

Chapter 11 Complex Sequences and Series

11-1 Sequence and Series

Sequence $\{z_n\}$: $z_0, z_1, z_2, z_3, \dots, z_n, \dots$

Series: $\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \dots + z_n + \dots$

Absolute convergence: $\sum_{n=0}^{\infty} |z_n|$ converges. (In this case, $\sum_{n=0}^{\infty} z_n$ converges also.)

Eg. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ is an absolutely convergent series because

$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n+1)^2} \right| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ are both convergent.

Conditional convergence: $\sum_{n=0}^{\infty} z_n$ converges but $\sum_{n=0}^{\infty} |z_n|$ diverges.

Eg. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is conditionally convergent because $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

is convergent but $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ is divergent.

Radius of convergence for $\sum_{n=0}^{\infty} a_n (z - z_0)^n$: There exists R (possibly $R = \infty$) such that the series converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0| > R$.

(Proof) $\frac{|a_{n+1} (z - z_0)^{n+1}|}{|a_n (z - z_0)^n|} < 1 \Rightarrow |z - z_0| < \frac{|a_n|}{|a_{n+1}|}$

Define $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$, so the proof is complete.

Eg. Find the radius of convergence for $\sum_{n=0}^{\infty} \frac{n^n}{n!} (z - i)^n$.

(Sol.) $\left| \frac{(n+1)^{n+1} (z - i)^{n+1}}{(n+1)!} \right| = \left| \left(\frac{n+1}{n} \right)^n (z - i) \right| < 1$

$|z - i| < \left(1 + \frac{1}{n} \right)^{-n} \rightarrow e^{-1}$ as $n \rightarrow \infty$, $\therefore R = e^{-1}$

11-2 Complex Taylor's Series & Laurent's Series

Taylor's series: $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$. In case $z_0=0$, it is called Maclaurin's series.

Eg. Find the Maclaurin's expansion of $f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw$.

$$\begin{aligned} \text{(Sol.) } e^{-w^2} &= 1 - w^2 + \frac{(-w^2)^2}{2!} + \frac{(-w^2)^3}{3!} + \frac{(-w^2)^4}{4!} + \frac{(-w^2)^5}{5!} + \dots \\ &= 1 - w^2 + \frac{w^4}{2!} - \frac{w^6}{3!} + \frac{w^8}{4!} - \frac{w^{10}}{5!} + \dots + (-1)^n \frac{w^{2n}}{n!} \\ \int_0^z e^{-w^2} dw &= z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)n!} + \dots \\ f(z) &= \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)n!} + \dots \right) \end{aligned}$$

Laurent's series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \dots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots,$$

where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}} dw$, and $C: |z - z_0| = \rho, r_1 < \rho < r_2$

Eg. Find the Laurent's series of $f(z) = \frac{z - \sin z}{z^3}$ about $z_0=0$, indicate the type of singularity and the region of convergence of the series. 【清大電研】

$$\begin{aligned} \text{(Sol.) } f(z) &= \frac{z - \sin z}{z^3} = \frac{1}{z^3} \left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right] \\ &= \frac{1}{z^3} \left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \frac{z^9}{9!} + \dots \right] = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \frac{z^6}{9!} + \dots \end{aligned}$$

$\therefore z_0 = 0$ is a removable singularity.

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \frac{z^6}{9!} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot z^{2n-2}}{(2n+1)!} = \sum_{n=1}^{\infty} a_n$$

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{(-1)^n \cdot z^{2n}}{(2n+3)!} \right| \left/ \left| \frac{(-1)^{n-1} \cdot z^{2n-2}}{(2n+1)!} \right| \right| < 1$$

$$z^2 < (2n+3)(2n+2) \Rightarrow z < R \rightarrow \infty$$

Eg. Expand $f(z) = \frac{z}{(z-1)(2-z)}$ by Laurent's series in case: (a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$, and (e) $0 < |z-2| < 1$.

(Sol.) $f(z) = \frac{z}{(z-1)(2-z)} = -\frac{z}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{2}{z-2}$

(a) $|z| < 1 \Rightarrow 1 > \frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1} = \frac{-1}{1-z} = (-1)(1+z+z^2+z^3+\dots) \\ \qquad \qquad \qquad = -1-z-z^2-z^3-z^4-\dots \\ \frac{-2}{z-2} = \frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \end{cases}, \therefore f(z) = -\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \dots$$

(b) $1 < |z| < 2 \Rightarrow 1 > \frac{1}{|z|}, 1 > \frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right) \\ \qquad \qquad \qquad = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\ \frac{-2}{z-2} = \frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \end{cases}, \therefore f(z) = \dots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \dots$$

