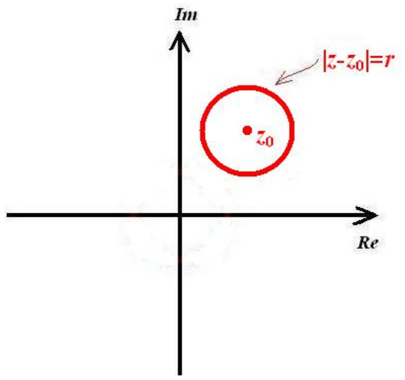
Chapter 10 Integration in the Complex Plane

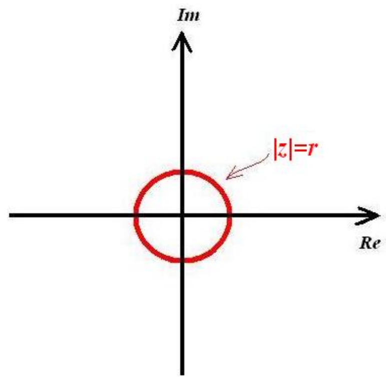
10-1 Complex Line Integrals and Some Integral Theorems

For smooth curve C: $z=z(t)$ for $a \leqq t \leqq b$, then $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$

Special case 1 C: $|z-z_0|=r \Leftrightarrow z(t)=z_0+re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$

Special case 2 C: $|z|=r \Leftrightarrow z(t)=re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$





Eg. Find $\oint_c \frac{1}{z}dz$, **C: $z=e^{it}$, $0 \leqq t \leqq 2\pi$.**

(Sol.) $z(t)=e^{it}$, $z'(t)= ie^{it}$, $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$

Eg. Evaluate $\oint_C \frac{dz}{z-3i}$, **C: $|z-3i|=\frac{1}{3}$.**

(Sol). $z(t)=3i+\frac{1}{3}e^{it}$, $z'(t)=\frac{1}{3}ie^{it}$, $\oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3}e^{it}dt = 2\pi i$

Eg. Evaluate $\oint_c \bar{z}dz$, **C: $|z|=1$.**

(Sol.) $z(t)=e^{it}$, $\bar{z}=e^{-it}$, $z'(t)= ie^{it}$, $\oint_c \bar{z}dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}dt = 2\pi i$

Eg. Evaluate $\oint_c [z-R_e(z)]dz$, **C: $|z|=2$.**

(Sol.) $z(t)=2e^{it}$, $z'(t)= i2e^{it}$

$$R_e(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(2e^{it}+2e^{-it}),\quad z-R_e(z) = \frac{1}{2}(z-\bar{z}) = \frac{1}{2}(2e^{it}-2e^{-it})$$

$$\oint_c [z-R_e(z)]dz = \int_0^{2\pi} [z(t)-R_e(z(t))] \cdot z'(t)dt$$

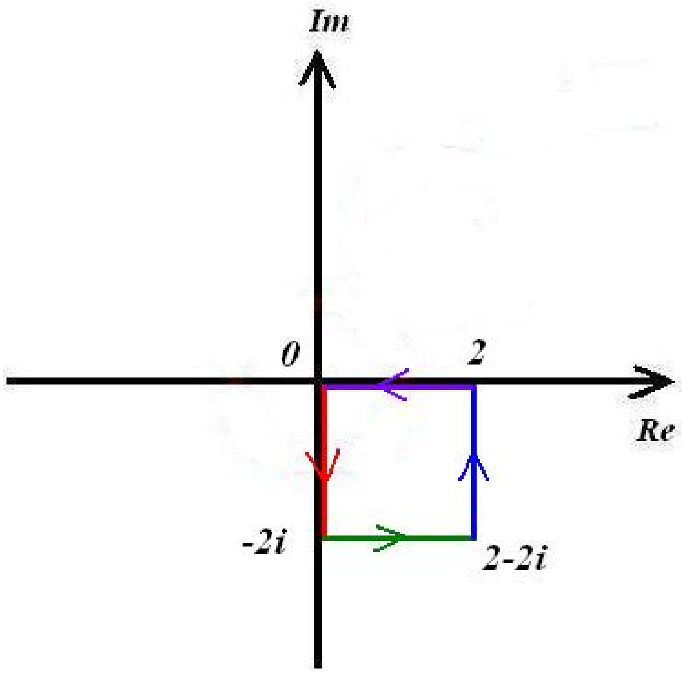
$$= \int_0^{2\pi} \frac{2}{2}(e^{it}-e^{-it}) \cdot 2ie^{it}dt = 2i\int_0^{2\pi}(e^{2it}-1)dt = -4\pi i$$

