Chapter 1 The First-Order Ordinary Differential Equations (ODE)

1-1 Separable Differential Equation $A(x)dx=B(y)dy$

Solution: $\int A(x)dx=\int B(y)dy+C$

Eg. Solve (a) $y'=3x^2+1$, $y(1)=4$, (b) $6x-2yy'=0$, and (c) $2\cdot\frac{dy}{dx}-\frac{1}{y}=\frac{2x}{y}$, $y(0)=0$.

(Sol.) (a) $dy=(3x^2+1)dx$, $y=x^3+x+c$, $y(1)=4\Rightarrow c=2$, $\therefore$ $y=x^3+x+2$.

(b) $2ydy=6xdx\Rightarrow y^2=3x^2+c$.

(c) $2ydy=(1+2x)dx$, $y^2+c=x+x^2$, $y(0)=0\Rightarrow c=0$, $\therefore$ $y^2=x+x^2$.

Eg. Solve $y'=xe^{x-y}$ with the boundary condition: $y=\ln 2$ at $x=0$. [2005 台大電研]

(Sol.) $e^y dy=xe^x dx\Rightarrow e^y=xe^x-e^x+c$, $y(x=0)=\ln 2\Rightarrow c=3$, $\therefore$ $e^y=xe^x-e^x+3$.

Eg. Solve $y'=y^2e^{2t}$ with $y(0)=2$. [2010 台大生醫電資所]

(Sol.) $\frac{1}{y^2}dy=e^{2t}dt\Rightarrow \frac{-1}{y}=\frac{e^{2t}}{2}+c$, $y(t=0)=2\Rightarrow c=-1$, $\therefore$ $\frac{-1}{y}=\frac{e^{2t}}{2}-1$.

1-2 The first-order Linear Differential Equation $y'+p(x)y=q(x)$

Solution: $y'\cdot e^{\int p(x)dx}+p(x)y\cdot e^{\int p(x)dx}=q(x)\cdot e^{\int p(x)dx}$

$$\Rightarrow\left[y\cdot e^{\int p(x)dx}\right]'=q(x)\cdot e^{\int p(x)dx}\Rightarrow ye^{\int p(x)dx}=\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c$$

$$\Rightarrow y(x)=e^{-\int p(x)dx}\cdot\left\{\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c\right\}$$

Eg. Solve $xy'+2y=3x^3$.

(Sol.) $y'+\frac{2}{x}y=3x^2$, $p(x)=\frac{2}{x}$, $\int p(x)dx=2\ln(x)$, $e^{2\ln(x)}=x^2\Rightarrow x^2y'+2xy=3x^4$

$\Rightarrow(x^2y)'=3x^4$, $x^2y=\frac{3x^5}{5}+c\Rightarrow y(x)=\frac{3x^3}{5}+\frac{c}{x^2}$

Eg. Solve $y'+y=\sin(x)$.

(Sol.) $p(x)=1$, $\int p(x)dx=x\Rightarrow y'\cdot e^x+e^x\cdot y=e^x\cdot\sin(x)\Rightarrow(e^x\cdot y)'=e^x\cdot\sin(x)$,

$e^x\cdot y=\frac{e^x[\sin(x)-\cos(x)]}{2}+c\Rightarrow y(x)=\frac{\sin(x)-\cos(x)}{2}+c\cdot e^{-x}$

Eg. Solve $y'+y\cot(x)=5e^{\cos(x)}$. [2013 師大應用電子所]

(Sol.) $p(x)=\cot(x)$, $\int p(x)dx=\ln|\sin(x)|$, $e^{\ln|\sin(x)|}=\sin(x)\Rightarrow\sin(x)y'+y\cos(x)=5\sin(x)e^{\cos(x)}$

$\Rightarrow[y\sin(x)]'=5\sin(x)e^{\cos(x)}\Rightarrow y\sin(x)=-5e^{\cos(x)}+c$, $\therefore$ $y(x)=\frac{-5e^{\cos(x)}+c}{\sin(x)}$

