Chapter 12 Conformal Mapping

12-1 Introduction

Complex image transformation: $w=f(z)=f(x+iy)=u+iv$

$$|det(\text{Jacobian matrix})| = \left|\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right| = |f'(z)|^2$$

Area magnification factor: $|f'(z)|^2$, **Rotation angle at z_0:** $\arg[f'(z_0)]$

Critical point z_c: z_c if $f'(z_c)=0$

Theorem If $f(z)$ is analytic in R and $f'(z_0)\neq 0$, then $w= f(z)$ is a conformal mapping.

(Proof) $f(z)$ is analytic $\Leftrightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$$\left|\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix}\right| = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$ is called the area magnification factor

C: $z=z(t)$, C': $w=w(t)=f(z)$, $\frac{dw}{dt}=\frac{dw}{dz}\cdot\frac{dz}{dt}=f'(z)\frac{dz}{dt}$

Let $\frac{dw}{dt}=\rho e^{i\phi}$, $\frac{dz}{dt}=re^{i\theta}$, $f'(z)=Re^{i\Psi}$, then

$$\rho e^{i\phi} = Rre^{i(\theta+\psi)} \Rightarrow \phi=\theta+\psi=\theta+\arg(f'(z))$$

$c_1 \to c_1'$, $c_2 \to c_2'$ and $\phi_1=\theta_1+\arg(f'(z_0))$, $\phi_2=\theta_2+\arg(f'(z_0))$

$$\Rightarrow \phi_1-\phi_2=\theta_1-\theta_2$$

12-2 Conformal Mapping

Linear transformation: $w=Az+B$, $A\neq 0$

This transformation can magnify a closed region or rotate a straight line.

Reciprocal transformation: $w=k/z$

This conformal mapping is from the exterior/interior region of a circle to the interior/exterior region of a circle.

Eg. Map $|z|<1$ to $|w|>4$. (Sol.) $w=\frac{4}{z}$

Linear fractional transformation: $w=\dfrac{Az+B}{Cz+D}$

This conformal mapping is from a circle/line to another circle/line. For $z_1\to w_1$, $z_2\to w_2$, $z_3\to w_3$, then $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\Rightarrow w=\dfrac{Az+B}{Cz+D}$

Eg. Find a linear fractional transformation $w=f(z)$ with $3\to i$, $1-i\to 4$, and $2-i\to 6+2i$.

(Sol.) $\dfrac{w-i}{w-(6+2i)}\cdot\dfrac{-2-2i}{4-i}=\dfrac{z-3}{z-2+i}\cdot\dfrac{-1}{-2-i}\Rightarrow w=\dfrac{(20+4i)z-(16i+68)}{(6+5i)z-(22+7i)}$

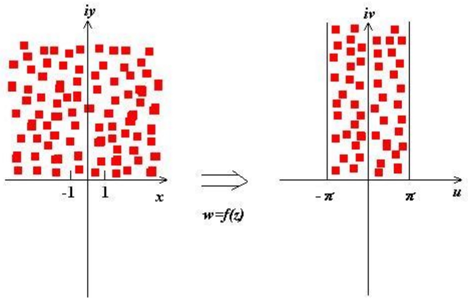
Eg. Find a linear fractional transformation $w=f(z)$ with $i\to 4i$, $1\to 3-i$, and $2+i\to\infty$.

(Sol.) $\dfrac{w-4i}{w-\infty}\cdot\dfrac{3-i-\infty}{3-5i}=\dfrac{z-i}{z-2-i}\cdot\dfrac{-1-i}{1-i}\Rightarrow w=\dfrac{-1+3i+(5-i)z}{2+i-z}$

Eg. Find a fixed point (invariant) for $w=\dfrac{2z-5}{z+4}$.

(Sol.) $w=\dfrac{2z-5}{z+4}=z\Rightarrow z^2+2z+5=0\Rightarrow z=-1\pm 2i$

Schwartz-Christoffel transformation: This conformal mapping is from the upper half plane to a polygon. For $x_1\to w_1$ (inner angle θ_1), $x_2\to w_2$ (inner angle θ_2), …, $x_n\to w_n$ (inner angle θ_n), then $w=A\int_0^z (z-x_1)^{\frac{\theta_1}{\pi}-1}(z-x_2)^{\frac{\theta_2}{\pi}-1}\cdots(z-x_n)^{\frac{\theta_n}{\pi}-1}dz+B$.

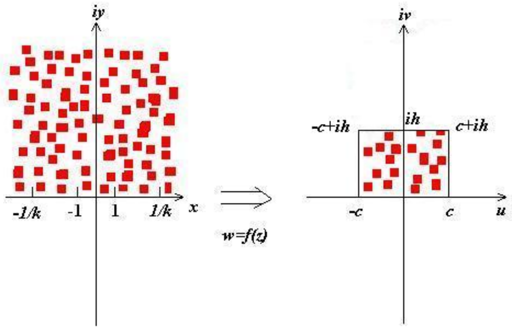


Eg. Find a conformal mapping $w=f(z)$ from the upper half plane to an infinitely deep rectangular well and $-\infty\to-\pi+i\infty$, $-1\to-\pi$, $1\to\pi$, and $\infty\to\pi+i\infty$.

