Chapter 1 The First-Order Ordinary Differential Equations (ODE)

1-1 Separable Differential Equation $A(x)dx=B(y)dy$

Solution: $\int A(x)dx=\int B(y)dy+C$

Eg. Solve (a) $y'=3x^2+1$, $y(1)=4$, (b) $6x-2yy'=0$, and (c) $2\cdot\frac{dy}{dx}-\frac{1}{y}=\frac{2x}{y}$, $y(0)=0$.

(Sol.) (a) $dy=(3x^2+1)dx$, $y=x^3+x+c$, $y(1)=4\Rightarrow c=2$, $\therefore$ $y=x^3+x+2$.

(b) $2ydy=6xdx\Rightarrow y^2=3x^2+c$.

(c) $2ydy=(1+2x)dx$, $y^2+c=x+x^2$, $y(0)=0\Rightarrow c=0$, $\therefore$ $y^2=x+x^2$.

Eg. Solve $y'=xe^{x-y}$ with the boundary condition: $y=\ln 2$ at $x=0$. [台大電研]

(Sol.) $e^y dy=xe^x dx\Rightarrow e^y=xe^x-e^x+c$, $y(x=0)=\ln 2\Rightarrow c=3$, $\therefore$ $e^y=xe^x-e^x+3$.

1-2 The first-order Linear Differential Equation $y'+p(x)y=q(x)$

Solution: $y'\cdot e^{\int p(x)dx}+p(x)y\cdot e^{\int p(x)dx}=q(x)\cdot e^{\int p(x)dx}$

$$\Rightarrow\left[y\cdot e^{\int p(x)dx}\right]'=q(x)\cdot e^{\int p(x)dx}\Rightarrow ye^{\int p(x)dx}=\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c$$

$$\Rightarrow y(x)=e^{-\int p(x)dx}\cdot\left\{\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c\right\}$$

Eg. Solve $y'+y=\sin(x)$.

(Sol.) $p(x)=1$, $\int p(x)dx=x\Rightarrow y'\cdot e^x+e^x\cdot y=e^x\cdot\sin(x)$

$$\Rightarrow\left(e^x\cdot y\right)'=e^x\cdot\sin(x),\quad e^x\cdot y=\frac{e^x[\sin(x)-\cos(x)]}{2}+c$$

$$\Rightarrow y(x)=\frac{\sin(x)-\cos(x)}{2}+c\cdot e^{-x}$$

Eg. Solve $xy'+2y=3x^3$.

(Sol.) $y'+\frac{2}{x}y=3x^2$, $p(x)=\frac{2}{x}$, $\int p(x)dx=2\ell n(x)$

$$\Rightarrow y'\cdot e^{2\ln(x)}+\frac{2}{x}y\cdot e^{2\ln(x)}=x^2y'+2xy=3x^2\cdot e^{2\ln(x)}=3x^4$$

$$\Rightarrow\left(x^2y\right)'=3x^4,\ x^2y=\frac{3x^5}{5}+c\Rightarrow y(x)=\frac{3x^3}{5}+\frac{c}{x^2}$$

1-3 Bernoulli Differential Equations $y'+p(x)y=r(x)y^{\alpha}$

(It is nonlinear in case of $\alpha \neq 1$)

Solution: Set $z = y^{1-\alpha}$, $y = zy^{\alpha}$, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{y^{\alpha}}{1-\alpha}\cdot\frac{dz}{dx}$

$$\Rightarrow \frac{y^{\alpha}}{1-\alpha}\cdot\frac{dz}{dx} + p(x)\cdot zy^{\alpha} = r(x)y^{\alpha} \Rightarrow \frac{dz}{dx} + (1-\alpha)p(x)\cdot z = r(x)\cdot(1-\alpha)$$

$$\Rightarrow z(x) = e^{-\int(1-\alpha)p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)\cdot e^{\int(1-\alpha)p(x)dx}dx + c\right]$$

$$\Rightarrow [y(x)]^{1-\alpha} = e^{-(1-\alpha)\int p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)\cdot e^{(1-\alpha)\int p(x)dx}dx + c\right]$$

