Chapter 13 Calculus of Variations and Lagrange's Multiplier Method

13-1 Euler Equation and Calculus of Variation

How to find the extrema of $I=\int_a^b f(x,y,y')dx$ under $y(a)=y_1$ and $y(b)=y_2$?

Assume $y(x)=u(x)+\varepsilon\eta(x)$, $\eta(a)=\eta(b)=0$, and

$$I(\varepsilon)=\int_a^b f(x,u(x)+\varepsilon\eta(x),u'(x)+\varepsilon\eta'(x))dx$$

Necessary condition for the extrema of $I(\varepsilon)$:

$$\delta I(\varepsilon)=0 \Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon}\cdot\delta\varepsilon=0 \Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon}=0$$

$$\Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon}=\int_a^b\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial\varepsilon}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\varepsilon}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial\varepsilon}\right]dx=\int_a^b\left[\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)\right]dx$$

$$=\int_a^b\frac{\partial f}{\partial y}\eta(x)dx+\frac{\partial f}{\partial y'}\eta(x)\Big|_a^b-\int_a^b\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx=\int_a^b\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx=0$$

$\Rightarrow$ Euler equation for $I=\int_a^b f(x,y,y')dx$ is $\dfrac{\partial f(x,y,y')}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial f(x,y,y')}{\partial y'}\right)=0$.

Note: The function $y(x)$ which maximizes or minimizes $I=\int_a^b f(x,y,y')dx$ is called the stationary solution.

Eg. Find the stationary solution $y(x)$ to maximize or minimize a functional $I=\int_1^3[x(y')^2+y]dx$, $y(1)=3$ and $y(3)=4$.

(Sol.) $f(x,y,y')=x(y')^2+y$, $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\dfrac{\partial f}{\partial y'}=1-\dfrac{d}{dx}(2xy')=1-2y'-2xy''=0$

$\Rightarrow x^2y''+xy'=\dfrac{x}{2}$ (Euler's differential equation) $\Rightarrow y=\dfrac{x}{2}+c\ln|x|+d$

$y(1)=3=\dfrac{1}{2}+d$, $y(3)=4=\dfrac{3}{2}+c\ell n(3)+d \Rightarrow c=0$, $d=\dfrac{5}{2}$, $\therefore$ $y(x)=\dfrac{x}{2}+\dfrac{5}{2}$

Eg. Find the stationary solution $y(x)$ of $I=\int_0^1[(y')^2-x^2y]dx$, $y(0)=-2$ and $y(1)=3$.

(Sol.) $f(x,y,y')=(y')^2-x^2y$, $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\dfrac{\partial f}{\partial y'}=-x^2-\dfrac{d}{dx}(2y')=-x^2-2y''=0$

$\Rightarrow y=-\dfrac{1}{24}x^4+c_1x+c_2$, $y(0)=-2=c_2$, $y(1)=3=-\dfrac{1}{24}+c_1-2$, $c_1=\dfrac{121}{24}$,

$\therefore$ $y(x)=-\dfrac{1}{24}x^4+\dfrac{121}{24}x-2$

13-2 Ritz Method

For $I=\int_a^b f(x,y,y')dx$, the stationary solution $y(x)$ is approximate to $y(x)\approx c_0\phi_0(x)+c_1\phi_1(x)+c_2\phi_2(x)+\cdots\cdots+c_n\phi_n(x)$ and fulfills $\delta I=0$.

Eg. Apply the Ritz method to solve $I=\int_0^1[(y')^2+xy]dx$, $y(0)=0$ and $y(1)=1$.

(Sol.) Let $y(x)=\sum_{n=0}^{\infty}c_n\phi_n(x)=c_0+c_1x+c_2x^2+c_3x^3+\cdots$, $\because$ $y(0)=0$, $\therefore$ $c_0=0$

It is found that $y=c_1x$ or $y=c_1x+c_2x^2$ are not exact solutions.

Consider $y(x)=c_1x+c_2x^2+c_3x^3$, $y(1)=1\Rightarrow c_3=1-c_1-c_2$.

$$y'(x)=c_1+2c_2x+3c_3x^2=c_1+2c_2x+3(1-c_1-c_2)x^2$$

$$[y'(x)]^2=c_1^{\ 2}+4c_2^{\ 2}x^2+9(1-c_1-c_2)^2x^4+4c_1c_2x$$
$$+6c_1(1-c_1-c_2)x^2+12c_2(1-c_1-c_2)x^3$$