1-3 Bernoulli Differential Equations $y'+p(x)y=r(x)y^{\alpha}$ (It is nonlinear if $\alpha\neq 1$)

Solution: Set $z=y^{1-\alpha}$, $y=zy^{\alpha}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{y^{\alpha}}{1-\alpha}\cdot\frac{dz}{dx}$

$$\Rightarrow \frac{y^{\alpha}}{1-\alpha}\cdot\frac{dz}{dx}+p(x)\cdot zy^{\alpha}=r(x)y^{\alpha}\Rightarrow \frac{dz}{dx}+(1-\alpha)p(x)\cdot z=r(x)\cdot(1-\alpha)$$

$$\Rightarrow z(x)=e^{-\int(1-\alpha)p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)\cdot e^{\int(1-\alpha)p(x)dx}dx+c\right]$$

$$\Rightarrow [y(x)]^{1-\alpha}=e^{-(1-\alpha)\int p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)\cdot e^{(1-\alpha)\int p(x)dx}dx+c\right]$$

Eg. Solve $y'=y(xy^3-1)$. [2004 台大電研]

(Sol.) $y'+y=xy^4$. Set $z=y^{1-4}=y^{-3}$, $y=z^{-1/3}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{3}z^{-4/3}\cdot z'$

$-\frac{1}{3}z^{-4/3}\cdot z'+z^{-1/3}=xz^{-4/3}$, $(-3z^{4/3})\times[-\frac{1}{3}z^{-4/3}\cdot z'+z^{-1/3}=xz^{-4/3}]$

$\Rightarrow z'-3z=-3x\Rightarrow\left[e^{-3x}\cdot z\right]'=-3x\cdot e^{-3x}\Rightarrow z=x+\frac{1}{3}+ce^{3x}\Rightarrow y^{-3}=x+\frac{1}{3}+ce^{3x}$

Eg. Solve $y'+\frac{1}{3}y=\frac{(1-2x)}{3}\cdot y^4$. [2013 師大應用電子所]

(Sol.) Set $z=y^{1-4}=y^{-3}$, $y=z^{-1/3}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{3}z^{-4/3}\cdot z'$,

$-\frac{1}{3}z^{-4/3}\cdot z'+\frac{1}{3}z^{-1/3}=\frac{(1-2x)}{3}\cdot z^{-4/3}$, $(-3z^{4/3})\times[-\frac{1}{3}z^{-4/3}\cdot z'+\frac{1}{3}z^{-1/3}=\frac{(1-2x)}{3}\cdot z^{-4/3}]$

$\Rightarrow z'-z=2x-1$, $[e^{-x}\cdot z]'=(2x-1)e^{-x}\Rightarrow e^{-x}\cdot z=(-2x-1)e^{-x}+c$,

$z=-2x-1+c\,e^{x}$, $\therefore\ y^{-3}=-2x-1+c\,e^{x}$

Eg. Solve $\frac{dP(t)}{dt}=P(t)\cdot(c_1-c_2P(t))$. [2003 台科大電研]

(Sol.) $\frac{dP}{dt}-c_1P=-c_2P^2$. Set $z=P^{1-2}=P^{-1}$, $P=z^{-1}$, $\frac{dP}{dt}=\frac{dP}{dz}\cdot\frac{dz}{dt}=-z^{-2}\cdot z'$,

$-z^{-2}\cdot z'-c_1z^{-1}=-c_2z^{-2}$, $(-z^2)\times[-z^{-2}\cdot z'-c_1z^{-1}=-c_2z^{-2}]$

$z'+c_1z=c_2\Rightarrow\left[e^{c_1t}\cdot z\right]'=c_2e^{c_1t}\Rightarrow z=\frac{c_2}{c_1}+De^{-c_1t}\Rightarrow P(t)^{-1}=\frac{c_2}{c_1}+De^{-c_1t}$