Eg. Evaluate $\oint_c [z^2 + I_m(z)]dz$ **, where** C **is the square with 0, -2*i*, 2-2*i*, and 2.**

(Sol.) $$\oint_c [z^2+I_m(z)]dz = \int_0^{-2}(-t^2+t)i\,dt + \int_0^2[(t-2i)^2-2]dt + \int_{-2}^0[(2+it)^2+t]i\,dt$$

$$+\int_2^0(t^2+0)dt = i\left(-\frac{t^3}{3}+\frac{t^2}{2}\right)\Bigg|_0^{-2} + \left(\frac{t^3}{3}-2it^2-6t\right)\Bigg|_0^2$$

$$+i\left(4t+2it^2-\frac{t^3}{3}+\frac{t^2}{2}\right)\Bigg|_{-2}^0 + \frac{t^3}{3}\Bigg|_2^0 = -4$$



Cauchy's integral theorem **Let** $f(z)$ **be analytic in a simply-connected domain** D**,** C **is a simple closed curve in** D**, then** $\oint_c f(z)dz = 0$.

Eg. Evaluate $\oint_C \frac{z}{\sin(z)(z-2i)^3}dz$ **,** C**: |z-8*i*|=1.**

(Sol.) $f(z) = \frac{z}{\sin(z)(z-2i)^3}$ is analytic except $z=2i$, $n\pi$ ($n=0, \pm1, \pm2, \ldots$)

No poles are within C, ∴ $\oint f(z)dz = 0$

Eg. Evaluate $\oint_c \frac{1}{z}dz$ **,** C**: |z-2|=1.**

(Sol.) $f(z)=\frac{1}{z}$ is analytic except $z=0$. No poles are within C, ∴ $\oint_c \frac{dz}{z} = 0$

Eg. Evaluate $\oint_c \frac{2z+1}{z^3-iz^2+6z}dz$ **, C: |z-3i|=** $\frac{1}{3}$ **.**

(Sol). $\oint_c \frac{(2z+1)dz}{z^3-iz^2+6z} = \frac{1}{6}\oint_c \frac{dz}{z} + \frac{-1+4i}{10}\oint_c \frac{dz}{z+2i} - \frac{1+6i}{15}\oint_c \frac{dz}{z-3i}$

$$= -\frac{1+6i}{15}\int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3}e^{it}dt = -\frac{1+6i}{15}(2\pi i) = \frac{\pi}{15}(12-2i)$$

Eg. Show that $\int_0^{2\pi} e^{\cos\theta} \cdot \cos(\theta+\sin\theta)d\theta = \int_0^{2\pi} e^{\cos\theta} \cdot \sin(\theta+\sin\theta)d\theta = 0$ **.**

(Proof) Let *C*: |*z*|=1 ⇒ $z(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$, ∴ $f(z) = e^z$ is analytic within *C*.

$$\oint_c e^z dz = 0 = \int_0^{2\pi} e^{\cos\theta + i\sin\theta} de^{i\theta} = \int_0^{2\pi} ie^{\cos\theta} \cdot e^{i(\theta+\sin\theta)}d\theta$$

$$= i\int_0^{2\pi} e^{\cos\theta}\{[\cos(\theta+\sin\theta) + i[\sin(\theta+\sin\theta)]\}d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} \cdot \{-\sin(\theta+\sin\theta) + i\cos(\theta+\sin\theta)\}d\theta$$

∴ *Re*(⋯)=*Im*(⋯)=0

Cauchy's integral formulae Let *f(z)* be an analytic is a simply-connected region *D*, *C* is a simple curve enclosing z_0 in *D*, then $\oint_c \frac{f(z)}{z-z_0}dz$ **=2πi • f(z₀) and** $\oint_c \frac{f(z)dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{(n-1)}(z_0)$ **.**

Eg. Let z_0 be within *C*, find $\oint_c \frac{dz}{z-z_0}$ **and** $\oint_c \frac{dz}{(z-z_0)^n}$ **, *n*≧2.**

(Sol.) Let $f(z_0)=1, f^{(n-1)}(z_0)=0 \Rightarrow \oint \frac{dz}{z-z_0} = 2\pi i$ and $\oint_c \frac{dz}{(z-z_0)^n} = 0$.