$$xy(x)=c_1x^2+c_2x^3+(1-c_1-c_2)x^4$$

$$I=\int_0^1[(y')^2+xy]dx=\int_0^1\{c_1^{\ 2}+4c_1c_2x+[c_1+4c_2^{\ 2}+6c_1(1-c_1-c_2)]x^2$$
$$+[c_2+12c_2(1-c_1-c_2)]x^3+[(1-c_1-c_2)+9(1-c_1-c_2)^2]x^4\}dx$$
$$=c_1^{\ 2}+2c_1c_2+\frac{c_1+4c_2^{\ 2}+6c_1-6c_1^{\ 2}-6c_1c_2}{3}+\frac{c_2+12c_2-12c_1c_2-12c_2^{\ 2}}{4}$$
$$+\frac{(1-c_1-c_2)+9(1+c_1^{\ 2}+c_2^{\ 2}-2c_1-2c_2+2c_1c_2)}{5}$$

$$\delta I=0$$

$$\Rightarrow 2c_1\delta c_1+2c_1\delta c_2+2c_2\delta c_1+\frac{7}{3}\delta c_1+\frac{8}{3}c_2\delta c_2-4c_1\delta c_1-2c_1\delta c_2-2c_2\delta c_1$$
$$+\frac{13}{4}\delta c_2-3c_1\delta c_2-3c_2\delta c_1-6c_2\delta c_2-\frac{\delta c_1}{5}-\frac{\delta c_2}{5}+\frac{18}{5}c_1\delta c_1+\frac{18}{5}c_2\delta c_2$$
$$-\frac{18}{5}\delta c_1-\frac{18}{5}\delta c_2+\frac{18}{5}c_1\delta c_2+\frac{18}{5}c_2\delta c_1=0$$

$$\Rightarrow\begin{cases}\delta c_1\cdot\left(2c_1+2c_2+\frac{7}{3}-4c_1-2c_2-3c_2-\frac{1}{5}+\frac{18}{5}c_1-\frac{18}{5}+\frac{18}{5}c_2\right)=0\\ \delta c_2\cdot\left(2c_1+\frac{8}{3}c_2-2c_1+\frac{13}{4}-3c_1-6c_2-\frac{1}{5}+\frac{18}{5}c_2-\frac{18}{5}+\frac{18}{5}c_1\right)=0\end{cases}$$

$$\Rightarrow\begin{cases}c_1=\frac{11}{12}\\ c_2=0\end{cases},\quad c_3=1-c_1-c_2=\frac{1}{12}\Rightarrow y(x)=\frac{x^3}{12}+\frac{11x}{12}$$

If we set $y(x)=c_1x+c_2x^2+c_3x^3+c_4x^4$, we also obtain $c_1=\frac{11}{12}$, $c_2=0=c_4$, $c_3=\frac{1}{12}$. $\therefore$

It has been convergent to an exact solution!

Eg. Apply the Ritz method to solve $I=\int_0^1 [(y')^2 + xy]dx$, $y(0)=0$ and $y(1)=1$.

(Sol.) Let $y(x) \approx c_0 x + c_1 x(x-1) + \cdots\cdots + c_n x^n (x-1)$

$\because y(1)=1$, $\therefore \ c_0 = 1$

1. Assume $n=1$, $y(x) \approx x + c_1 x(x-1)$, $y'(x) = 2c_1 x - c_1 + 1$

$$\delta I = \delta\int_0^1 [(y')^2 + xy]dx = \delta\int_0^1 [(1+2c_1 x - c_1)^2 + x^2 + c_1 x^3 - c_1 x^2]dx$$

$$= \delta\left[1 + \frac{1}{3}c_1^{\ 2} + \frac{1}{3} - \frac{1}{12}c_1\right] = \frac{2}{3}c_1 \delta c_1 - \frac{1}{12}\delta c_1$$

$$= \left(\frac{2}{3}c_1 - \frac{1}{12}\right)\cdot \delta c_1 = 0 \ \forall \delta c_1 \neq 0$$

$$\Rightarrow c_1 = \frac{1}{8}, \ \therefore \ \ y(x) \approx x + \frac{1}{8}x(x-1)$$

2. Assume $n=2$, $y(x) \approx x + c_1 x(x-1) + c_2 x^2 (x-1)$

$$\Rightarrow c_1 = \frac{1}{12}, \ \ c_2 = \frac{1}{12}$$

$$\Rightarrow y(x) = x + \frac{1}{12}[x(x-1) + x^2(x-1)] = \frac{11}{12}x + \frac{1}{12}x^3$$

3. Assume $n=3$, $y(x) \approx x + c_1 x(x-1) + c_2 x^2 (x-1) + c_3 x^3 (x-1)$

$$\Rightarrow c_1 = \frac{1}{12}, \ \ c_2 = \frac{1}{12}, \ \ c_3 = 0$$

$$\Rightarrow y(x) = x + \frac{1}{12}[x(x-1) + x^2(x-1)] = \frac{11}{12}x + \frac{1}{12}x^3$$

$\therefore \ \ y(x) = \frac{11}{12}x + \frac{1}{12}x^3$ is an exact solution!