Eg. Solve $y'=y(xy^3-1)$. [台大電研]

(Sol.) $y' + y = xy^4$. Set $z = y^{1-4} = y^{-3}$, $y = z^{-1/3}$, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = -\frac{1}{3}z^{-4/3}\cdot z'$

$$\Rightarrow z' - 3z = -3x \Rightarrow \left[e^{-3x}\cdot z\right]' = -3x\cdot e^{-3x} \Rightarrow z = x + \frac{1}{3} + ce^{3x} \Rightarrow y^{-3} = x + \frac{1}{3} + ce^{3x}$$

Eg. Solve $xy'+2y=xy^3$. [清大電研]

(Sol.) $y' + \frac{2}{x}y = y^3$, let $z = y^{1-3} = y^{-2}$, $y = \frac{1}{\sqrt{z}} = z^{-\frac{1}{2}}$, $y^3 = z^{-\frac{3}{2}}$

$$\left(\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = -\frac{1}{2}\cdot z^{-\frac{3}{2}}\frac{dz}{dx}\right) \Rightarrow -\frac{1}{2}z^{-\frac{3}{2}}\frac{dz}{dx} + \frac{2}{x}\cdot z\cdot z^{-\frac{3}{2}} = z^{-\frac{3}{2}}$$

$$\frac{dz}{dx} - \frac{4}{x}z = -2 \text{ , } e^{-\int\frac{4}{x}dx} = e^{-4\ln(x)} = x^{-4}$$

$$x^{-4}\frac{dz}{dx} - 4x^{-5}z = -2x^{-4} \Rightarrow \left(x^{-4}z\right)' = -2x^{-4}$$

$$x^{-4}\cdot z = \frac{2}{3}x^{-3} + c \Rightarrow z = \frac{2}{3}x + cx^4 \Rightarrow y^{-2} = \frac{2}{3}x + cx^4$$

Eg. Solve $\frac{dP(t)}{dt} = P(t)\cdot(c_1 - c_2P(t))$. [台科大電研]

(Sol.) $\frac{dP}{dt} - c_1P = -c_2P^2$. Set $z = P^{1-2} = P^{-1}$, $P = z^{-1}$, $\frac{dP}{dt} = \frac{dP}{dz}\cdot\frac{dz}{dt} = -z^{-2}\cdot z'$

$$-z^{-2}\cdot z' - c_1z^{-1} = -c_2z^{-2}, \quad z' + c_1z = c_2 \Rightarrow \left[e^{c_1t}\cdot z\right]' = c_2e^{c_1t} \Rightarrow z = \frac{c_2}{c_1} + De^{-c_1t}$$

$$\Rightarrow P(t)^{-1} = \frac{c_2}{c_1} + De^{-c_1t}$$

1-4 Homogeneous & Quasi-homogeneous Differential Equations

The first-order homogeneous differential equation: $y' = f(y/x)$

Solution: Set $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow x\frac{du}{dx} + u = f(u) \Rightarrow \frac{du}{f(u) - u} = \frac{dx}{x}$ is a separable differential equation.

Eg. Solve $x\frac{dy}{dx} = \frac{y^2}{x} + y$.

(Sol.) Set $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx} \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} \Rightarrow u^2 + u = u + x\frac{du}{dx}$

$$\Rightarrow \quad u^2 = x\frac{du}{dx},\ \frac{dx}{x} = \frac{du}{u^2},\ \ell n\,|\,x\,| + c = -\frac{1}{u}$$

$$\Rightarrow \quad u = \frac{-1}{\ell n\,|\,x\,| + c},\ y = xu = \frac{-x}{\ell n\,|\,x\,| + c}$$

Eg. Solve $\frac{dy}{dx} = \frac{x + y}{x - y}$.