Eg. Evaluate $\oint_c \frac{\sin^6 z}{z-\frac{\pi}{6}}dz$ **and** $\oint_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}dz$ **if** $C: \left|z-\frac{\pi}{6}\right| = \delta > 0$ **.**

(Sol.) $\oint_c \frac{\sin^6 z}{z-\frac{\pi}{6}}dz = 2\pi i \cdot \sin^6\left(\frac{\pi}{6}\right) = \frac{\pi i}{32}$, $\oint_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}dz = \frac{2\pi i}{2!} \cdot [\sin^6 z]''\Big|_{z_0=\frac{\pi}{6}} = \frac{21\pi i}{16}$

Eg. Evaluate $\oint_c \frac{e^{iz}}{z^3} dz$**, *C*: |z|=3.**

(Sol.) $\oint_c \frac{e^{iz}}{z^3} dz = i2\pi \cdot \frac{(e^{iz})''}{(3-1)!}\bigg|_{z_0=0} = -i\pi$

Eg. Evaluate $\oint_c \frac{2\sin(z^2)}{(z-1)^4} dz$**, *C* is a closed curve not passing 1.**

(Sol.) If *C* does not enclose 1, $\frac{2\sin(z^2)}{(z-1)^4}$ is analytic within *C*, $\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4} dz = 0$

If *C* encloses 1, let $f(z) = 2\sin(z^2) \Rightarrow \frac{2\sin(z^2)}{(z-1)^4} = \frac{f(z)}{(z-1)^4}$, n=4, n-1=3

$f^{(3)}(z) = -24z\sin(z^2) - 16z^3\cos(z^2)$

$\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4} dz = \frac{2\pi i}{3!}[-24\sin(1) - 16\cos(1)] = \frac{\pi i}{3}[-24\sin(1) - 16\cos(1)]$

Eg. Evaluate $\oint_{|z|=1} \frac{dz}{z^2-4}$ **and** $\oint_{|z|=3} \frac{e^{2z}}{z-2}$**.** 【交大土木所】

(Ans.) 0, $2\pi i e^4$

10-2 Laurent's Theorem & the Residue Theorem

If $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\underbrace{\cdots+\frac{a_{-3}}{(z-z_0)^3}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}}_{(principal\ part)}$

$\underbrace{+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots}_{(analytic\ part)}$

Laurent's theorem $a_n=\frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$

Residue: $a_{-1}=\frac{1}{2\pi i}\oint_c f(z)dz$

Residue theorem **Let $f(z)$ be analytic in D except $z_1, z_2, \ldots, z_n$ and C encloses $z_1, z_2, \ldots, z_n$ within D. Then we have** $\oint_c f(z)dz=2\pi i\cdot\sum_{j=1}^{n}\underset{z_j}{\mathrm{Res}}(f)$ **and**

$\underset{z_j}{\mathrm{Res}}(f)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$, **where m is the order of a pole $z=z_j$.**

In case of m=1, $\underset{z_j}{\mathrm{Res}}(f)=\lim_{z\to z_j}[(z-z_j)\cdot f(z)]$.

Eg. Find the residue of $f(z)=\frac{\sin(z)}{(z-i)^3}$ **and evaluate** $\oint_c\frac{\sin(z)}{(z-i)^3}dz$ **, C: $|z-i|=2$.**

(Sol.) m=3, $\underset{i}{\mathrm{Res}}(f)=\frac{1}{(3-1)!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\cdot\frac{\sin(z)}{(z-i)^3}]=\frac{-1}{2}\sin(i)=-\frac{1}{2}i\sinh(1)$

$$\therefore\ \oint\frac{\sin(z)}{(z-i)^2}dz=2\pi i\cdot\left(-\frac{1}{2}i\sinh(1)\right)=\pi\sinh(1)$$

Eg. Evaluate $\oint_c\frac{z^2+1}{z^2-1}dz$ **, C: $|z-1|=1$.** 【交大科管所】

(Sol.) There is only one pole 1 within C.

$$\underset{1}{\mathrm{Res}}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z^2+1}{z^2-1}]=\lim_{z\to 1}\frac{z^2+1}{z+1}=1$$

$$\oint_c\frac{z^2+1}{z^2-1}dz=2\pi i\cdot 1=2\pi i$$

Eg. Evaluate $\oint_c \tan z dz$ **, *C*: |*z*|=2.** 【中山電研】

(Sol.) There are two poles ±π/2 within *C*.

$$\operatorname{Re}_{\frac{\pi}{2}} s(f) = \lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\tan(z)] = \lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}] = \lim_{z\to\frac{\pi}{2}}\frac{z\sin(z)-\frac{\pi}{2}\sin(z)}{\cos(z)} =$$

$$\lim_{z\to\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)-\frac{\pi}{2}\cos(z)}{-\sin(z)} = -1$$

$$\operatorname{Re}_{-\frac{\pi}{2}} s(f) = \lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\tan(z)] = \lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}] = \lim_{z\to-\frac{\pi}{2}}\frac{z\sin(z)+\frac{\pi}{2}\sin(z)}{\cos(z)} =$$

$$\lim_{z\to-\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)+\frac{\pi}{2}\cos(z)}{-\sin(z)} = -1$$

$$\oint_c \tan z dz = 2\pi i \cdot [(-1)+(-1)] = -4\pi i$$

Eg. Evaluate $\oint_c \frac{\sin(z)}{z^2(z^2+4)}dz$ **, *C* is any piecewise-smooth curve enclosing 0, 2*i*, and –2*i*.**