Eg. Solve $\frac{dy(x)}{dx}-y(x)+e^{2x}y^2(x)=0$. [2015 台大電研]

(Sol.) $y'-y=-e^{2x}y^2$. Set $z=y^{1-2}=y^{-1}$, $y=z^{-1}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{z^2}\cdot z'$

$-z^{-2}\cdot z'-z^{-1}=-e^{2x}z^{-2}$, $(-z^2)\times[-z^{-2}\cdot z'-z^{-1}=-e^{2x}z^{-2}]$

$\Rightarrow z'+z=e^{2x}\Rightarrow\left[e^{x}\cdot z\right]'=e^{3x}\Rightarrow z=\frac{e^{2x}}{3}+ce^{-x}\Rightarrow y^{-1}=\frac{e^{2x}}{3}+ce^{-x}$

1-4 Homogeneous & Quasi-homogeneous Differential Equations

The first-order homogeneous differential equation: $y'=f(y/x)$

Solution: Set $u=\frac{y}{x}$, $y=ux$, $\frac{dy}{dx}=u+x\frac{du}{dx}\Rightarrow x\frac{du}{dx}+u=f(u)\Rightarrow\frac{du}{f(u)-u}=\frac{dx}{x}$ is a separable differential equation.

Eg. Solve $x\frac{dy}{dx}=\frac{y^2}{x}+y$.

(Sol.) Set $u=\frac{y}{x}$, $y=ux$, $\frac{dy}{dx}=u+x\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+\frac{y}{x}\Rightarrow u^2+u=u+x\frac{du}{dx}$

$$\Rightarrow\quad u^2=x\frac{du}{dx},\ \frac{dx}{x}=\frac{du}{u^2},\ \ell n\,|\,x\,|+c=-\frac{1}{u}$$

$$\Rightarrow\quad u=\frac{-1}{\ell n\,|\,x\,|+c},\ y=xu=\frac{-x}{\ell n\,|\,x\,|+c}$$

Eg. Solve $\frac{dy}{dx}=\frac{x+y}{x-y}$.

(Sol.) $$\frac{dy}{dx}=\frac{1+\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)}\Rightarrow\frac{1+u}{1-u}=u+x\frac{du}{dx}\Rightarrow x\frac{du}{dx}=\frac{1+u-u+u^2}{1-u}=\frac{1+u^2}{1-u}$$

$$\Rightarrow\quad\left(\frac{1-u}{1+u^2}\right)du=\frac{dx}{x},\ \tan^{-1}(u)-\frac{1}{2}\ell n\,|\,1+u^2\,|=\ell n\,|\,x\,|+c$$

$$\Rightarrow\quad\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ell n\left|1+\left(\frac{y}{x}\right)^2\right|=\ell n\,|\,x\,|+c\Rightarrow\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ell n\left|x^2+y^2\right|=c$$

Eg. Solve $\left(x-\sqrt{xy}\right)y'=y$. **[1990 中山電研]**

(Sol.) $(1-\sqrt{\frac{y}{x}})\frac{dy}{dx}=\frac{y}{x}$. Let $u=\frac{y}{x}$, $y=ux$, $\frac{dy}{dx}=u+x\frac{du}{dx}$

$$\Rightarrow\frac{dx}{x}=\left(\frac{1-\sqrt{u}}{u\sqrt{u}}\right)du\Rightarrow\ell n|x|+C=\int\frac{1}{u\sqrt{u}}du-\int\frac{1}{u}du=\int u^{\frac{-3}{2}}du-\ell n|u|=-2u^{\frac{-1}{2}}-\ell n|u|$$

$$\Rightarrow\ell n|x|+2\sqrt{\frac{x}{y}}+\ell n\left(\frac{y}{x}\right)=C\Rightarrow\ell n|y|+2\sqrt{\frac{x}{y}}=C$$