(Sol.) $$\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)} \Rightarrow \frac{1+u}{1-u} = u + x\frac{du}{dx} \Rightarrow x\frac{du}{dx} = \frac{1 + u - u + u^2}{1 - u} = \frac{1 + u^2}{1 - u}$$

$$\Rightarrow \quad \left(\frac{1-u}{1+u^2}\right)du = \frac{dx}{x},\ \tan^{-1}(u) - \frac{1}{2}\ell n\,|\,1 + u^2\,| = \ell n\,|\,x\,| + c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ell n\left|1 + \left(\frac{y}{x}\right)^2\right| = \ell n\,|\,x\,| + c \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ell n\left|\,x^2 + y^2\,\right| = c$$

Eg. Solve $y' = \frac{x - y}{x + y}$. **[交大電信所]**

(Sol.) $y' = \frac{x - y}{x + y} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$. Set $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$

$$x\frac{du}{dx} + u = \frac{1-u}{1+u} \Rightarrow x\frac{du}{dx} = \frac{1 - 2u - u^2}{1 + u},\ \int \frac{u+1}{-u^2 - 2u + 1}du = \int \frac{dx}{x} = \ell n\,|\,x\,| + c$$

$$-\frac{1}{2}\ell n\,|-u^2 - 2u + 1| = \ell n\,|\,x\,| + c,\ -\frac{1}{2}\ell n\,\left|-\left(\frac{y}{x}\right)^2 - 2\left(\frac{y}{x}\right) + 1\right| = \ell n\,|\,x\,| + c$$

$$-\frac{1}{2}\ell n\left(-y^2 - 2xy + x^2\right) = c,\ \ell n\left(-y^2 - 2xy + x^2\right) = c,\ e^{\ell n\left(-y^2 - 2xy + x^2\right)} = e^c$$

$$-y^2 - 2xy + x^2 = C \Rightarrow y^2 - x^2 + 2xy = C$$

Eg. Solve $\left(x-\sqrt{xy}\right)y'=y$. **[中山電研]**

(Sol.) $(1-\sqrt{\frac{y}{x}})\frac{dy}{dx}=\frac{y}{x}$. Let $u=\frac{y}{x}$, $y=ux$, $\frac{dy}{dx}=u+x\frac{du}{dx}$

$$\Rightarrow \frac{dx}{x}=\left(\frac{1-\sqrt{u}}{u\sqrt{u}}\right)du \Rightarrow \ell n|x|+C=\int\frac{1}{u\sqrt{u}}du-\int\frac{1}{u}du=\int u^{\frac{-3}{2}}du-\ell n|u|=-2u^{\frac{-1}{2}}-\ell n|u|$$

$$\Rightarrow \ell n|x|+2\sqrt{\frac{x}{y}}+\ell n\left(\frac{y}{x}\right)=C \Rightarrow \ell n|y|+2\sqrt{\frac{x}{y}}=C$$

Eg. Solve *y'*=4*y*/(4*x*-*y*). [文化電機轉學考]

(Ans.) $\ln(|y|)=-\frac{4x}{y}+c$

Quasi-homogeneous differential equation: $\frac{dy}{dx}=f\left(\frac{ax+by+c}{dx+ey+h}\right)$

Case 1 ae-bd≠0

Solution: Let A and B fulfill $\begin{cases}aA+bB+c=0\\ dA+eB+h=0\end{cases}$, and set $x=X+A$, $y=Y+B$

$$\frac{dy}{dx}=\frac{dY}{dX}=f\left(\frac{ax+by+c}{dx+ey+h}\right)=f\left(\frac{a(X+A)+b(Y+b)+c}{d(X+A)+e(Y+B)+h}\right)$$

$$=f\left(\frac{aX+bY+aA+bB+c}{dX+eY+dA+eB+h}\right)$$

$$\Rightarrow \frac{dY}{dX}=f\left(\frac{aX+bY}{dX+eY}\right)$$ is a homogeneous equation.