(Sol.) $f(z)=\frac{\sin(z)/z}{z(z+2i)(z-2i)}$, $\because \lim_{z\to 0}[\sin(z)/z]=1$, ∴ *f*(*z*) has a removable singularity at 0 ⇒ *m* of the pole *z*=0 in *f*(*z*) is 1.

$$\operatorname{Re}_{0} s(f) = \lim_{z\to 0} z\cdot\frac{\sin(z)}{z^2(z^2+4)} = \frac{1}{4}$$

$$\operatorname{Re}_{2i} s(f) = \lim_{z\to 2i}(z-2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)} = \frac{i}{16}\sin(2i) = -\frac{1}{16}\sinh(2)$$

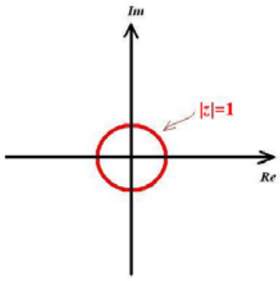
$$\operatorname{Re}_{-2i} s(f) = \lim_{z\to -2i}(z+2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)} = \frac{i}{16}\sin(2i) = -\frac{1}{16}\sinh(2)$$

$$\oint_c f(z)dz = 2\pi i\left[\frac{1}{4}-\frac{1}{16}\sinh(2)-\frac{1}{16}\sinh(2)\right] = \frac{\pi i}{2}-\frac{\pi i}{4}\sinh(2)$$

Eg. Evaluate $\oint_c \frac{\cos z}{z^2(z-1)}dz$ **for (a) *C*: |*z*|=$\frac{1}{3}$, (b) *C*: |*z*-1|=$\frac{1}{3}$, (c) *C*: |*z*|=2.** 【台大機研】

(Ans.) $-2\pi i$, $2\pi i\cos(1)$, $2\pi i[-1+\cos(1)]$

10-3 Evaluation of Real Integrals



Case 1 $\int_0^{2\pi} K[\cos\theta,\sin\theta]d\theta$

Choose C: $|z|=1$, $z=e^{i\theta}$ $\Rightarrow \sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$,

$\cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$, $d\theta=\frac{dz}{iz}$

$$\Rightarrow I=\int_0^{2\pi} K[\cos\theta,\sin\theta]d\theta=\oint_c K\left[\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\cdot\frac{dz}{iz}$$

Eg. Evaluate $\int_0^{2\pi}\frac{\cos\theta}{1+\frac{1}{4}\cos\theta}d\theta$ **.**

(Sol.) $$\int_0^{2\pi}\frac{\cos\theta}{1+\frac{1}{4}\cos\theta}d\theta=\oint_c\frac{\frac{1}{2}\left[z+\frac{1}{z}\right]}{1+\frac{1}{4}\cdot\frac{1}{2}\left[z+\frac{1}{z}\right]}\cdot\frac{dz}{iz}=\oint_c\frac{4(z^2+1)}{iz(z^2+8z+1)}dz$$

Poles: 0, $-4+\sqrt{15}$ (within C), $-4-\sqrt{15}$ (exterior to C)

$$\operatorname{Re}\underset{0}{s}(f)=\lim_{z\to0}z\cdot\frac{4z^2+4}{iz(z^2+8z+1)}=-4i$$

$$\operatorname{Re}\underset{-4+\sqrt{15}}{s}(f)=\lim_{z\to-4+\sqrt{15}}[z-(-4+\sqrt{15})]\cdot\frac{4(z^2+1)}{iz(z^2+8z+1)}=\frac{16i}{\sqrt{15}}$$

$$\therefore\int_0^{2\pi}\frac{\cos\theta}{1+\frac{1}{4}\cos\theta}d\theta=2\pi i\left(-4i+\frac{16}{\sqrt{15}}i\right)=8\pi-\frac{32\pi}{\sqrt{15}}$$

Eg. Show that $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$ **if $a>|b|$.**

