$\begin{cases}g(I)=I^5=I=(a+b)I\\ g'(I)=5I^4=5I=aI\end{cases}\Rightarrow\begin{cases}a=5\\ b=-4\end{cases}$, $\therefore B^5=5\begin{bmatrix}-1&-1\\4&3\end{bmatrix}-4\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}-9&-5\\20&11\end{bmatrix}$

6-2 Minimum Polynomials

Minimum polynomial: A monic polynomial of least positive degree for which $p(A) = 0$.

Theorem

(a) If $g(\lambda)$ is any polynomial for $g(A)=0$, then $p(\lambda)$ divides $g(\lambda)$. In particular, $p(\lambda)$ divides the characteristic polynomial of A.

(b) The minimum polynomial and the characteristic polynomial have the same zeros.

(c) The minimum polynomial is unique.

(Proof) (a) If $g(\lambda)=p(\lambda)q(\lambda)+r(\lambda)$, $\deg(r(\lambda))<\deg(p(\lambda))$. Since $g(A)=p(A)q(A)+r(A)=0$ and $p(A)=0 \Rightarrow r(A) = 0$, but $p(\lambda)$ is the minimum-degree polynomial and $\deg(r(A)) < \deg(p(A))$

$\therefore$ They are contradictory to each other, $\therefore r(\lambda)=0$

(b) Let characteristic polynomial$=g(\lambda)=p(\lambda)q(\lambda)$. If $p(\lambda)=0$, then $g(\lambda)=0$.

Let $\begin{vmatrix} x \\ \lambda \end{vmatrix}$ be the $\begin{vmatrix} eigenvector \\ eigenvalue \end{vmatrix}$ of A, then $p(\lambda)x=g(\lambda)x=0$.

For $x \neq 0$, $0 = p(A)(x) = p(\lambda)x \Rightarrow p(\lambda) = 0$, $\therefore$ $p(\lambda)$ and $g(\lambda)$ has the same zeros.

Eg. For $A=\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$, find the minimum polynomial for A.

(Sol.) $\because$ $det(A-\lambda I) =-(\lambda-2)^2(\lambda-3)$, $\therefore$ the minimum polynomial for A is either $(\lambda-2)(\lambda-3)$ or $(\lambda-2)^2(\lambda-3)$. Substitute A into $(\lambda-2)(\lambda-3)$ and shows that $p(A) = (A-2I)(A-3I) = 0$.

Theorem A is a diagonalizable matrix. $\Leftrightarrow$ The minimum polynomial for A is of the form $p(A)=(A-\lambda_1 I)\cdots(A-\lambda_k I)$.

Theorem Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct eigenvalues of A, and p_i is the order of the largest Jordan block corresponding to λ_i, and then the minimum polynomial of A is $p(\lambda)=(\lambda-\lambda_1)^{p1}(\lambda-\lambda_2)^{p2}\cdots(\lambda-\lambda_k)^{pk}$.

Eg. If the Jordan canonical form of A is $\left[\begin{array}{ccc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$, find the minimum polynomial.

(Sol.) For $\lambda_1=2$, $p_1=3$. For $\lambda_2=3$, $p_2=2$, $\therefore$ $p(A)=(A-2I)^3(A-3I)^2$, but the characteristic polynomial of A is $(\lambda-2)^5(\lambda-3)^2$.

Eg. If the dot diagrams of A are as follows: $\begin{array}{c}\lambda_1=2\\ \begin{array}{lll}\bullet&\bullet&\bullet\\ \bullet&\bullet&\\ \bullet&&\end{array}\end{array}$, $\begin{array}{c}\lambda_2=4\\ \begin{array}{ll}\bullet&\bullet\\ \bullet&\\ \bullet&\end{array}\end{array}$, $\begin{array}{c}\lambda_3=-3\\ \begin{array}{ll}\bullet&\bullet\end{array}\end{array}$, **find the minimum polynomial.**

(Sol.) p_1 is 3, p_2 is 3, and p_3 is 1, $\therefore$ $p(A)=(A-2I)^3(A-4I)^3(A+3I)$.

Application of the Minimal Polynomials: Computing matrix functions

Eg. For $A=\begin{bmatrix}3&0&1\\2&2&2\\-1&0&1\end{bmatrix}$, find A^3-3A^2+3A.

(Sol.) $f(\lambda)=(\lambda-2)^3=0$, $\lambda_1=2,\ m_1=3$, $A-2I=\begin{bmatrix}1&0&1\\2&0&2\\-1&0&-1\end{bmatrix}$, $rank(A-2I)=1$

$r_1=3-rank(A-2I)=3-1=2$, dot diagram: $\begin{array}{ll}\bullet&\bullet\\ \bullet&\end{array}$

$r_2=1\Rightarrow J=\left[\begin{array}{cc:c}2&1&0\\0&2&0\\ \hdashline 0&0&2\end{array}\right]$ $\Rightarrow$ The minimal polynomial is $(\lambda-2)^2$, $\therefore$ $(A-2I)^2=A^2-4A+4I=0$

$g(A)=A^3-3A^2+3A=aA+bI$, $g'(A)=3A^2-6A+3I=aI$

$g(A=2I)=8I^3-12I+6I=2I=(2a+b)I\Rightarrow 2a+b=2$

$g'(A=2I)=12I^2-12I+3I=aI\Rightarrow a=3\Rightarrow b=-4$, $\therefore$ $A^3-3A^2+3A=3A-4I=\begin{bmatrix}5&0&3\\6&2&6\\-3&0&-1\end{bmatrix}$.