Eg. Solve $\frac{dy}{dx}=\frac{2x+y-1}{x-2}$.

(Sol.) $a=2$, $b=1$, $c=-1$, $d=1$, $e=0$, $h=-2$

$ae-bd=0-1=-1\neq 0$

$$\begin{cases}2A+B-1=0\\ A-2=0\end{cases} \Rightarrow \begin{matrix}A=2\\ B=-3\end{matrix} \Rightarrow x=X+2,\ y=Y-3$$

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{2(X+2)+(Y-3)-1}{X}=\frac{2X+Y}{X}=2+\left(\frac{Y}{X}\right)=u+2$$

$$X\frac{du}{dX}+u=u+2,\quad u=\ln(cX^2)\Rightarrow\frac{y+3}{x-2}=\ln[c(x-2)^2]$$

Eg. Solve $\frac{dy}{dx}=\frac{x+y+1}{x-y-1}$. **[文化電機轉學考]**

Case 2 ae-bd=0

Solution: Set $v=\dfrac{ax+by}{a}=\dfrac{dx+ey}{d}$, $y=\dfrac{a}{b}(v-x)\Rightarrow\dfrac{dy}{dx}=\dfrac{a}{b}\left(\dfrac{dv}{dx}-1\right)$

$\therefore\ \dfrac{dy}{dx}=f\left(\dfrac{ax+by+c}{dx+ey+h}\right)\Rightarrow\dfrac{a}{b}\left(\dfrac{dv}{dx}-1\right)=f\left(\dfrac{av+c}{dv+h}\right)\Rightarrow\dfrac{dv}{dx}=1+\dfrac{b}{a}f\left(\dfrac{av+c}{dv+h}\right)$ is a separable equation for v and x

Eg. Solve $\dfrac{dy}{dx}=\dfrac{2x+y-1}{4x+2y-4}$.

(Sol.) $a=2$, $b=1$, $c=-1$, $d=4$, $e=2$, $h=-4$

$\because\ ae-bd=0$, $\therefore\ v=\dfrac{2x+y}{2}=\dfrac{4x+2y}{4}$

$$\Rightarrow\frac{dv}{dx}=1+\frac{1}{2}\left(\frac{2v-1}{4v-4}\right)\Rightarrow\left(\frac{8v-8}{10v-9}\right)dv=dx\Rightarrow\frac{4v}{5}-\frac{2}{25}\ell n\,|10v-9|+c=x$$

$$\frac{2}{5}(2x+y)-\frac{2}{25}\ell n\,|10x+5y-9|+c=x$$

1-5 Exact Differential Equations and Integrating Factors

Exact differential equation: $M(x,y)dx+N(x,y)dy=0$ if $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

Solution: $\exists F(x,y)$ fulfills $\frac{\partial^2 F(x,y)}{\partial x\partial y}=\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

$$\Rightarrow dF=\frac{\partial F(x,y)}{\partial x}\cdot dx+\frac{\partial F(x,y)}{\partial y}dy=M(x,y)dx+N(x,y)dy=0$$

Solve $\frac{\partial F(x,y)}{\partial x}=M(x,y)$ and $\frac{\partial F(x,y)}{\partial y}=N(x,y)$

$\Rightarrow F(x,y)=C$ is its solution.

Eg. Solve $\frac{dy}{dx}=\frac{-2xy^3-2}{3x^2y^2+e^y}$.