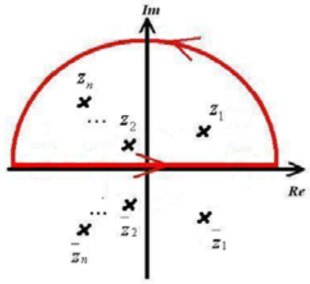
(Proof) $$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\oint_c\frac{dz/iz}{a+b\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)}=\oint_c\frac{2dz}{bz^2+2iaz-b}=\oint_c\frac{2dz}{b(z-z_1)(z-z_2)}$$

Poles: $z_1=\frac{i}{b}\left(-a+\sqrt{a^2-b^2}\right)$ is within C: $|z|=1$, but $z_2=\frac{i}{b}\left(-a-\sqrt{a^2-b^2}\right)$ is not.

$$\operatorname{Re}\underset{z_1}{s}(f)=\lim_{z\to z_1}(z-z_1)\cdot\frac{2}{b(z-z_1)(z-z_2)}=\frac{2}{b}\cdot\frac{1}{z_1-z_2}$$

$$=\frac{2}{b}\cdot\frac{b}{2i\sqrt{a^2-b^2}}=\frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore\int_0^{2\pi}\frac{d\theta}{a^2+b\sin\theta}=2\pi i\cdot\frac{1}{i\sqrt{a^2-b^2}}=\frac{2\pi}{\sqrt{a^2-b^2}}$$



Case 2 $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx$ or $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$

Choose C as a semi-circle with infinite radius enclosing the upper half-plane. Poles: z_1, z_2, …, z_n are in the upper half-plane, $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n$ are in the lower half-plane.

Assume $deg(q) \geq deg(p)+2$, then $\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \oint_c f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j} (f)$

Eg. Evaluate (a) $\int_{-\infty}^{\infty} \frac{1}{x^2+64} dx$ **and (b)** $\int_{-\infty}^{\infty} \frac{1}{x^6+64} dx$.

(Sol.) (a) Poles: $8i$ (upper half-plane), $-8i$ (lower half-plane)

$$\operatorname{Res}_{8i}(f) = \lim_{z\to 8i} (z-8i) \cdot \frac{1}{(z+8i)(z-8i)} = \frac{1}{16i}, \quad \int_{-\infty}^{\infty} \frac{1}{x^2+64} dx = 2\pi i \cdot \frac{1}{16i} = \frac{\pi}{8}$$

(b) Poles: $2e^{\frac{\pi i}{6}}$, $2i$, $2e^{\frac{5\pi i}{6}}$ (upper half-plane), $2e^{-\frac{\pi i}{6}}$, $-2i$, $2e^{-\frac{5\pi i}{6}}$ (lower half-plane)

$$\operatorname{Res}_{2e^{i\pi/6}}(f) = \frac{1}{6\left(2e^{\pi i/6}\right)^5} = \frac{1}{192} e^{-5\pi i/6} = \frac{1}{192}\left[-\frac{\sqrt{3}}{2} - \frac{i}{2}\right], \quad \operatorname{Res}_{2i}(f) = \frac{1}{6(2i)^5} = -\frac{i}{192}$$

$$\operatorname{Res}_{2e^{i5\pi/6}}(f) = \frac{1}{6\left(2e^{5\pi i/6}\right)^5} = \frac{1}{192} e^{-\pi i/6} = \frac{1}{192}\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right], \int_{-\infty}^{\infty} \frac{dx}{x^6+64} = 2\pi i\left[\frac{1}{192}(-i-i)\right] = \frac{\pi}{48}$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+9)} dx$ **and** $\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x^2+9)} dx$.

(Sol.) Poles: $2i$, $3i$ (upper half-plane), $-2i$, $-3i$ (lower half-plane)

$$f(z) = \frac{e^{iz}}{(z^2+4)(z^2+9)}, \quad \operatorname{Res}_{2i}(f) = \frac{e^{-2}}{20i}, \quad \operatorname{Res}_{3i}(f) = \frac{-e^{-3}}{30i}$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x^2+9)} dx = 2\pi i\left(\frac{e^{-2}}{20i} - \frac{e^{-3}}{30i}\right) = \frac{\pi}{5}\left(\frac{e^{-2}}{2} - \frac{e^{-3}}{3}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+9)} dx = \frac{\pi}{5}\left(\frac{e^{-2}}{2} - \frac{e^{-3}}{3}\right), \quad \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x^2+9)} dx = 0$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{x\sin(\sqrt{3}x)}{x^2+16} dx$.