13-3 Relationship between Differential Equations and Calculus of Variations

For $y''+P(x)y'+Q(x)y=F(x)$, set $p(x)=e^{\int P(x)dx}$, $q(x)=-Q(x)\,e^{\int P(x)dx}=-Q(x)p(x)$, and $f(x)=-F(x)\,e^{\int P(x)dx}=-F(x)p(x)$. It is equivalent to the Sturm-Liouville form $-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=f(x)$. And then it can be transformed into $\delta I=0$, where $I=\int_0^1\{p(x)[y'(x)]^2+q(x)y^2(x)-2f(x)y(x)\}dx$.

Eg. Transform $xy''+y'=1/2$ into $\delta I=0$.

(Sol.) $y''+\frac{y'}{x}+0y=\frac{1}{2x}$, $p(x)=\exp[\int\frac{1}{x}dx]=x$, $q(x)=-0\cdot x=0$, and $f(x)=-\frac{1}{2x}\cdot x=-1/2$,

$\therefore$ $xy''+y'=1/2$ is equivalent to $-\frac{d}{dx}\left[x\frac{dy}{dx}\right]=-\frac{1}{2}$, so it can be transformed into

$\delta I=\delta\int_0^1[x(y')^2+y]dx=0$.

Eg. Transform $y''-y=x/2$ into $\delta I=0$.

(Sol.) $y''+0y'-y=x/2$, $p(x)=\exp[\int 0dx]=1$, $q(x)=-(-1)\cdot 1=1$, and $f(x)=-x/2\cdot 1=-x/2$,

$\therefore$ $y''-y=x/2$ is equivalent to $-\frac{d}{dx}\left[\frac{dy}{dx}\right]+y=-\frac{x}{2}$, so it can be transformed into

$\delta I=\delta\int_0^1[(y')^2+y^2+xy]dx=0$.

Eg. Transform $y''=-x^2/2$ into $\delta I=0$.

(Sol.) $y''+0y'+0y=-x^2/2$, $p(x)=\exp[\int 0dx]=1$, $q(x)=-0\cdot 1=0$, and $f(x)=-(-\frac{x^2}{2})\cdot 1=\frac{x^2}{2}$,

$\therefore$ $y''=-x^2/2$ is equivalent to $-\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{x^2}{2}$, so it can be transformed into

$\delta I=\int_0^1[(y')^2-x^2y]dx=0$.

13-4 Lagrange's Multiplier Method

To find the relative extrema of $F(x_1,x_2,\ldots,x_n)$ subject to k conditions $\varphi_1(x_1,x_2,\ldots,x_n)=0$, $\varphi_2(x_1,x_2,\ldots,x_n)=0$, $\varphi_3(x_1,x_2,\ldots,x_n)=0$, $\varphi_4(x_1,x_2,\ldots,x_n)=0$, …, $\varphi_k(x_1,x_2,\ldots,x_n)=0$, we have $G(x_1,x_2,\ldots,x_n;\lambda_1,\lambda_2,\ldots,\lambda_k)=F(x_1,x_2,\ldots,x_n)+\lambda_1\varphi_1(x_1,x_2,\ldots,x_n)+\ldots+\lambda_k\varphi_k(x_1,x_2,\ldots,x_n)$

subject to the necessary conditions:
$$\begin{cases}\dfrac{\partial G}{\partial x_1}=0\\ \dfrac{\partial G}{\partial x_2}=0\\ \vdots\\ \dfrac{\partial G}{\partial x_n}=0\end{cases}\Rightarrow\begin{cases}x_1=x_1(\lambda_1,\lambda_2,\cdots,\lambda_k)\\ x_2=x_2(\lambda_1,\lambda_2,\cdots,\lambda_k)\\ \vdots\\ x_n=x_n(\lambda_1,\lambda_2,\cdots,\lambda_k)\end{cases}\Rightarrow\begin{cases}\lambda_1=?\\ \lambda_2=?\\ \vdots\\ \lambda_k=?\end{cases}$$

$\Rightarrow F(x_1,x_2,\ldots,x_n)$=Max or Min.

Eg. Find the extrema of $F(x,y,z)=x^2+y^2+z^2$ subject to the conditions $x^2/4+y^2/5+z^2/25=1$ and $z=x+y$.