(Sol.) $(2xy^3+2)dx+(3x^2y^2+e^y)dy=0$

$$\frac{\partial(2xy^3+2)}{\partial y}=6xy^2=\frac{\partial(3x^2y^2+e^y)}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x}=2xy^3+2\Rightarrow F(x,y)=x^2y^3+2x+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=3x^2y^2+e^y\Rightarrow F(x,y)=x^2y^3+e^y+c_2(x)$$

$$\Rightarrow F(x,y)=x^2y^3+2x+e^y+c,\ \therefore\ x^2y^3+2x+e^y=C$$

Eg. Solve $y=(y^2-x)y'$. [台大電研]

(Sol.) $y\,dx+(x-y^2)dy=0$

$$\Rightarrow \frac{\partial(y)}{\partial y}=\frac{\partial(x-y^2)}{\partial x}=1,\ F(x,y)=\int y\,dx+c_1=\int(x-y^2)dy+c_2=xy-\frac{y^3}{3}+c_2,$$

$$\therefore\ xy-\frac{y^3}{3}=C$$

Eg. Solve $y'=-\frac{y+y^2}{x(1+2y)}$.

(Sol.) $(y+y^2)dx+(x+2xy)dy=0$, $\frac{\partial(y+y^2)}{\partial y}=1+2y=\frac{\partial(x+2xy)}{\partial x}$

$$\Rightarrow F(x,y)=\int(y+y^2)\,dx+c_1=\int(x+2xy)dy+c_2=xy+xy^2+c_1,$$

$$\therefore\ xy+xy^2=C$$

Integrating factor *u*(*x*,*y*): For $M(x,y)dx+N(x,y)dy=0$, in case

$$\frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x} \text{ but } \frac{\partial[u(x,y)M(x,y)]}{\partial y}=\frac{\partial[u(x,y)N(x,y)]}{\partial x}, \text{ and then } u(x,y) \text{ is}$$

called the integrating factor.

Eg. Solve $(y^2-6xy)dx+(3xy-6x^2)dy=0$.

(Sol.) $\frac{\partial(y^2-6xy)}{\partial y}=2y-6x\neq 3y-12x=\frac{\partial(3xy-6x^2)}{\partial x}$

Choose $u(x,y)=y\Rightarrow(y^3-6xy^2)dx+(3xy^2-6x^2y)dy=0$

$$\frac{\partial(y^3-6xy^2)}{\partial y}=3y^2-12xy=\frac{\partial(3xy^2-6x^2y)}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x}=y^3-6xy^2\Rightarrow F(x,y)=xy^3-3x^2y^2+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=3xy^2-6x^2y\Rightarrow F(x,y)=xy^3-3x^2y^2+c_2(x)$$

$$\therefore\ xy^3-3x^2y^2=C$$

Another Method: $\frac{dy}{dx}=-\frac{y^2-6xy}{3xy-6x^2}$ is the *first-order homogeneous equation*.

$$\frac{dy}{dx}=-\frac{y^2-6xy}{3xy-6x^2}=-\frac{\left(\frac{y}{x}\right)^2-6\left(\frac{y}{x}\right)}{3\left(\frac{y}{x}\right)-6}. \text{ Let } u=\frac{y}{x}\Rightarrow y'=u+x\frac{du}{dx}$$

$$\frac{3u-6}{-4u^2+12u}du=\frac{dx}{x}\Rightarrow-\frac{3}{4}\int\frac{u-2}{u(u-3)}du=\ell n|x|+c$$

$$\Rightarrow-\frac{1}{4}\left[\int\frac{2du}{u}+\int\frac{du}{u-3}\right]=\ln|x|+c\Rightarrow\ell n\left|u^2(u-3)\right|=-4\ell n|x|-4c$$

$$\Rightarrow\left(\frac{y}{x}\right)^2\left(\frac{y}{x}-3\right)=\frac{A}{x^4},\ xy^3-3x^2y^2=A,\ \ xy^3-3x^2y^2=A$$

Eg. Solve $(x+e^y)dy-dx=0$.