(c) $|z| > 2 \Rightarrow \frac{2}{|z|} < 1 \Rightarrow 1 > \frac{1}{|z|}$

$$\Rightarrow \begin{cases} \frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) \\ \qquad \qquad \qquad = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \\ \frac{-2}{z-2} = -\frac{2}{z} \cdot \frac{1}{1-\frac{2}{z}} = -\frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\dots\right) \\ \qquad \qquad \qquad = -\frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \dots \end{cases}, \therefore f(z) = -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \dots$$

(d) $|z-1| > 1 \Rightarrow \frac{1}{|z-1|} < 1$

$$\frac{-2}{z-2} = \frac{-2}{(z-1)-1} = -\frac{2}{z-1} \cdot \frac{1}{1-\frac{1}{z-1}} = -\frac{2}{z-1} \cdot [1+(z-1)^{-1}+(z-1)^{-2}+\dots] = -\frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \dots$$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-2} = \frac{-1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \frac{2}{(z-1)^4} - \dots$$

(e) $0 < |z-2| < 1 \Rightarrow \frac{1}{z-1} = \frac{1}{1+(z-2)} = 1 - (z-2) + (z-2)^2 - (z-2)^3 + \dots$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-2} = -\frac{2}{z-2} + 1 - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4 - \dots$$

Eg. Find the Laurent's series of $f(z) = \frac{1}{z^2 - 3z + 2}$ for $1 < |z| < 2$. 【1991 台大機研】

$$\begin{aligned}
 \text{(Sol.) } \frac{1}{z^2 - 3z + 2} &= \frac{-1}{z-1} + \frac{1}{z-2} \\
 1 < |z| < 2 &\Rightarrow 1 > \frac{1}{|z|}, \frac{-1}{z-1} = \frac{-1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{-1}{z} \cdot \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] \\
 &\Rightarrow 1 > \frac{|z|}{2}, \frac{1}{z-2} = \frac{-1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{-1}{2} \left[1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right] \\
 &\Rightarrow f(z) = -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) - \frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right)
 \end{aligned}$$

Eg. (a) $\frac{1}{(z-1)(z-3)}$ for $1 < |z| < 3$ about $z_0=0$, (b) $\frac{e^z}{(z-1)^2}$ for $|z-1| > 0$ about $z_0=1$.

Find their respective Laurent's series. 【1991 交大電信】

$$\begin{aligned}
 \text{(Sol.) (a) } \frac{1}{(z-1)(z-3)} &= \frac{-\frac{1}{2}}{z-1} + \frac{\frac{1}{2}}{z-3} \\
 1 < |z| < 3 &\Rightarrow \frac{1}{|z|} < 1, \frac{-\frac{1}{2}}{z-1} = \frac{-\frac{1}{2}}{z} \cdot \frac{1}{(1-\frac{1}{z})} = \frac{-\frac{1}{2}}{z} \cdot \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] \\
 1 < |z| < 3 &\Rightarrow 1 > \frac{|z|}{3}, \frac{\frac{1}{2}}{z-3} = \frac{-\frac{1}{2}}{3} \cdot \frac{1}{(1-\frac{z}{3})} = \frac{-1}{6} \left[1 + \frac{z}{3} + \frac{z^2}{9} + \dots \right] \\
 &\Rightarrow f(z) = -\frac{1}{2} \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) - \frac{1}{6} \left(1 + \frac{z}{3} + \frac{z^2}{9} + \dots \right) \\
 \text{(b) } \frac{e^z}{(z-1)^2} &= \frac{e \cdot e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2} \cdot \left[1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \dots \right] \\
 &= e \cdot \left[\frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \frac{(z-1)}{3!} + \dots \right]
 \end{aligned}$$

Eg. Find the Laurent's series of $f(z) = \frac{5z+j}{z^2+jz}$ for $1 < |z-j| < 2$. 【2013 中山電研】

11-3 Summation of Series by the Residue Theorem

Theorem Let z_1, \dots, z_m be the poles of $f(z)$, then $\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{j=1}^m \operatorname{Res}_{z_j} [\cot(\pi z) \cdot f(z)]$

and $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{j=1}^m \operatorname{Res}_{z_j} [\csc(\pi z) \cdot f(z)]$.

Eg. Evaluate $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. [1991 清大電研]

$$\text{(Sol.) } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = (-1) \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} &= \sum_{n=-\infty}^0 \frac{(-1)^n}{2n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \left(\dots + \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 \right) + \left(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots \right) \\ &= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} \end{aligned}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{2} \operatorname{Res}_{\frac{1}{2}} \left[\frac{\csc(\pi z)}{2z-1} \right]$$

$$= \frac{\pi}{4} \operatorname{Res}_{\frac{1}{2}} \left[\frac{\csc(\pi z)}{z - \frac{1}{2}} \right] = \frac{\pi}{4} \lim_{z \rightarrow \frac{1}{2}} \frac{\csc \pi z}{z - \frac{1}{2}} \times \left(z - \frac{1}{2} \right) = \frac{\pi}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

Eg. Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$.