Eg. Solve $y'=4y/(4x-y)$. [文化電機轉學考]

(Ans.) $\ln(|y|)=-\frac{4x}{y}+c$

Quasi-homogeneous differential equation: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+h}\right)$

Case 1 ae-bd≠0

Solution: Let A and B fulfill $\begin{cases} aA+bB+c=0 \\ dA+eB+h=0 \end{cases}$, and set $x = X + A$, $y = Y + B$

$$\frac{dy}{dx} = \frac{dY}{dX} = f\left(\frac{ax+by+c}{dx+ey+h}\right) = f\left(\frac{a(X+A)+b(Y+b)+c}{d(X+A)+e(Y+B)+h}\right)$$

$$= f\left(\frac{aX+bY+aA+bB+c}{dX+eY+dA+eB+h}\right)$$

$$\Rightarrow \frac{dY}{dX} = f\left(\frac{aX+bY}{dX+eY}\right)$$ is a homogeneous equation.

Eg. Solve $\frac{dy}{dx} = \frac{2x+y-1}{x-2}$.

(Sol.) $a=2$, $b=1$, $c=-1$, $d=1$, $e=0$, $h=-2$, $ae-bd=0-1=-1\neq 0$

$$\begin{cases} 2A+B-1=0 \\ A-2=0 \end{cases} \Rightarrow \begin{matrix} A=2 \\ B=-3 \end{matrix} \Rightarrow x = X+2,\ y = Y-3$$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{2(X+2)+(Y-3)-1}{X} = \frac{2X+Y}{X} = 2+\left(\frac{Y}{X}\right) = u+2$$

$$X\frac{du}{dX}+u = u+2,\quad u = \ln(cX^2) \Rightarrow \frac{y+3}{x-2} = \ln[c(x-2)^2]$$

Case 2 ae-bd=0

Solution: Set $v = \frac{ax+by}{a} = \frac{dx+ey}{d}$, $y = \frac{a}{b}(v-x) \Rightarrow \frac{dy}{dx} = \frac{a}{b}\left(\frac{dv}{dx}-1\right)$

$\therefore\ \frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+h}\right) \Rightarrow \frac{a}{b}\left(\frac{dv}{dx}-1\right) = f\left(\frac{av+c}{dv+h}\right) \Rightarrow \frac{dv}{dx} = 1+\frac{b}{a}f\left(\frac{av+c}{dv+h}\right)$ is a separable equation for v and x

Eg. Solve $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y-4}$.

(Sol.) $a=2$, $b=1$, $c=-1$, $d=4$, $e=2$, $h=-4$

$$\because\ ae-bd=0,\ \therefore\ v = \frac{2x+y}{2} = \frac{4x+2y}{4}$$

$$\Rightarrow \frac{dv}{dx} = 1+\frac{1}{2}\left(\frac{2v-1}{4v-4}\right) \Rightarrow \left(\frac{8v-8}{10v-9}\right)dv = dx \Rightarrow \frac{4v}{5}-\frac{2}{25}\ell n\,|10v-9|+c = x$$

$$\frac{2}{5}(2x+y)-\frac{2}{25}\ell n\,|10x+5y-9|+c = x$$

1-5 Exact Differential Equations and Integrating Factors

Exact differential equation: *M(x,y)dx+N(x,y)dy*=0 if $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

Solution: $\exists F(x,y)$ fulfills $\frac{\partial^2 F(x,y)}{\partial x\partial y}=\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

$$\Rightarrow dF=\frac{\partial F(x,y)}{\partial x}\cdot dx+\frac{\partial F(x,y)}{\partial y}dy=M(x,y)dx+N(x,y)dy=0$$

Solve $\frac{\partial F(x,y)}{\partial x}=M(x,y)$ and $\frac{\partial F(x,y)}{\partial y}=N(x,y)$

$\Rightarrow$ *F(x,y)=C* is its solution.

