Chapter 2 The Second-Order Ordinary Differential Equations

2-1 Introduction

Initial-value problem

Eg. $y''-2y=x^2-1,\ y(1)=3,\ y'(1)=-5.$

Boundary-value problem

Eg. $y''+y=0,\ y(0)=y(\pi)=0.$

Theorem Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of $y''+P(x)y'+Q(x)y=0$, then $y(x)=c_1y_1(x)+c_2y_2(x)$ is its general solution.

2-2 The 2-order Linear Constant-coefficient Ordinary Differential Equation $y''+Ay'+By=F(x)$

Homogeneous equation: $y''+Ay'+By=0$

Let $y=e^{rx}$, $r^2+Ar+B=0 \Rightarrow r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}$

Case 1 $A^2-4B>0 \Rightarrow r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}=r_1,\ r_2,\ r_1\neq r_2,\ \therefore\ y(x)=c_1e^{r_1x}+c_2e^{r_2x}$

Eg. Solve $y''+4y'-2y=0.$

(Sol.) $r^2+4r-2=0,\ r=-2\pm\sqrt{6},\ \therefore\ y(x)=c_1e^{(-2+\sqrt{6})x}+c_2e^{(-2-\sqrt{6})x}$

Case 2 $A^2-4B=0 \Rightarrow r=\dfrac{-A}{2},\ y_1(x)=e^{-\frac{Ax}{2}},\ y_2=u(x)e^{-\frac{Ax}{2}}$

$$y_2''(x)+Ay_2'(x)+By_2(x)=0 \Rightarrow u(x)=c_1x+c_2$$

Choose $c_1=1,\ c_2=0,\ y_2(x)=xe^{-\frac{Ax}{2}},\ \therefore\ y(x)=c_1e^{-\frac{Ax}{2}}+c_2xe^{-\frac{Ax}{2}}$

Eg. Solve $y''+4y'+4y=0.$

(Sol.) $r^2+4r+4=0$, $r=-2,\ -2$, $\therefore\ y(x)=c_1e^{-2x}+c_2xe^{-2x}$

Case 3 $A^2-4B<0 \Rightarrow r=\dfrac{-A\pm i\sqrt{4B-A^2}}{2}=p\pm iq$

$$y(x)=c_1e^{(p+iq)x}+c_2e^{(p-iq)x}=d_1e^{px}\cdot\cos(qx)+d_2e^{px}\cdot\sin(qx)$$

Eg. Solve $y''+9y=0.$

(Sol.) $r^2+9=0$, $r=\pm i3,\ \therefore\ y(x)=c_1\cos(3x)+c_2\sin(3x)$

Eg. Solve $y''+2y'+6y=0.$

(Sol.) $r^2+2r+6=0$, $r=-1\pm i\sqrt{5},\ \therefore\ y(x)=c_1e^{-x}\cos(\sqrt{5}x)+c_2e^{-x}\sin(\sqrt{5}x)$

Eg. Two students solve $y''+ay'+by=0$, $y(0)=A$ and $y'(0)=B$. Using wrong constants for b and B, one student obtain the solution $y_A=e^{-2x}(\cos 3x+2\sin 3x)$. Using wrong constants for a and A, one student obtain the solution $y_B=-3e^x+2e^{3x}$. Find the correct constants for a, b, A, and B and solve the initial value problem. [台大電研]

(Sol.) $y_A=e^{-2x}(\cos 3x+2\sin 3x)\Rightarrow r=-2\pm i3\Rightarrow r^2+4r+13=0=r^2+ar+b$,

$\because$ b is wrong but a is correct, $\therefore$ $a=4$

$\because$ $y(0)=A$ is correct, $\therefore$ $y_A(0)=e^0(\cos 0+2\sin 0)=1=A$

$y_B=-3e^x+2e^{3x}\Rightarrow r=1, 3\Rightarrow r^2-4r+3=0=r^2+ar+b$, $\because$ a is wrong but b is correct, $\therefore$ $b=3$

$y_B'(x)=-3e^x+6e^{3x}$, $\because$ $y'(0)=B$ is correct, $\therefore$ $y_B'(0)=-3+6=3=B$

The correct $r^2+ar+b=r^2+4r+3=0\Rightarrow r=-1, -3\Rightarrow$ The correct $y(x)=ce^{-x}+de^{-3x}$

$y(0)=A=1\Rightarrow c+d=1$ and $y'(0)=B=3\Rightarrow -c-3d=3$, we obtain $c=3$, $d=-2$

$$\Rightarrow y(x)=3e^{-x}-2e^{-3x}$$

Non-homogeneous equation: $y''+Ay'+By=F(x)$

1. Find the homogeneous solution y_h of $y''+Ay'+By=0$,
2. Find a particular solution y_p of $y''+Ay'+By=F(x)$,
3. General solution is y_h+y_p.

