Chapter 7 Special Applications of Matrices

7-1 Condition Numbers of Matrices

Norm of a matrix: $\|A\|=\max\limits_{x\neq 0}\dfrac{\|Ax\|}{\|x\|}$, where A is an $n\times n$ matrix.

Rayleigh quotient: $R(x)=\dfrac{<Bx,x>}{<x,x>}$ for $x\neq 0$, and $B=B^*$.

Theorem (a) $B=B^*\Rightarrow\begin{cases}\max\limits_{x\neq 0}R(x)\\ \min\limits_{x\neq 0}R(x)\end{cases}$ **are the** $\begin{matrix}\text{largest absolute value}\\ \text{smallest absolute value}\end{matrix}$ **among the eigenvalues of B, respectively. (b) $A\in M_{n\times n}(F)$, $\|A\|=\sqrt{\lambda}$, where λ is the largest absolute value among the eigenvalues of A^*A.**

(Proof of (b)) $\|A\|^2=\max\dfrac{<Ax,Ax>}{\|x\|^2}=\max\dfrac{<A^*Ax,x>}{<x,x>}=\max\dfrac{<Bx,x>}{<x,x>}=\max R(x)$.

According to (a)$\Rightarrow\|A\|^2$=the largest absolute value among the eigenvalues of $B=A^*A$.

In **Matlab** language, we can use the following instructions to obtain the Euclidean norm of a matrix:

```
>> A=[1 1; 2 3]; C=norm(A)
C =
    3.8643
```

Condition number: $Cond(A)=\|A\|\cdot\|A^{-1}\|\geqq 1$.

Theorem For $Ax=b$, where A is an invertible matrix and $b\neq 0$.

(a) $\dfrac{1}{\text{Cond(A)}}\cdot\dfrac{\|\delta b\|}{\|b\|}\leq\dfrac{\|\delta x\|}{\|x\|}\leq\text{Cond(A)}\cdot\dfrac{\|\delta b\|}{\|b\|}$**, where δb and δx satisfy $A(x+\delta x)=b+\delta b$.**

(b) $Cond(A)=\sqrt{\lambda_1/\lambda_n}$**, where** $\begin{cases}\lambda_1\\ \lambda_n\end{cases}$ **are the** $\begin{matrix}\text{largest absolute value}\\ \text{smallest absolute value}\end{matrix}$ **among the eigenvalues of A^*A, respectively.**

(Proof)

(a) $Ax=b$, $A(x+\delta x)=b+\delta b\Rightarrow A(\delta x)=\delta b$, $\delta x=A^{-1}\cdot\delta b$

$$\|b\|=\|Ax\|\leq\|A\|\cdot\|x\|\Rightarrow\|x\|\geq\frac{\|b\|}{\|A\|},\quad \|\delta x\|=\|A^{-1}\cdot\delta b\|\leq\|A^{-1}\|\cdot\|\delta b\|$$

$$\therefore\ \frac{\|\delta x\|}{\|x\|}\leq\|A\|\cdot\|A^{-1}\|\cdot\frac{\|\delta b\|}{\|b\|}=\text{Cond(A)}\cdot\frac{\|\delta b\|}{\|b\|}$$

On the other hand, $x = A^{-1}b \Rightarrow \|x\| \le \|A^{-1}\| \cdot \|b\|$, $\delta b = A \cdot (\delta x) \Rightarrow \|\delta b\| \le \|A\| \cdot \|\delta x\| \Rightarrow \|\delta x\| \ge \frac{\|\delta b\|}{\|A\|}$

$$\therefore \quad \frac{\|\delta x\|}{\|x\|} \ge \frac{1}{\|A^{-1}\| \cdot \|A\|} \cdot \frac{\|\delta b\|}{\|b\|} = \frac{1}{\text{Cond(A)}} \cdot \frac{\|\delta b\|}{\|b\|}$$

(b) It is known that $\|A\|=\sqrt{\lambda_1}$, where λ_1 is the largest absolute value among the eigenvalues of A^*A. Let λ_n be the smallest absolute value among eigenvalues of A^*A (and so does AA^*). Moreover, $\|A^{-1}\|^2 = \max \frac{< A^{-1}x, A^{-1}x >}{< x,x >} = \frac{< (A^{-1})^* A^{-1}x, x >}{< x,x >} = \frac{< (AA^*)^{-1} x, x >}{< x,x >} =$ the largest absolute value among the eigenvalues of $(AA^*)^{-1}$ or $(A^*A)^{-1} = \frac{1}{\lambda_n} \Rightarrow \|A^{-1}\| = \sqrt{1/\lambda_n}$, $Cond(A)=\|A\| \cdot \|A^{-1}\|=\sqrt{\frac{\lambda_1}{\lambda_n}}$.

In **Matlab** language, we can use the following instructions to obtain the condition number of a matrix:

```
>> A=[1 1; 2 3]; C=cond(A)
C =

    14.9330

>> A=[0.003   59.14; 5.29   -6.13]; C=cond(A)
C =

    11.3000
```

Eg. For an ill-condition system of linear equations: $\begin{cases} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.13x_2 = 46.78 \end{cases}$ **with** $A=\begin{bmatrix} 0.003 & 59.14 \\ 5.29 & -6.13 \end{bmatrix}$**, it is found that *Cond*(*A*) is very small. The exact solution of the system of linear equations is** $\begin{cases} x_1 = 10 \\ x_2 = 1 \end{cases}$**.**

(Check) $\frac{5.291}{0.003} = 1763.66\cdots \approx 1764$

$\Rightarrow (-6.13 - 59.14 \times 1764)x_2 = 46.78 - 1764 \times 59.17 \Rightarrow -104300x_2 \approx -104400$

$\Rightarrow x_2 \approx 1.001 \approx 1 \Rightarrow x_1 \approx [59.17 - 59.14 \times 1.001]/0.003 \approx -10 \leftarrow$ **inaccurate**

7-2 Minimum Solutions of Linear Equations

Minimum solution: For $Ax=b$, $b\in F^n$, if $Rank(A)<n$, then $Ax=b$ has at least one solution. One solution s is called a minimum solution if $\|s\|\leqq\|u\|$ for all other solution u of $Au=b$.