Eg. Solve (6*xy*-*y*³)*dx*+(4*y*+3*x*²-3*xy*²)*dy*=0. [2011 中正電研]

(Sol.) $\frac{\partial(6xy-y^3)}{\partial y}=6x-3y^2=\frac{\partial(4y+3x^2-3xy^2)}{\partial x}$

$$\frac{\partial F(x,y)}{\partial x}=6xy-y^3\Rightarrow F(x,y)=3x^2y-xy^3+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=4y+3x^2-3xy^2\Rightarrow F(x,y)=2y^2+3x^2y-xy^3+c_2(x)$$

$$\Rightarrow F(x,y)=2y^2+3x^2y-xy^3+c\text{ , }\therefore 2y^2+3x^2y-xy^3=C$$

Eg. Solve $\frac{dy}{dx}=\frac{-2xy^3-2}{3x^2y^2+e^y}$.

(Sol.) $(2xy^3+2)dx+(3x^2y^2+e^y)dy=0$, $\frac{\partial(2xy^3+2)}{\partial y}=6xy^2=\frac{\partial(3x^2y^2+e^y)}{\partial x}$

$$\frac{\partial F(x,y)}{\partial x}=2xy^3+2\Rightarrow F(x,y)=x^2y^3+2x+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=3x^2y^2+e^y\Rightarrow F(x,y)=x^2y^3+e^y+c_2(x)$$

$$\Rightarrow F(x,y)=x^2y^3+2x+e^y+c\text{ , }\therefore\ x^2y^3+2x+e^y=C$$

Eg. Solve *y*=(*y*²-*x*)*y*'.

(Sol.) $y\,dx+(x-y^2)dy=0$

$$\Rightarrow\frac{\partial(y)}{\partial y}=\frac{\partial(x-y^2)}{\partial x}=1\text{ , }F(x,y)=\int y\,dx+c_1=\int(x-y^2)dy+c_2=xy-\frac{y^3}{3}+c_2\text{ ,}$$

$$\therefore\ xy-\frac{y^3}{3}=C$$

Integrating factor $u(x,y)$: For $M(x,y)dx+N(x,y)dy=0$, in case $\frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x}$ but $\frac{\partial[u(x,y)M(x,y)]}{\partial y}=\frac{\partial[u(x,y)N(x,y)]}{\partial x}$, and then $u(x,y)$ is called the integrating factor.

Eg. Solve $(y^2-6xy)dx+(3xy-6x^2)dy=0$.

(Sol.) $\frac{\partial(y^2-6xy)}{\partial y}=2y-6x\neq 3y-12x=\frac{\partial(3xy-6x^2)}{\partial x}$

Choose $u(x,y)=y\Rightarrow(y^3-6xy^2)dx+(3xy^2-6x^2y)dy=0$

$\frac{\partial(y^3-6xy^2)}{\partial y}=3y^2-12xy=\frac{\partial(3xy^2-6x^2y)}{\partial x}$

$\frac{\partial F(x,y)}{\partial x}=y^3-6xy^2\Rightarrow F(x,y)=xy^3-3x^2y^2+c_1(y)$

$\frac{\partial F(x,y)}{\partial y}=3xy^2-6x^2y\Rightarrow F(x,y)=xy^3-3x^2y^2+c_2(x)$. $\therefore\ xy^3-3x^2y^2=C$

Another Method: $\frac{dy}{dx}=-\frac{y^2-6xy}{3xy-6x^2}$ is the *first-order homogeneous equation*.