Eg. Solve $y''-4y=8x^2-2x$.

(Sol.) $r^2-4=0$, $r=2,-2$, $\therefore$ $y_h=c_1e^{-2x}+c_2e^{2x}$,

$y_p=ax^2+bx+c$, $y_p'=2ax+b$, $y_p''=2a$, $\therefore$ $y_p''-4y_p=-4ax^2-4bx+2a-4c=8x^2-2x$

$\Rightarrow a=-2, b=\frac{1}{2}, c=-1$, $\therefore$ $y(x)=c_1e^{-2x}+c_2e^{2x}-2x^2+\frac{1}{2}x-1$

Eg. Solve $y''+2y'-3y=4e^{2x}$.

(Sol.) $r^2+2r-3=0$, $r=1,-3$, $\therefore$ $y_h=c_1e^x+c_2e^{-3x}$,

$y_p=Ae^{2x}$, $y_p'=2Ae^{2x}$, $y_p''=4Ae^{2x}$, $\therefore$ $y_p''+2y_p'-3y_p=4e^{2x}\Rightarrow A=\frac{4}{5}$,

$\therefore$ $y=c_1e^x+c_2e^{-3x}+\frac{4}{5}e^{2x}$

Eg. Solve $y''+2y'-3y=4e^x$.

(Sol.) $r^2+2r-3=0$, $r=1,-3\Rightarrow y_h=c_1e^x+c_2e^{-3x}$, $\therefore$ $y_p\neq Ae^x$,

Try $y_p=Axe^x$, $y_p'=Ae^x+Axe^x$, $y_p''=2Ae^x+Axe^x$,

$\therefore$ $y_p''+2y_p'-3y_p=4e^x\Rightarrow A=1$, $\therefore$ $y=c_1e^x+c_2e^{-3x}+xe^x$

Eg. Solve $y''+4y=\cos(x)$.

(Sol.) $r^2+4=0,\ r=\pm 2i$, $\therefore$ $y_h=c_1\cos(2x)+c_2\sin(2x)$,

and $y_p=A\cos(x)+B\sin(x)$, $y_p'=-A\sin(x)+B\cos(x)$, $y_p''=-A\cos(x)-B\sin(x)$,

$y''+4y=-A\cos(x)-B\sin(x)+4A\cos(x)+4B\sin(x)=3A\cos(x)+3B\sin(x)=\cos(x)\Rightarrow A=\frac{1}{3}$, $B=0$

$$\therefore\ y=c_1\cos(2x)+c_2\sin(2x)+\frac{1}{3}\cos(x)$$

Eg. Solve $y''+4y=\cos(2x)$.

(Sol.) $r^2+4=0,\ r=\pm 2i$, $\therefore$ $y_h=c_1\cos(2x)+c_2\sin(2x)$ but $y_p\neq A\cos(2x)+B\sin(2x)$

Try $y_p=Ax\cos(2x)+Bx\sin(2x)$, $y_p'=A\cos(2x)-2Ax\sin(2x)+B\sin(2x)+2Bx\cos(2x)$,

$y_p''=-2A\sin(2x)-2A\sin(2x)-4Ax\cos(2x)+2B\cos(2x)+2B\cos(2x)-4Bx\sin(2x)$,

$$y_p''+4y_p=\cos(2x)\Rightarrow A=0,\ B=\frac{1}{4},\ \therefore\ y=c_1\cos(2x)+c_2\sin(2x)+\frac{x}{4}\sin(2x)$$

Eg. For $y''-3y'+7y=x-\cos(2x)$, find y_p.