$$\begin{aligned} \text{(Sol.) } \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} &= \sum_{n=-\infty}^0 \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \\ &= \left(\dots + \frac{1}{5^2} + \frac{1}{3^2} + 1 \right) + \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \end{aligned}$$

$$\therefore 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{-\pi}{2} \operatorname{Res}_{\frac{1}{2}} \left[\frac{\cot(\pi z)}{(2z-1)^2} \right]$$

$$= \frac{-\pi}{8} \operatorname{Res}_{\frac{1}{2}} \left[\frac{\cot(\pi z)}{\left(z - \frac{1}{2}\right)^2} \right] = \frac{-\pi}{8} \lim_{z \rightarrow \frac{1}{2}} \frac{1}{1!} \frac{d}{dz} \left[\frac{\cot(\pi z)}{\left(z - \frac{1}{2}\right)^2} \times \left(z - \frac{1}{2}\right)^2 \right]$$

$$= \frac{-\pi}{8} \cdot (-\pi) \cdot \csc^2\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}, \therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

Eg. Evaluate $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$.

$$\text{(Sol.) } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2} = -\frac{\pi}{2} \operatorname{Res}_0 \left(\frac{\cot(\pi z)}{z^2} \right)$$

$$\frac{\cot(\pi z)}{z^2} = \frac{\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{\left(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - + \dots \right)}{z^2 \left(\pi z - \frac{\pi^3 z^3}{3!} + - \dots \right)} = \frac{1}{\pi} \cdot \frac{1}{z^3} - \frac{\pi}{3} \cdot \frac{1}{z} + \dots$$

$$\therefore \operatorname{Res}_0 \left(\frac{\cot(\pi z)}{z^2} \right) = -\frac{\pi}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left(-\frac{\pi}{2} \right) \left(-\frac{\pi}{3} \right) = \frac{\pi^2}{6}$$

11-4 Infinite products

Theorem If $\prod_{n=1}^{\infty} a_n$ ($\neq 0$) converges, then $\lim_{n \rightarrow \infty} a_n = 1$.

(Proof) Let $p_N = \prod_{n=1}^N a_n$, $1 = \lim_{N \rightarrow \infty} p_N / \lim_{N \rightarrow \infty} p_{N-1} = \lim_{N \rightarrow \infty} \left(\prod_{n=1}^N a_n / \prod_{n=1}^{N-1} a_n \right) = \lim_{N \rightarrow \infty} a_N$

Theorem $\prod_{n=1}^{\infty} (1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} \log(1 + a_n)$ converges.

Theorem $\prod_{n=1}^{\infty} (1 + a_n)$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Eg. Determine the domain D of convergence for $\prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n} \right)^{n^2} \cdot z^n \right]$.

(Sol.) $\prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n} \right)^{n^2} z^n \right]$ diverges if $\lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n} \right)^{n^2} z^n \right| > 1$

$$\left(1 + \frac{1}{n} \right)^{n^2} = e^{n^2 \ln \left(1 + \frac{1}{n} \right)} = e^{n \ln \left(1 + \frac{1}{n} \right)^n} \rightarrow e^{n \ln e} = e^n$$

$$\lim_{n \rightarrow \infty} |e^n \cdot z^n| > 1 \Rightarrow |z| > e^{-1}, \quad \prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n} \right)^{n^2} z^n \right] \text{ diverges}$$

$\therefore D = \{z : |z| < e^{-1}\}$ for convergence.

Weierstrass' theorem Let $f(z)$ be analytic. All zeros $a_1, a_2, a_3, \dots, a_n, \dots$ are single and $0 < |a_1| < |a_2| < |a_3| < \dots$, $\lim_{n \rightarrow \infty} |a_n| = \infty$, then $f(z) = f(0) e^{zf'(0)/f(0)} \cdot \prod_{k=1}^{\infty} \left[\left(1 - z/a_k \right) e^{z/a_k} \right]$.

Eg. Show that $\cos(z) = \left[1 - \frac{z^2}{(\pi/2)^2} \right] \left[1 - \frac{z^2}{(3\pi/2)^2} \right] \left[1 - \frac{z^2}{(5\pi/2)^2} \right] \dots$

(Proof) Zeros of $\cos(z) = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

$$\cos(0) = 1, \quad z[\cos(z)]'/\cos(z)|_{z=0} = 0, \quad e^{z/a_k} \cdot e^{z/(-a_k)} = 1$$

$$\therefore \cos(z) = \left[1 - \frac{z^2}{(\pi/2)^2} \right] \cdot \left[1 - \frac{z^2}{(3\pi/2)^2} \right] \dots$$

Note: $\sin(z) = z \left[1 - \frac{z^2}{\pi^2} \right] \cdot \left[1 - \frac{z^2}{(2\pi)^2} \right] \cdot \left[1 - \frac{z^2}{(3\pi)^2} \right] \dots$