$$\frac{dy}{dx}=-\frac{y^2-6xy}{3xy-6x^2}=-\frac{\left(\frac{y}{x}\right)^2-6\left(\frac{y}{x}\right)}{3\left(\frac{y}{x}\right)-6}$$. Let $u=\frac{y}{x}\Rightarrow y'=u+x\frac{du}{dx}$

$$\frac{3u-6}{-4u^2+12u}du=\frac{dx}{x}\Rightarrow -\frac{3}{4}\int\frac{u-2}{u(u-3)}du=\ell n|x|+c$$

$$\Rightarrow -\frac{1}{4}\left[\int\frac{2du}{u}+\int\frac{du}{u-3}\right]=\ln|x|+c\Rightarrow \ell n\left|u^2(u-3)\right|=-4\ell n|x|-4c$$

$$\Rightarrow\left(\frac{y}{x}\right)^2\left(\frac{y}{x}-3\right)=\frac{A}{x^4},\ xy^3-3x^2y^2=A,\ \ xy^3-3x^2y^2=A$$

Eg. Solve $(x+e^y)dy-dx=0$.

(Sol.) $M(x,y)=-1,\ N(x,y)=x+e^y,\ \frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x}$

Choose integrating factor: $e^{-y}\Rightarrow -e^{-y}dx+(xe^{-y}+1)dy=0$

$\frac{\partial(-e^{-y})}{\partial y}=e^{-y}=\frac{\partial(xe^{-y}+1)}{\partial x}$

$F(x,y)=-xe^{-y}+c_1(x,y)=-xe^{-y}+y+c_2(x,y)$, $\therefore$ $-xe^{-y}+y=C$

Another method: $\frac{dx}{dy}-x=e^y$ is the *first-order linear differential equation* for $x(y)$.

$\frac{dx}{dy}-x=e^y,\ p(y)=-1,\ e^{\int p(y)dy}=e^{-y},\ e^{-y}\frac{dx}{dy}-e^{-y}x=e^{-y}e^y=1$

$(e^{-y}x)'=1,\ e^{-y}x=y+c,\ -xe^{-y}+y=C$

1-6 Riccati's Equation $y'=P(x)y^2+Q(x)y+R(x)$

Suppose that there exists one specific solution $y=S(x)$, then a general solution can be obtained as follows

$$y = S(x) + \frac{1}{z},\quad y' = S'(x) - \frac{1}{z^2}\cdot z'$$

$$S'(x) - \frac{1}{z^2}\cdot z' = P(x)\cdot\left[S^2(x) + \frac{2S(x)}{z} + \frac{1}{z^2}\right] + Q(x)\cdot\left[S(x) + \frac{1}{z}\right] + \mathrm{R}(\mathrm{x})$$

$$-\frac{1}{z^2}\cdot z' = P(x)\cdot\frac{1}{z^2} + 2P(x)S(x)\frac{1}{z} + Q(x)\cdot\frac{1}{z}$$

$z' + [2P(x)S(x) + Q(x)]z = -P(x)$ is the 1st-order linear differential equation.

Eg. Solve $y' = e^{-3x}y^2 - y + 3e^{3x}$.

(Sol.) $y = e^{3x}$ is a solution $\Rightarrow y = e^{3x} + \frac{1}{z}, y' = 3e^{3x} - \frac{z'}{z^2}$

$$3e^{3x} - \frac{z'}{z^2} = e^{-3x}\cdot\left(e^{6x} + \frac{2e^{3x}}{z} + \frac{1}{z^2}\right) - \left(e^{3x} + \frac{1}{z}\right) + 3e^{3x}$$

$$z' = -2z - e^{-3x} + z, z' + z = -e^{-3x},\quad z'e^x + e^x\cdot z = -e^{-2x}$$

$$\left(z\cdot e^x\right)' = -e^{-2x},\quad ze^x = \frac{1}{2}e^{-2x} + c,\ \therefore\quad y = e^{3x} + \frac{2}{e^{-3x} + 2ce^{-x}}$$

1-7 Solutions of the First-order Ordinary Differential Equations by Matlab language

In **Matlab** language, we can use the following instructions to obtain the solution of the first-order ordinary differential equation:

```
>>soln=dsolve('Dy=3*y+exp(2*x)','y(0) = 3') % solve y'=3y+exp(2x), y(0)=3

ans=-exp(2*x)+4*exp(3*x)
```

1-8 Some Theorems on the First-order Ordinary Differential Equations

A family of curves $F(x,y,k)=0$ is a solution of $y'=f(x,y)$.