(Sol.) $y_p=ax+b+h\cos(2x)+k\sin(2x)$, $y_p'=a-2h\sin(2x)+2k\cos(2x)$,

$y_p''=-4h\cos(2x)-4k\sin(2x)$, $y_p''-3y_p'+7y_p=x-\cos(2x)$

$$\Rightarrow a=\frac{1}{7},\ b=\frac{3}{49},\ h=-\frac{1}{15},\ k=\frac{2}{15}\Rightarrow y_p=\frac{1}{7}x+\frac{3}{49}-\frac{1}{15}\cos(2x)+\frac{2}{15}\sin(2x)$$

Eg. Solve $y''+2y'+y=\sum_{n=1}^{10}\frac{1}{n\pi}\sin(nt)$. [台大電研]

(Sol.) $r^2+2r+1=0,\ r=-1,-1,\ y_h=c_1e^{-t}+c_2te^{-t}$

$$\text{Set}\ \ y_p=\sum_{n=1}^{10}\left[a_n\cos(nt)+b_n\sin(nt)\right],\ \ y_p'=\sum_{n=1}^{10}\left[-na_n\sin(nt)+nb_n\cos(nt)\right],$$

$$y_p''=\sum_{n=1}^{10}\left[-n^2a_n\cos(nt)-n^2b_n\sin(nt)\right]$$

$$\Rightarrow\sum_{n=1}^{10}\left[\left(-n^2a_n+2nb_n+a_n\right)\cdot\cos(nt)+\left(-n^2b_n-2na_n+b_n\right)\cdot\sin(nt)\right]=\sum_{n=1}^{10}\frac{\sin(nt)}{n\pi}$$

$$\Rightarrow\begin{cases}-n^2a_n+2nb_n+a_n=0\\ -n^2b_n-2na_n+b_n=\dfrac{1}{n\pi}\end{cases}\Rightarrow\begin{cases}a_n=\dfrac{-2}{\pi\left(n^2+1\right)^2}\\ b_n=\dfrac{1-n^2}{n\pi\left(n^2+1\right)^2}\end{cases}$$

$$\Rightarrow y(t)=y_h+y_p=c_1e^{-t}+c_2te^{-t}+\sum_{n=1}^{10}\frac{\left(1-n^2\right)\sin(nt)-2n\cos(nt)}{n\pi\left(n^2+1\right)^2}$$

Eg. Solve $y''-6y'+9y=6x^2+2-12e^{3x}$. [台大電研]

Eg. Solve $y''-2y'+y=x^2e^x$. [文化電機轉學考、台大電研]

Eg. Solve $y''-3y'+2y=x+e^{2x}$ and $y''+3y'+2y=x^2(e^x+e^{-x})$ [交大電研]

Eg. Solve $y''+4y=x^2\cos(2x)$ [交大電子研究所]

Variation of parameters to find the particular solution y_p:

Let y_1 and y_2 be linearly independent solutions of $y''+Ay'+By=0$, then a particular solution y_p is $y_p=u(x)y_1(x)+v(x)y_2(x)$, and

$$y_p'=u'y_1+v'y_2+uy_1'+vy_2'=uy_1'+vy_2'$$

Impose the condition: $u'y_1+v'y_2=0\cdots\cdots(1)$

$$y_p''=u'y_1'+v'y_2'+uy_1''+vy_2''$$

$$\Rightarrow u'y_1'+v'y_2'+uy_1''+vy_2''+A\left(uy_1'+vy_2'\right)+B(uy_1+vy_2)=F(x)$$

$$\Rightarrow u\left[y_1''+Ay_1'+By_1\right]+v\left[y_2''+Ay_2'+by_2\right]+u'y_1'+v'y_2'=F(x)$$

$$u'y_1'+v'y_2'=F(x)\cdots\cdots(2)$$

$$(1),\ (2)\Rightarrow u'=\frac{-y_2F(x)}{y_1y_2'-y_2y_1'},\quad v'=\frac{y_1F(x)}{y_1y_2'-y_2y_1'}$$

Wronskian determinant: $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$

$$u(x)=\int\frac{-y_2F(x)}{W(y_1,y_2)}dx,\quad v(x)=\int\frac{y_1F(x)}{W(y_1,y_2)}dx\Rightarrow y_p=u(x)y_1(x)+v(x)y_2(x)$$

Eg. Solve $y''+4y=\tan(2x)$.