(Sol.) $M(x,y)=-1,\ N(x,y)=x+e^y,\ \dfrac{\partial M(x,y)}{\partial y}\neq\dfrac{\partial N(x,y)}{\partial x}$

Choose integrating factor: $e^{-y}\Rightarrow -e^{-y}dx+(xe^{-y}+1)dy=0$

$$\frac{\partial(-e^{-y})}{\partial y}=e^{-y}=\frac{\partial(xe^{-y}+1)}{\partial x}$$

$F(x,y)=-xe^{-y}+c_1(x,y)=-xe^{-y}+y+c_2(x,y),\ \therefore\ -xe^{-y}+y=C$

Another method: $\dfrac{dx}{dy}-x=e^y$ is the *first-order linear differential equation* for $x(y)$.

$$\frac{dx}{dy}-x=e^y,\ p(y)=-1,\ e^{\int p(y)dy}=e^{-y},\ e^{-y}\frac{dx}{dy}-e^{-y}x=e^{-y}e^y=1$$

$$(e^{-y}x)'=1,\ e^{-y}x=y+c,\ -xe^{-y}+y=C$$

Eg. Find the integrating factor of $(2y^2-9xy)dx+(3xy-6x^2)dy=0$ and solve it.

(Sol.) $\dfrac{\partial(2y^2-9xy)}{\partial y}=4y-9x\neq 3y-12x=\dfrac{\partial(3xy-6x^2)}{\partial x}$

Suppose $u(x,y)=x^a y^b,\ \dfrac{\partial(2x^a y^{2+b}-9x^{a+1}y^{b+1})}{\partial y}=\dfrac{\partial(3x^{a+1}y^{b+1}-6x^{a+2}y^b)}{\partial x}$

$$\Rightarrow (1+2b-3a)y+(3-9b+6a)x=0\quad \forall x,y$$

$$\Rightarrow\begin{cases}1+2b-3a=0\\3-9b+6a=0\end{cases}\Rightarrow\begin{matrix}a=1\\b=1\end{matrix}\Rightarrow u(x,y)=xy$$

$$\frac{\partial(2xy^3-9x^2y^2)}{\partial y}=6x^2y-18x^2y=\frac{\partial(3x^2y^2-6x^3y)}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x}=2xy^3-9x^2y^2\Rightarrow F(x,y)=x^2y^3-3x^3y^2+c_1(x)$$

$$\frac{\partial F(x,y)}{\partial y}=3x^2y^2-6x^3y\Rightarrow F(x,y)=x^2y^3-3x^3y^2+c_2(x)$$

$$\therefore\ x^2y^3-3x^3y^2=C$$

1-6 Riccati's Equation ***y'=P(x)y²+Q(x)y+R(x)***

Suppose that there exists one specific solution $y=S(x)$, then a general solution can be obtained as follows

$$y = S(x) + \frac{1}{z},\quad y' = S'(x) - \frac{1}{z^2}\cdot z'$$

$$S'(x) - \frac{1}{z^2}\cdot z' = P(x)\cdot\left[S^2(x) + \frac{2S(x)}{z} + \frac{1}{z^2}\right] + Q(x)\cdot\left[S(x) + \frac{1}{z}\right] + \mathrm{R(x)}$$

$$-\frac{1}{z^2}\cdot z' = P(x)\cdot\frac{1}{z^2} + 2P(x)S(x)\frac{1}{z} + Q(x)\cdot\frac{1}{z}$$

$$z' + [2P(x)S(x) + Q(x)]z = -P(x)$$ is the 1st-order linear differential equation.

Eg. Solve $y' = e^{-3x}y^2 - y + 3e^{3x}$.

(Sol.) $y = e^{3x}$ is a solution.

$$\Rightarrow y = e^{3x} + \frac{1}{z}, y' = 3e^{3x} - \frac{z'}{z^2}$$

$$3e^{3x} - \frac{z'}{z^2} = e^{-3x}\cdot\left(e^{6x} + \frac{2e^{3x}}{z} + \frac{1}{z^2}\right) - \left(e^{3x} + \frac{1}{z}\right) + 3e^{3x}$$

$$z' = -2z - e^{-3x} + z, z' + z = -e^{-3x},\quad z'e^x + e^x\cdot z = -e^{-2x}$$

$$\left(z\cdot e^x\right)' = -e^{-2x},\quad ze^x = \frac{1}{2}e^{-2x} + c,\ \therefore\quad y = e^{3x} + \frac{2}{e^{-3x} + 2ce^{-x}}$$