Eg. A family of circles $x^2+y^2-k^2=0$ is a solution of $y'=-x/y$.

Theorem An oblique trajectory intersecting $y'=f(x,y)$ at an angle α is $y' = \frac{f(x,y) + \tan(\alpha)}{1 - f(x,y)\tan(\alpha)}$; particularly, if $\alpha=\pi/2$, then the orthogonal trajectory is $y'=-1/f(x,y)$.

Eg. Find the families of oblique trajectories intersecting the circle $x^2+y^2=k^2$ at angles of 45° and 90°.

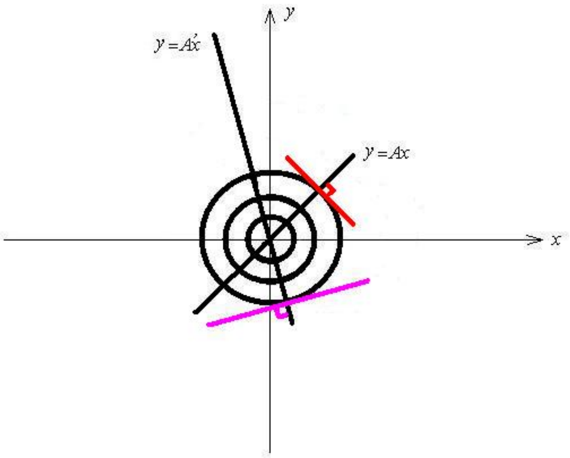
(Sol.) $x^2+y^2=k^2 \leftrightarrow y'=-\dfrac{x}{y}=f(x,y)$

1. $\tan(45°)=1$, $y'=\dfrac{-\frac{x}{y}+1}{1+\frac{x}{y}}=\dfrac{y-x}{y+x}=\dfrac{\left(\frac{y}{x}\right)-1}{\left(\frac{y}{x}\right)+1}=\dfrac{v-1}{v+1}$

$$v+xv'=\frac{v-1}{v+1},\quad v'=\frac{v-1-v^2-v}{(v+1)x}=-\frac{1+v^2}{v+1}\cdot\frac{1}{x}$$

$$\left(\frac{1+v}{1+v^2}\right)dv=-\frac{dx}{x}\Rightarrow\frac{1}{2}\ell n\,|\,1+v^2\,|+\tan^{-1}(v)=-\ell n\,|\,x\,|+c$$

$$\therefore\ \frac{1}{2}\ell n\left|1+\left(\frac{y}{x}\right)^2\right|+\tan^{-1}\left(\frac{y}{x}\right)=-\ell n\,|\,x\,|+c$$



2. $y'=-\dfrac{1}{\left(-\frac{x}{y}\right)}=\dfrac{y}{x},\ \dfrac{dy}{y}=\dfrac{dx}{x}$,

$\therefore\ y=Ax$

Theorem **A family of curves $F(\theta,r,k)=0$, of which differential equation is $f(\theta,r,r')=0$. Then the family of orthogonal trajectories has differential equation $f(\theta,r,-r^2/r')=0$, where $r'=dr/d\theta$.**

Eg. Find the family of trajectories orthogonal to $r=k\cos(\theta)$.

(Sol.) $r=k\cos(\theta)$, $r'=-k\sin(\theta)\Rightarrow r=-\cot(\theta)r'$

Family of orthogonal trajectories: $r=-\cot(\theta)\cdot\left(-\dfrac{r^2}{r'}\right)\Rightarrow\dfrac{r}{r'}=\tan(\theta)\Rightarrow r=k'\sin(\theta)$