(Sol.) $y''+4y=0\Rightarrow y_1=\cos(2x)$, $y_2=\sin(2x)$

$$W(y_1,y_2)=y_1y_2'-y_2y_1'=2\cos^2(2x)-\sin(2x)[-2\sin(2x)]=2$$

$$u(x)=\int\frac{-\sin(2x)\cdot\tan(2x)}{2}dx=\frac{1}{4}\sin(2x)-\frac{1}{4}\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|$$

$$v(x)=\int\frac{\cos(2x)\cdot\tan(2x)}{2}dx=-\frac{1}{4}\cos(2x)$$

$$y_p=u(x)y_1(x)+v(x)y_2(x)$$

$$=\frac{1}{4}\sin(2x)\cos(2x)-\frac{\cos(2x)}{4}\cdot\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|-\frac{1}{4}\cos(2x)\sin(2x)$$

$$=-\frac{1}{4}\cos(2x)\cdot\ell n\left|\tan\left(\frac{\pi}{4}+x\right)\right|$$

Eg. Solve $4y''+36y=\csc(3x)$.

(Ans.) $y=a\cos(3x)+b\sin(3x)-\dfrac{x\cos(3x)}{12}+\dfrac{\sin(3x)\cdot\ln[|\sin(3x)|]}{36}$

Eg. Solve $y''+4y=\sec(2x)$. [中原電子所]

Eg. Find y_p of $y''-\frac{4}{x}y'+\frac{4}{x^2}y=x^2+1$. **[清大電研]**

(Sol.) $y_1=x,\ \ y_2=x^4,\ \ W(y_1,y_2)=x\cdot 4x^3-x^4\cdot 1=3x^4,$

$$u(x)=\int\frac{-x^4(x^2+1)}{3x^4}dx=-\frac{x^3}{9}-\frac{x}{3},\quad v(x)=\int\frac{x(x^2+1)}{3x^4}dx=\frac{1}{3}\ell n\,|\,x\,|-\frac{1}{6x^2}$$

$$y_p=u(x)y_1+v(x)y_2=-\frac{x^4}{9}-\frac{x^2}{2}+\frac{x^4}{3}\ell n\,|\,x\,|$$

$\therefore\ y(x)=y_h+y_p=cx+dx^4-\frac{x^4}{9}-\frac{x^2}{2}+\frac{x^4}{3}\ell n\,|\,x\,|=c_1x+c_2x^4-\frac{x^2}{2}+\frac{x^4}{3}\ell n\,|\,x\,|$

2-3 Euler Equation $x^2y''+Axy'+By=F(x)$

Solution: $z=\ln(x)$, $y'=\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=\dfrac{1}{x}\dfrac{dy}{dz}$,

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}\text{, and }\ x^2y''=-\frac{dy}{dz}+\frac{d^2y}{dz^2},\quad xy'=\frac{dy}{dz}$$

$$\Rightarrow \frac{d^2y}{dz^2}+(A-1)\frac{dy}{dz}+By=F(e^z)$$ is the second-order linear ODE.

Eg. Solve $x^2y''-5xy'+8y=2\ln(x)+x^3$.

(Sol.) $z=\ell n(x)$, $y''=-\dfrac{1}{x^2}\dfrac{dy}{dz}+\dfrac{1}{x^2}\dfrac{d^2y}{dz^2}$, $y'=\dfrac{1}{x}\dfrac{dy}{dz}$, $\dfrac{d^2y}{dz^2}-6\dfrac{dy}{dz}+8y=2z+e^{3z}$

$y_h(z)=c_1e^{2z}+c_2e^{4z}=c_1x^2+c_2x^4$, $y_p(z)=\dfrac{1}{4}z+\dfrac{3}{16}-e^{3z}=\dfrac{1}{4}\ell n(x)+\dfrac{3}{16}-x^3$

$$y(x)=c_1x^2+c_2x^4+\frac{1}{4}\ell n(x)+\frac{3}{16}-x^3$$

Eg. Solve $x^2y''-2y=1/x$. [文化電機轉學考]

(Sol.) $z=\ell n(x)$, $\dfrac{d^2y}{dz^2}-\dfrac{dy}{dz}-2y=e^{-z}$, $y_h(z)=ce^{2z}+de^{-z}=cx^2+\dfrac{d}{x}$,

$y_p(z)=\dfrac{-ze^{-z}}{3}=\dfrac{-\ln(x)}{3x}\Rightarrow y(x)=cx^2+\dfrac{d}{x}-\dfrac{\ln(x)}{3x}$

Eg. Solve (a) $x^2y''-4xy'+4y=0$ and (b) $x^2y''+5xy'+4y=0$.