(Sol.) Poles: $4i$ (upper half-plane), $-4i$ (lower half-plane)

$$f(z) = \frac{ze^{i\sqrt{3}z}}{z^2+16}, \quad \operatorname{Res}_{4i}(f) = \frac{4ie^{-4\sqrt{3}}}{8i} = \frac{e^{-4\sqrt{3}}}{2}$$

$$\int_{-\infty}^{\infty} \frac{xe^{i\sqrt{3}x}}{x^2+16} dx = 2\pi i \cdot \frac{e^{-4\sqrt{3}}}{2} = \pi i e^{-4\sqrt{3}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\cos(\sqrt{3}x)}{x^2+16} dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{x\sin(\sqrt{3}x)}{x^2+16} dx = \pi e^{-4\sqrt{3}}$$

Eg. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx$.

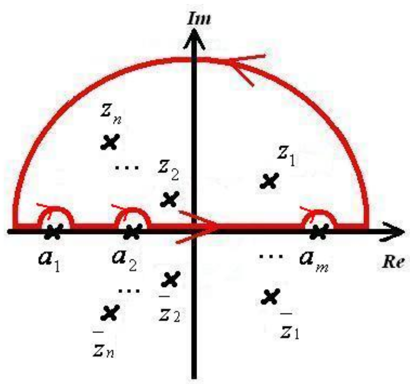
(Sol.) $\int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$. Poles: i (upper half-plane), $-i$ (lower half-plane). $f(z) = \frac{z^2}{(z^2+1)^2} = \frac{z^2}{(z+i)^2(z-i)^2}$, m of $(z-i)^2$ is 2.

$$\lim_{z\to i} \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-i)^2 \cdot \frac{z^2}{(z+i)^2(z-i)^2}\right] = \frac{2z(z+i)^2 - 2z^2(z+i)}{(z+i)^4}\Bigg|_{z=i} = \frac{-8i+4i}{16} = -\frac{i}{4}$$

$$\therefore \int_0^{\infty} \frac{x^2}{(x^2+1)^2} dx = \frac{2\pi i}{2}\left(-\frac{i}{4}\right) = \frac{\pi}{4}$$

Eg. Evaluate $\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2} dx$, $\begin{matrix} a \ge 0 \\ b > 0 \end{matrix}$. 【台大機研】

Eg. Evaluate $\int_0^{\infty} \frac{x\sin(x)}{x^2+4} dx$. 【交大電信】 (Ans.) $\frac{\pi e^{-2}}{2}$



Case 3 **$\int_{-\infty}^{\infty} H(x)dx$, where $H(z)=O\left(\frac{1}{x^p}\right)$, $p>1$.**

Some poles of $H(z)$ are located on the real axis.

Choose C as a semi-circle with infinite radius enclosing the upper half-plane, but excluding the poles on the real axis. Let z_k $(1 \le k \le n)$ be the pole on the upper half-plane and a_j $(1 \le j \le m)$ be the pole on the real axis. Then we have

$$\int_{-\infty}^{\infty} H(x)dx = 2\pi i \cdot \sum_{k=1}^{n} \underset{z_k}{\mathrm{Res}}(H) + \pi i \cdot \sum_{j=1}^{m} \underset{a_j}{\mathrm{Res}}(H).$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x(x^2-4x+5)} dx$. [交大電信研究所]

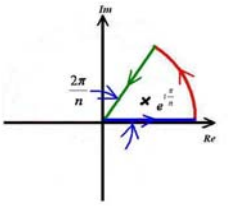
(Sol.) $f(z) = \frac{1}{z(z^2-4z+5)} = \frac{1}{z[z-(2+i)][z-(2-i)]}$ has 3 poles:

0 (on the real axis), $2+i$ (upper half-plane), $2-i$ (lower half-plane)

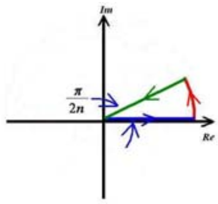
$$\int_{-\infty}^{\infty} \frac{1}{x(x^2-4x+5)} dx = 2\pi i \cdot \underset{2+i}{\mathrm{Res}}(f) + \pi i \cdot \underset{0}{\mathrm{Res}}(f)$$

$$= 2\pi i \cdot \lim_{z\to 2+i}\left[[z-(2+i)] \cdot \frac{1}{z[z-(2+i)][z-(2-i)]}\right] + \pi i \cdot \lim_{z\to 0}\left[z \cdot \frac{1}{z(z^2-4z+5)}\right]$$

$$= \frac{2\pi i}{(2+i)\cdot 2i} + \frac{\pi i}{5} = \frac{\pi(2-i)}{5} + \frac{\pi i}{5} = \frac{2\pi}{5}$$



Case 4 $\int_0^\infty \begin{Bmatrix}\sin(x^n)\\ \cos(x^n)\end{Bmatrix}dx$ **or** $\int_0^\infty G(x^n)dx$



Choose C as a sector with angle $\frac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$ or a sector with angle $\frac{\pi}{2n}$ enclosing no poles.