1-7 Solutions of the First-order Ordinary Differential Equations by Matlab language

In **Matlab** language, we can use the following instructions to obtain the solution of the first-order ordinary differential equation:

```
>>soln=dsolve('Dy=3*y+exp(2*x)','y(0) = 3') % solve y'=3y+exp(2x), y(0)=3

ans=

-exp(2*x)+4*exp(3*x)
```

1-8 Some Theorems on the First-order Ordinary Differential Equations

A family of curves $F(x,y,k)=0$ is a solution of $y'=f(x,y)$.

Eg. A family of circles $x^2+y^2-k^2=0$ is a solution of $y'=-x/y$.

Theorem An oblique trajectory intersecting $y'=f(x,y)$ at an angle α is $y'=\dfrac{f(x,y)+\tan(\alpha)}{1-f(x,y)\tan(\alpha)}$; particularly, if $\alpha=\pi/2$, then the orthogonal trajectory is $y'=-1/f(x,y)$.

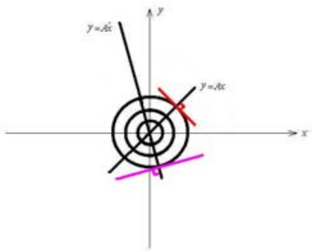
Eg. Find the families of oblique trajectories intersecting the circle $x^2+y^2=k^2$ at angles of 45° and 90°.

(Sol.) $x^2+y^2=k^2 \leftrightarrow y'=-\dfrac{x}{y}=f(x,y)$

$$1.\quad \tan(45°)=1,\quad y'=\frac{-\frac{x}{y}+1}{1+\frac{x}{y}}=\frac{y-x}{y+x}=\frac{\left(\frac{y}{x}\right)-1}{\left(\frac{y}{x}\right)+1}=\frac{v-1}{v+1}$$

$$v+xv'=\frac{v-1}{v+1},\quad v'=\frac{v-1-v^2-v}{(v+1)x}=-\frac{1+v^2}{v+1}\cdot\frac{1}{x}$$

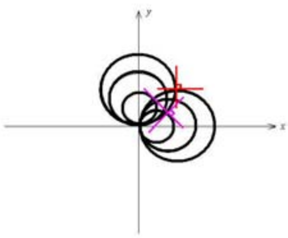
$$\left(\frac{1+v}{1+v^2}\right)dv=-\frac{dx}{x}\Rightarrow\frac{1}{2}\ell n\,|1+v^2|+\tan^{-1}(v)=-\ell n\,|x|+c$$



$$\therefore\ \frac{1}{2}\ell n\left|1+\left(\frac{y}{x}\right)^2\right|+\tan^{-1}\left(\frac{y}{x}\right)=-\ell n\,|x|+c$$

$$2.\quad y'=-\frac{1}{\left(-\frac{x}{y}\right)}=\frac{y}{x},\quad \frac{dy}{y}=\frac{dx}{x},\quad \therefore\ y=Ax$$

Theorem A family of curves $F(\theta,r,k)=0$, of which differential equation is $f(\theta,r,r')=0$. Then the family of orthogonal trajectories has differential equation $f(\theta,r,-r^2/r')=0$, where $r'=dr/d\theta$.



Eg. Find the family of trajectories orthogonal to $r=k\cos(\theta)$.

(Sol.) $r=k\cos(\theta)$, $r'=-k\sin(\theta)\Rightarrow r=-\cot(\theta)r'$

Family of orthogonal trajectories:

$$r=-\cot(\theta)\cdot\left(-\frac{r^2}{r'}\right)\Rightarrow\frac{r}{r'}=\tan(\theta)\Rightarrow r=k'\sin(\theta)$$