(Sol.) (a) Let $y=x^r$, $y'=rx^{r-1}$, $y''=(r^2-r)x^{r-2}$

$$\Rightarrow x^2y''-4xy'+4y=x^r(r^2-r-4r+4)=0$$

$$r^2-5r+4=0,\quad r=1,\ 4\Rightarrow y=c_1x+c_2x^4$$

(b) $y=x^r\Rightarrow r^2+4r+4=0$, $r=-2$, -2

$$\Rightarrow y_1=x^{-2},\quad y_2=x^{-2}\ell n(x)\Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ell n(x)$$

Another method: $z=\ln(x)$, $y'=\dfrac{1}{x}\dfrac{dy}{dz}$, $y''=-\dfrac{1}{x^2}\dfrac{dy}{dz}+\dfrac{1}{x^2}\dfrac{d^2y}{dz^2}$

$$\Rightarrow \frac{d^2y}{dz^2}+4\frac{dy}{dz}+4y=0\Rightarrow y_1(z)=e^{-2z},\quad y_2(z)=ze^{-2z}$$

$$\Rightarrow y_1(x)=x^{-2},\quad y_2(x)=x^{-2}\ell n(x)\Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ell n(x)$$

Eg. Solve $(x\text{-}2)^2y''+4(x\text{-}2)y'+6y=0$. [文化電機轉學考]

(Sol.) $z=\ln(x\text{-}2)$, $y'=\frac{1}{(x-2)}\frac{dy}{dz}$, $y''=\frac{-1}{(x-2)^2}\frac{dy}{dz}+\frac{1}{(x-2)^2}\cdot\frac{d^2y}{dz^2}$

$$\frac{d^2y}{dz^2}+3\frac{dy}{dz}+6y=0,\quad y_1(z)=e^{-\frac{3}{2}z}\cdot\cos\left(\frac{\sqrt{15}}{2}z\right),\quad y_2(z)=e^{-\frac{3z}{2}}\cdot\sin\left(\frac{\sqrt{15}}{2}z\right)$$

$$\Rightarrow y_1(x)=(x-2)^{-\frac{3}{2}}\cdot\cos\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right],\ y_2(x)=(x-2)^{-\frac{3}{2}}\cdot\sin\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right]$$

$$\therefore y(x)=c_1(x-2)^{-\frac{3}{2}}\cdot\cos\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right]+c_2(x-2)^{-\frac{3}{2}}\cdot\sin\left[\frac{\sqrt{15}}{2}\ell n(x-2)\right]$$

Eg. Solve (a) $x^2y''-xy'+y=\ln(x)$ and (b) $x^2y''-4xy'+4y=x^4+x^2$. [交大電子所]

(Ans.) (a) $y(x)=c_1x+c_2x\ell nx+\ell nx+2$, (b) $y(x)=c_1x+c_2x^4+\frac{1}{3}x^4\ell nx-\frac{1}{2}x^2$

Eg. Solve $xy''+4y'=\frac{\ln x^3}{x}$. **[台大電研]** (Ans.) $y(x)=c_1+c_2x^{-3}+\frac{(\ln x)^2}{2}-\frac{\ln x}{3}$

2-4 Miscellaneous Problems

Eg. Solve $y'=y^2-2xy+x^2+1$ [交大控制研究所]

(Sol.) Let $u=y-x$, $y'=u'+1=u^2+1$, $\frac{du}{u^2}=dx$, $-1/u=x+c\Rightarrow\frac{-1}{y-x}=x+c$.

Eg. Solve $1+x^2y^2+y+xy'=0$. [交大電信研究所]

(Sol.) Let $u=xy$, $u'=y+xy'$, $1+u^2+u'=0$, $\frac{du}{1+u^2}=-dx$, $\tan^{-1}(u)=-x+c\Rightarrow\tan^{-1}(xy)=-x+c$.