Eg. Evaluate $\int_0^\infty \frac{dx}{1+x^n}$, $n>1$.

(Sol.) Choose C as a sector with angle $\frac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$.

$$\oint_c \frac{dz}{1+z^n} = 2\pi i \cdot \underset{e^{i\frac{\pi}{n}}}{\mathrm{Res}}(f) = 2\pi i \cdot \lim_{z\to e^{i\frac{\pi}{n}}}\left[\left(z-e^{i\frac{\pi}{n}}\right)\frac{1}{1+z^n}\right] = \frac{2\pi i}{nz^{n-1}}\Bigg|_{z=e^{i\frac{\pi}{n}}} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$= \int_0^R \frac{dx}{1+x^n} + \int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} + \int_R^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n e^{i2\pi}}$$

As $R\to\infty$, $\int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} \to 0$ ($\because n>1$)

$$\therefore \int_0^\infty \frac{dx}{1+x^n} + \int_\infty^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n} = \left(1-e^{i\frac{2\pi}{n}}\right)\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^n} = \frac{e^{i\frac{\pi}{n}}}{1-e^{i\frac{2\pi}{n}}}\cdot\frac{-2\pi i}{n} = \frac{1}{\frac{e^{i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}}}{2i}}\cdot\frac{\pi}{n} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$

Eg. Show that $\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$.

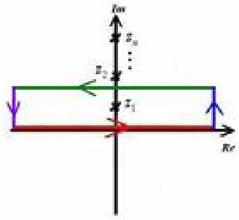
(Proof) Choose C as a sector with angle $\frac{\pi}{4}$ enclosing no poles.

$$\oint_c e^{iz^2}dz = 0 = \int_0^R e^{ix^2}dx + \int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta + \int_R^0 e^{iR^2e^{i\frac{\pi}{2}}}\cdot e^{i\frac{\pi}{4}}dR$$

As $R\to\infty$, $\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta = iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos2\theta+i\sin2\theta)}e^{i\theta}\,d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin2\theta}e^{i(\theta+R^2\cos2\theta)}\,d\theta \to 0$,

$$\int_0^\infty [\cos(x^2)+i\sin(x^2)]dx = \int_0^\infty\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right)e^{-R^2}dR = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{\pi}}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2}i$$

$$\therefore \int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$



Case 5 $\int_{-\infty}^{\infty} G(e^x)dx$, **where** $G(x) = \dfrac{p(x)}{x^n + q_{n-1}(x)}$

Choose C as an infinitely-wide rectangle and there is one pole on the imaginary axis.

Eg. Evaluate $\int_{-\infty}^{\infty} \dfrac{e^{mx}}{1+e^x}dx$, $0<m<1$. **【台大機研】**

(Sol.) Pole: $i\pi$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = \int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx + \int_0^{2\pi}\frac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy + \int_{\infty}^{-\infty}\frac{e^{mx}\cdot e^{i2m\pi}}{1+e^x\cdot e^{i2\pi}}dx + \int_{2\pi}^{0}\frac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy$$

$$= 2\pi i\cdot \operatorname*{Res}_{\pi i}(f) = 2\pi i\cdot \lim_{z\to\pi i}\frac{(z-\pi i)e^{mz}}{1+e^z}$$

$$= 2\pi i\cdot \lim_{z\to\pi i}\frac{e^{mz}+m(z-\pi i)e^{mz}}{e^z} = 2\pi i(-1)^{m-1} = 2\pi i e^{i(m-1)\pi}$$

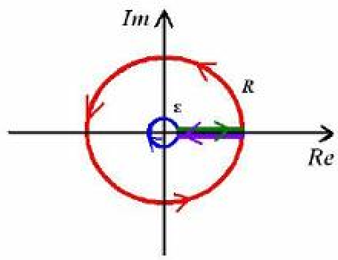
$\because\ 0<m<1,\ \therefore R\to\infty$, $\int_0^{2\pi}\dfrac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy \to 0$ and $\int_{2\pi}^{0}\dfrac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy \to 0$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = 2\pi i e^{i(m-1)\pi} = \int_{-\infty}^{\infty}(1-e^{i2m\pi})\cdot\frac{e^{mx}}{1+e^x}dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx = \frac{2\pi i}{1-e^{i2m\pi}}\cdot e^{im\pi}\cdot(-1) = \frac{\pi}{\dfrac{e^{im\pi}-e^{-im\pi}}{2i}} = \frac{\pi}{\sin(m\pi)}$$