(Sol.) $\varphi_1(x,y,z)=x^2/4+y^2/5+z^2/25-1=0$, $\varphi_2(x,y,z)=x+y-z=0$

$$G(x,y,z;\lambda_1,\lambda_2)=x^2+y^2+z^2+\lambda_1\left(x^2/4+y^2/5+z^2/25-1\right)+\lambda_2(x+y-z)$$

$$\begin{cases}\dfrac{\partial G}{\partial x}=2x+\dfrac{\lambda_1 x}{2}+\lambda_2=0\\ \dfrac{\partial G}{\partial y}=2y+\dfrac{2\lambda_1 y}{5}+\lambda_2=0\\ \dfrac{\partial G}{\partial z}=2z+\dfrac{2\lambda_1 z}{25}-\lambda_2=0\end{cases}\Rightarrow\begin{cases}x=\dfrac{-2\lambda_2}{\lambda_1+4}\\ y=\dfrac{-5\lambda_2}{2\lambda_1+10}\\ z=\dfrac{25\lambda_2}{2\lambda_1+50}\end{cases}$$

$$\phi_2(x,y,z)=x+y-z=0\Rightarrow\lambda_1=-10\quad\text{or}\quad-\frac{75}{17}$$

1. $\lambda_1=-10\Rightarrow x=\dfrac{\lambda_2}{3},\quad y=\dfrac{\lambda_2}{2},\quad z=\dfrac{5}{6}\lambda_2$

$$\phi_1(x,y,z)=x^2/4+y^2/5+z^2/25-1=0\Rightarrow\lambda_2=\pm 6\sqrt{\frac{5}{19}}$$

$$\Rightarrow(x,y,z)=\left(\pm 2\sqrt{\frac{5}{19}},\pm 3\sqrt{\frac{5}{19}},\pm 5\sqrt{\frac{5}{19}}\right)\Rightarrow F(x,y,z)=x^2+y^2+z^2=10$$

2. $\lambda_1=-\dfrac{75}{17}\Rightarrow x=\dfrac{34}{7}\lambda_2,\quad y=-\dfrac{17}{4}\lambda_2,\quad z=\dfrac{17}{28}\lambda_2\Rightarrow\lambda_2=\pm\dfrac{140}{17\sqrt{646}}$

$$\Rightarrow(x,y,z)=\left(\pm\frac{40}{\sqrt{646}},\mp\frac{35}{\sqrt{646}},\pm\frac{5}{\sqrt{646}}\right)\Rightarrow F(x,y,z)=\frac{75}{17}$$

∴ Max=10, Min=75/17

Eg. For a simple lens of focal length *f*, the object distance *p* and the image distance *q* are related by 1/*p*+1/*q*=1/*f*. Find the minimum object-image distance *p*+*q*. [中央光電所]

(Sol.) Let $G(p,q,\lambda)=p+q+\lambda(1/p+1/q-1/f)$

$$\begin{cases} \dfrac{\partial G}{\partial p}=0 \Rightarrow 1-\dfrac{\lambda}{p^2}=0 \Rightarrow p=\pm\sqrt{\lambda} \\ \dfrac{\partial G}{\partial q}=0 \Rightarrow 1-\dfrac{\lambda}{q^2}=0 \Rightarrow q=\pm\sqrt{\lambda} \end{cases}$$, $1/p+1/q=2/\sqrt{\lambda}=1/f \Rightarrow \lambda=4f^2 \Rightarrow p=q=2f \Rightarrow p+q=4f$

Eg. Apply Lagrange's multiplier method to find the extrema of $F(x,y)=x^2+y^2$ under the constraint $x^2/4+y^2/9=1$.

(Sol.) Let $G(x,y;\lambda)=x^2+y^2+\lambda\left(\dfrac{x^2}{4}+\dfrac{y^2}{9}-1\right)$

$$\begin{cases} \dfrac{\partial G}{\partial x}=0 \Rightarrow 2x+\dfrac{\lambda}{2}x=0 \Rightarrow x=0 \ or \ \lambda=-4 \\ \dfrac{\partial G}{\partial y}=0 \Rightarrow 2y+\dfrac{2\lambda}{9}y=0 \Rightarrow y=0 \ or \ \lambda=-9 \end{cases}$$

1. $x=0 \wedge \dfrac{x^2}{4}+\dfrac{y^2}{9}=1 \Rightarrow y=\pm 3$

   $\Rightarrow x^2+y^2=0+(\pm 3)^2=9$

2. $y=0 \wedge \dfrac{x^2}{4}+\dfrac{y^2}{9}=1 \Rightarrow x=\pm 2$

   $\Rightarrow x^2+y^2=(\pm 2)^2+0^2=4$, ∴ Max=9, Min=4

Another method:

1. $x=0 \wedge \lambda=-9$

   $G(x=0,y;\lambda=-9)=0^2+y^2-9\left(\dfrac{0^2}{4}+\dfrac{y^2}{9}-1\right)=9$

2. $y=0 \wedge \lambda=-4$

   $G(x,y=0;\lambda=-4)=x^2+0^2-4\left(\dfrac{x^2}{4}+\dfrac{0^2}{9}-1\right)=4$, ∴ Max=9, Min=4