Eg. Solve $x\cdot\frac{dy}{dx}-y=\frac{x^3}{y}\cdot\exp(\frac{y}{x})$. **[交大電子研究所]**

(Sol.) $\frac{dy}{dx}-\frac{y}{x}=\frac{x^2}{y}\cdot\exp(\frac{y}{x})$.

Let $u=\frac{y}{x}$, $y=ux$, $\frac{dy}{dx}=u+x\frac{du}{dx}$, $u+x\frac{du}{dx}-u=\frac{x^2}{ux}\exp(u)$, $\frac{du}{dx}=\frac{1}{u}\exp(u)$,

$ue^{-u}du=dx$, $-ue^{-u}-e^{-u}=x+c$, $[-(\frac{y}{x})-1]\cdot\exp(\frac{-y}{x})=x+c$.

Eg. Solve *xy"+2y'=4x*^{3}.

(Sol.) Let $u=y' \Rightarrow xu'+2u=4x^3 \Rightarrow u=\frac{4x^3}{5}+\frac{c}{x^2} \Rightarrow y(x)=\int u(x)dx=\frac{x^4}{5}-\frac{c}{x}+D$

Another method: $x^2y''+2xy'=4x^4$ (Euler's equation)

Eg. Solve *y"-2yy'=0.*

(Sol.) Set $u=y'$, $y''=\frac{dy'}{dx}=\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=u\frac{du}{dy} \Rightarrow u\frac{du}{dy}-2yu=0 \Rightarrow u=y^2+c$

$$\Rightarrow \frac{dy}{dx}=y^2+c \Rightarrow \frac{dy}{y^2+c}=dx \Rightarrow \frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{y}{\sqrt{c}}\right)=x+K$$

Eg. Solve $\frac{dy}{dx}=\frac{x}{x^2y+y^3}$.

(Sol.) Let $u=x^2$, $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2x\cdot\frac{dy}{du}=\frac{x}{uy+y^3}$

$$\frac{du}{dy}-2yu=2y^3 \Rightarrow [u\cdot e^{-y^2}]'=2y^3e^{-y^2}$$

$$u=e^{+y^2}\cdot[-y^2e^{-y^2}-e^{-y^2}+c]=-y^2-1+ce^{y^2} \Rightarrow x^2=-y^2-1+ce^{y^2}$$

Another method: $\frac{dx}{dy}=xy+\frac{y^3}{x}$ (Bernoulli's equation)

Given a solution $y_1(x)$ of $y''+P(x)y'+Q(x)y=0$, then a second solution $y_2(x)=v(x)y_1(x)$ is obtained by the following method:

$v''y_1(x)+v'[2y_1'(x)+P(x)y_1]=0$. Set $v'=u \Rightarrow u'+\left[\frac{2y_1'}{y_1}+P(x)\right]u=0$

Eg. Solve $y''+\frac{3}{x}y'+\frac{1}{x^2}y=0$ **if** $y_1=\frac{1}{x}$ **is given.**

(Sol.) Let $y_2=v(x)y_1(x)=\frac{v(x)}{x}$, $y_2'=v'\cdot y_1+v\cdot y_1'$

$$y_2''=(v'\cdot y_1+v\cdot y_1')'=v''\cdot y_1+v'\cdot y_1'+v'\cdot y_1'+vy_1''=v''\cdot y_1+2v'\cdot y_1'+vy_1''$$

$$v''\cdot y_1+2v'\cdot y_1'+vy_1''+\frac{3}{x}(v'\cdot y_1+vy_1')+\frac{vy_1}{x^2}=v''y_1+2v'y_1'+\frac{3}{x}v'y_1=0$$

$$v'=u \Rightarrow u'+\left(\frac{2y_1'}{y_1}+\frac{3}{x}\right)u=u'+\left(\frac{-2\cdot\frac{1}{x^2}}{\frac{1}{x}}+\frac{3}{x}\right)u=u'+\frac{1}{x}u=0$$

$$u=v'=\frac{1}{x} \Rightarrow v(x)=\ell n|x| \Rightarrow y_2(x)=\frac{\ell n|x|}{x}$$

Eg. Given *y(x)=x* is a solution of *y"-xy'+y=0*, find the other solution. [交大電信所]