***Case 6* Other types**

Eg. For 0<*p*<1, $\int_0^\infty \frac{x^p dx}{x(1+x)} = ?$



(Sol.) ∵ 0<*p*<1, ∴ Poles are 0 and -1.

$$\oint \frac{z^p dz}{z(1+z)} = 2\pi i \cdot \mathrm{Res}_{-1}(f) = 2\pi i \cdot \lim_{z\to -1}(z+1)\cdot\frac{z^p}{z(z+1)} = 2\pi i \cdot e^{i\pi(p-1)}$$, and

$$\oint \frac{z^p dz}{z(1+z)} = \int_\varepsilon^R \frac{x^p dx}{x(1+x)} + \int_0^{2\pi} \frac{i(\mathrm{R}e^{i\theta})^p d\theta}{1+\mathrm{R}e^{i\theta}} + \int_R^\varepsilon \frac{(xe^{i2\pi})^{p-1} d(xe^{i2\pi})}{1+xe^{i2\pi}} + \int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p d\theta}{1+\varepsilon e^{i\theta}}$$

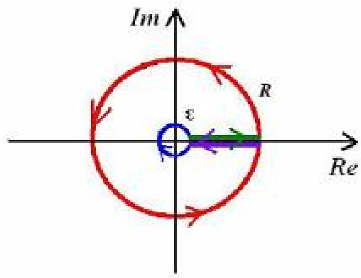
∵ 0<*p*<1, *R*→∞, ∴ $\int_0^{2\pi} \frac{i(\mathrm{R}e^{i\theta})^p d\theta}{1+\mathrm{R}e^{i\theta}} \to 0$

∵ $\varepsilon \to 0$, ∴ $\int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p d\theta}{1+\varepsilon e^{i\theta}} \to 0$

$$\Rightarrow \oint \frac{z^p dz}{z(1+z)} = \int_0^\infty \frac{x^p dx}{x(1+x)} + \int_\infty^0 \frac{(xe^{i2\pi})^{p-1} d(xe^{i2\pi})}{1+xe^{i2\pi}} = \int_0^\infty \frac{[1-e^{i2\pi(p-1)}]\cdot x^p dx}{x(1+x)}$$

$$\therefore \int_0^\infty \frac{x^p dx}{x(1+x)} = \frac{2\pi i \cdot e^{i\pi(p-1)}}{1-e^{i2\pi(p-1)}} = \frac{\pi}{(e^{ip\pi}-e^{-ip\pi})/2i} = \frac{\pi}{\sin(p\pi)}$$

Eg. For -1<*a*<1, $\int_0^\infty \frac{x^a dx}{(1+x)^2} = ?$ **【交大電信研究所、中央光電所】**



(Sol.) −1 is a multiple-order pole.

$$\oint \frac{z^a dz}{(1+z)^2} = 2\pi i \cdot \mathrm{Res}_{-1}(f) = 2\pi i \cdot \lim_{z\to -1}\frac{1}{1!}d[(z+1)^2\cdot\frac{z^a}{(z+1)^2}]/dz = 2\pi i \cdot a e^{i\pi(a-1)}$$, and

$$\oint \frac{z^a dz}{(1+z)^2} = \int_\varepsilon^R \frac{x^a dx}{(1+x)^2} + \int_0^{2\pi} \frac{(\mathrm{R}e^{i\theta})^a i\mathrm{R}e^{i\theta} d\theta}{(1+\mathrm{R}e^{i\theta})^2} + \int_R^\varepsilon \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a i\varepsilon e^{i\theta} d\theta}{(1+\varepsilon e^{i\theta})^2}$$

∵ -1<*a*<1, *R*→∞, ∴ $\int_0^{2\pi} \frac{(\mathrm{R}e^{i\theta})^a i\mathrm{R}e^{i\theta} d\theta}{(1+\mathrm{R}e^{i\theta})^2} \to 0$

∵ $\varepsilon \to 0$, ∴ $\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a i\varepsilon e^{i\theta} d\theta}{(1+\varepsilon e^{i\theta})^2} \to 0$

$$\Rightarrow \oint \frac{z^a dz}{(1+z)^2} = \int_0^\infty \frac{x^a dx}{(1+x)^2} + \int_\infty^0 \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} = \int_0^\infty \frac{[1-e^{i2\pi(a+1)}]\cdot x^a dx}{(1+x)^2}$$

$$\therefore \int_0^\infty \frac{x^a dx}{(1+x)^2} = \frac{2\pi i \cdot a e^{i\pi(a-1)}}{1-e^{i2\pi(a+1)}} = \frac{\pi a}{(e^{i\pi a}-e^{-i\pi a})/2i} = \frac{a\pi}{\sin(a\pi)}$$