Theorem $A\in M_{m\times n}(F)$, $b\in F^n$. **Suppose that** $Ax=b$ **has at least one solution. Then there exists one minimum solution** s **of** $Ax=b$ **and if** u **is the solution of** $AA^*x=b$, **then** $s=A^*u=A^*(AA^*)^{-1}b$.

Eg. Find the minimum solution of $\begin{cases} x+2y-z=1 \\ 2x+3y+z=2 \\ 4x+7y-z=4 \end{cases}$.

(Sol.) $A=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$, $A^*=\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ -1 & 1 & -1 \end{bmatrix}$, $AA^*=\begin{bmatrix} 6 & 7 & 19 \\ 7 & 14 & 28 \\ 19 & 28 & 66 \end{bmatrix} \Rightarrow \begin{cases} 6x+7y+19z=1 \\ 7x+14y+28z=2 \\ 19x+28y+66z=4 \end{cases}$

$$\Rightarrow u=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 1/7 \\ 0 \end{bmatrix}+c\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow s=A^*u=\begin{bmatrix} 2/7 \\ 3/7 \\ 1/7 \end{bmatrix}$$

Theorem For $A\in M_{m\times n}(F)$, $b\in F^n$. **There exists** $x_0\in F^n$ **is a solution of** $A^*Ax_0=A^*b$ **and** $\|Ax_0-b\|\le\|Ax-b\|$ **for all** $x\in F^n$. **Furthermore, if** $Rank(A)=n$, **then** $x_0=(A^*A)^{-1}A^*b$. (**Note:** $Ax=b$ may have no solutions but $Ax_0\fallingdotseq b$ in this case) **[1995 台大電研]**

(Proof) Define $W=\{Ax:\ x\in F^n\}$. Let T be the orthogonal projection on W. Choose $x_0\in F^n$ such that $T(b)=Ax_0$, and then $\|T(b)-b\|=\|Ax_0-b\|\le\|Ax-b\|$ for all $x\in F^n$.

$\because$ T is an orthogonal projection on W, $\therefore$ $T(b)-b\in W^\perp$

$\Rightarrow \langle Ax,T(b)-b\rangle=\langle Ax,Ax_0-b\rangle=\langle x,A^*Ax_0-A^*b\rangle=0\Rightarrow A^*Ax_0=A^*b\Rightarrow x_0=(A^*A)^{-1}A^*b$

Eg. Given the data (-3,9), (-2,6), (0,2) and (1,1), find the parabola to fit them by the least-square rule. [清大電研]

(Sol.) Let $y=ex^2+fx+g$, $Ax=b\Rightarrow\begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} e \\ f \\ g \end{bmatrix}=\begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}$, $A^*A=\begin{bmatrix} 9 & 4 & 0 & 1 \\ -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}=\begin{bmatrix} 98 & -34 & 14 \\ -34 & -2 & -4 \\ 14 & -4 & 4 \end{bmatrix}$,

$(A^*A)^{-1}=\begin{bmatrix} 1/9 & 2/9 & -1/6 \\ 2/9 & 49/90 & -7/30 \\ -1/6 & -7/30 & 3/5 \end{bmatrix}$, $x_0=(A^*A)^{-1}A^*b$

$$\Rightarrow\begin{bmatrix} e \\ f \\ g \end{bmatrix}=(A^*A)^{-1}A^*\begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1/3 \\ -4/3 \\ 2 \end{bmatrix}\Rightarrow y=\frac{x^2}{3}-\frac{4x}{3}+2.$$

Eg. $T\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}$, $T\left(\begin{bmatrix}2\\1\\1\\-1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\-1\\1\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}\right)=\begin{bmatrix}2\\-2\\0\\1\\-1\end{bmatrix}$, $T\left(\begin{bmatrix}2\\1\\3\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\2\\-1\\3\\1\end{bmatrix}$. **Please find the least square approximation for the solution of** $T(v)=\begin{bmatrix}3\\1\\2\\1\\3\end{bmatrix}$. [清大電研]

(Sol.) $A=\begin{bmatrix}-3/2 & -1 & 3/2 & -3/2\\ 1 & -1 & 1/2 & -1/2\\ -1/2 & -1 & 1/2 & -1/2\\ 0 & 0 & 1 & 0\\ 1 & 1 & -1 & 1\end{bmatrix}$, $b=\begin{bmatrix}3\\1\\2\\1\\3\end{bmatrix}$, $Ax=b\Rightarrow A^{*}Ax=A^{*}b$

$A^{*}A=\begin{bmatrix}9/2 & 2 & -7/2 & 9/2\\ 2 & 4 & -3 & 2\\ -7/2 & -3 & 9/2 & -7/2\\ 9/2 & 2 & -7/2 & 9/2\end{bmatrix}$, $A^{*}b=\begin{bmatrix}-3/2\\-3\\7/2\\-3/2\end{bmatrix}\Rightarrow x=(A^{*}A)^{-1}A^{*}b=\begin{bmatrix}23\\-7\\35\\0\end{bmatrix}$