(Sol.) $w=A\int(z+1)^{\frac{(\pi/2)}{\pi}-1}(z-1)^{\frac{(\pi/2)}{\pi}-1}dz+B$

$$=A\int\frac{dz}{\sqrt{z^2-1}}+B=\frac{A}{i}\int\frac{dz}{\sqrt{1-z^2}}+B=A'\sin^{-1}(z)+B$$

$-1\to-\pi$, $1\to\pi\Rightarrow\begin{cases}\pi=A'\dfrac{\pi}{2}+B\\-\pi=-A'\dfrac{\pi}{2}+B\end{cases}\Rightarrow\begin{cases}A'=2\\B=0\end{cases}$, $\therefore$ $w=2\sin^{-1}(z)$

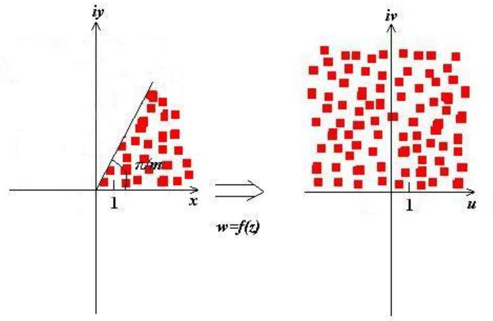


Eg. Find a conformal mapping *w=f*(*z*) from the upper half plane to a rectangular region and -∞→*ih*, -1/*k*→-*c*+*ih*, -1→-*c*, 0→0, 1→*c*, 1/*k*→*c*+*ih*, and ∞→*ih*. (*k*<1)

(Sol.) $$w = A\int\left(z+\frac{1}{k}\right)^{\frac{(\pi/2)}{\pi}-1}\cdot(z+1)^{\frac{(\pi/2)}{\pi}-1}\cdot(z-0)^{\frac{\pi}{\pi}-1}\cdot(z-1)^{\frac{(\pi/2)}{\pi}-1}\cdot\left(z-\frac{1}{k}\right)^{\frac{(\pi/2)}{\pi}-1}dz+B$$

$$= A\int\frac{dz}{(z^2-1)^{1/2}\left(z^2-\frac{1}{k^2}\right)^{1/2}}+B = kA\int\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}+B$$

$$\because\ z=0\rightarrow w=0,\ \therefore\ B=0\Rightarrow w=C\int_0^z\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

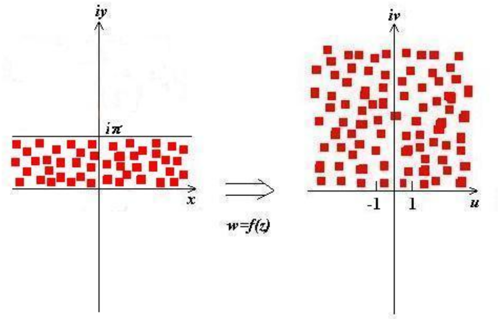


Eg. Find a conformal mapping *w=f*(*z*) from a sector to the upper half plane and $(\lim_{R\to\infty}\mathrm{Re}^{i\frac{\pi}{m}})$→-∞, 0→0, and 1→1.

(Sol.) $$z = f^{-1}(w) = A\int(w-0)^{\frac{(\pi/m)}{\pi}-1}\cdot(w-1)^{\frac{\pi}{\pi}-1}dw+B$$

$$= A\int w^{\frac{1}{m}-1}dw+B = Amw^{\frac{1}{m}}+B = A'w^{\frac{1}{m}}+B$$

$$\because\ z=0\rightarrow w=0\ \text{ and }\ z=1\rightarrow w=1,\ \therefore\ A'=1,\ B=0\Rightarrow z=w^{\frac{1}{m}}\Rightarrow w=z^m$$



Eg. Find a conformal mapping $w=f(z)$ from an infinitely long belt to the upper half plane and $\infty+\pi i\to-\infty$, $\pi i\to-1$, $-\infty+\pi i\to 0^-$, $-\infty\to 0^+$, $0\to 1$, and $\infty\to\infty$.

(Sol.) $z=f^{-1}(w)=A\int(w+1)^{\frac{\pi}{\pi}-1}(w-0^-)^{\frac{(\pi/2)}{\pi}-1}(w-0^+)^{\frac{(\pi/2)}{\pi}-1}(w-1)^{\frac{\pi}{\pi}-1}dw+B$

$$=A\int\frac{dw}{w}+B=A\ln(w)+B$$

$$\begin{matrix} z=\pi i\to w=-1 \\ z=0\to w=1 \end{matrix} \Rightarrow \begin{matrix} \pi i=A\ln(-1)+B=A\ln(e^{i\pi})+B=i\pi A+B \\ 0=A\ln(1)+B=B \end{matrix}$$

$\Rightarrow A=1,\ B=0\Rightarrow z=\ln(w)\Rightarrow w=f(z)=e^z$