

## Chapter 14 Calculus of Variations and Lagrange's Multiplier Method

### 14-1 Euler Equation and Calculus of Variation

**How to find the extrema of  $I = \int_a^b f(x, y, y') dx$  under  $y(a)=y_1$  and  $y(b)=y_2$ ?**

Assume  $y(x) = u(x) + \varepsilon \eta(x)$ ,  $\eta(a) = \eta(b) = 0$ , and

$$I(\varepsilon) = \int_a^b f(x, u(x) + \varepsilon \eta(x), u'(x) + \varepsilon \eta'(x)) dx$$

Necessary condition for the extrema of  $I(\varepsilon)$ :

$$\begin{aligned} \delta I(\varepsilon) = 0 &\Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon} \cdot \delta\varepsilon = 0 \Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon} = 0 \\ \Leftrightarrow \frac{dI(\varepsilon)}{d\varepsilon} &= \int_a^b \left[ \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial \varepsilon} \right] dx = \int_a^b \left[ \frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx \\ &= \int_a^b \frac{\partial f}{\partial y} \eta(x) dx + \frac{\partial f}{\partial y'} \eta(x) \Big|_a^b - \int_a^b \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) \eta(x) dx = \int_a^b \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) \right] \eta(x) dx = 0 \end{aligned}$$

$\Rightarrow$  Euler equation for  $I = \int_a^b f(x, y, y') dx$  is  $\frac{\partial f(x, y, y')}{\partial y} - \frac{d}{dx} \left( \frac{\partial f(x, y, y')}{\partial y'} \right) = 0$ .

**Note:** The function  $y(x)$  which maximizes or minimizes  $I = \int_a^b f(x, y, y') dx$  is called the stationary solution.

**Eg. Find the stationary solution  $y(x)$  to maximize or minimize a functional**

$I = \int_1^3 [x(y')^2 + y] dx$ ,  $y(1)=3$  and  $y(3)=4$ .

(Sol.)  $f(x, y, y') = x(y')^2 + y$ ,  $\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 1 - \frac{d}{dx} (2xy') = 1 - 2y' - 2xy'' = 0$

$$\Rightarrow x^2 y'' + xy' = \frac{x}{2} \quad (\text{Euler's differential equation}) \Rightarrow y = \frac{x}{2} + c \ln |x| + d$$

$$y(1) = 3 = \frac{1}{2} + d, \quad y(3) = 4 = \frac{3}{2} + c \ln(3) + d \Rightarrow c = 0, \quad d = \frac{5}{2}, \quad \therefore y(x) = \frac{x}{2} + \frac{5}{2}$$

**Eg. Find the stationary solution  $y(x)$  of  $I = \int_0^1 [(y')^2 - x^2 y] dx$ ,  $y(0)=-2$  and  $y(1)=3$ .**

(Sol.)  $f(x, y, y') = (y')^2 - x^2 y$ ,  $\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = -x^2 - \frac{d}{dx} (2y') = -x^2 - 2y'' = 0$

$$\Rightarrow y = -\frac{1}{24} x^4 + c_1 x + c_2, \quad y(0) = -2 = c_2, \quad y(1) = 3 = -\frac{1}{24} + c_1 - 2, \quad c_1 = \frac{121}{24},$$

$$\therefore y(x) = -\frac{1}{24} x^4 + \frac{121}{24} x - 2$$

## 14-2 Ritz Method

For  $I = \int_a^b f(x, y, y') dx$ , the stationary solution  $y(x)$  is approximate to  $y(x) \approx c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \dots + c_n \phi_n(x)$  and fulfills  $\delta I = 0$ .

**Eg. Apply the Ritz method to solve  $I = \int_0^1 [(y')^2 + xy] dx$ ,  $y(0)=0$  and  $y(1)=1$ .**

(Sol.) Let  $y(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$ ,  $\because y(0)=0, \therefore c_0=0$

It is found that  $y = c_1 x$  and  $y = c_1 x + c_2 x^2$  are not exact solutions.

Consider  $y(x) = c_1 x + c_2 x^2 + c_3 x^3$ ,  $y(1) = 1 \Rightarrow c_3 = 1 - c_1 - c_2$ .

$$y'(x) = c_1 + 2c_2 x + 3c_3 x^2 = c_1 + 2c_2 x + 3(1 - c_1 - c_2)x^2$$

$$[y'(x)]^2 = c_1^2 + 4c_2^2 x^2 + 9(1 - c_1 - c_2)^2 x^4 + 4c_1 c_2 x + 6c_1(1 - c_1 - c_2)x^2 + 12c_2(1 - c_1 - c_2)x^3$$

$$xy(x) = c_1 x^2 + c_2 x^3 + (1 - c_1 - c_2)x^4$$

$$\begin{aligned} I &= \int_0^1 [(y')^2 + xy] dx = \int_0^1 \{c_1^2 + 4c_1 c_2 x + [c_1 + 4c_2^2 + 6c_1(1 - c_1 - c_2)]x^2 \\ &\quad + [c_2 + 12c_2(1 - c_1 - c_2)]x^3 + [(1 - c_1 - c_2) + 9(1 - c_1 - c_2)^2]x^4\} dx \\ &= c_1^2 + 2c_1 c_2 + \frac{c_1 + 4c_2^2 + 6c_1 - 6c_1^2 - 6c_1 c_2}{3} + \frac{c_2 + 12c_2 - 12c_1 c_2 - 12c_2^2}{4} \\ &\quad + \frac{(1 - c_1 - c_2) + 9(1 + c_1^2 + c_2^2 - 2c_1 - 2c_2 + 2c_1 c_2)}{5} \end{aligned}$$

$$\delta I = 0$$

$$\Rightarrow 2c_1 \delta c_1 + 2c_1 \delta c_2 + 2c_2 \delta c_1 + \frac{7}{3} \delta c_1 + \frac{8}{3} c_2 \delta c_2 - 4c_1 \delta c_1 - 2c_1 \delta c_2 - 2c_2 \delta c_1$$

$$+ \frac{13}{4} \delta c_2 - 3c_1 \delta c_2 - 3c_2 \delta c_1 - 6c_2 \delta c_2 - \frac{\delta c_1}{5} - \frac{\delta c_2}{5} + \frac{18}{5} c_1 \delta c_1 + \frac{18}{5} c_2 \delta c_2$$

$$- \frac{18}{5} \delta c_1 - \frac{18}{5} \delta c_2 + \frac{18}{5} c_1 \delta c_2 + \frac{18}{5} c_2 \delta c_1 = 0$$

$$\Rightarrow \begin{cases} \delta c_1 \cdot \left( 2c_1 + 2c_2 + \frac{7}{3} - 4c_1 - 2c_2 - 3c_2 - \frac{1}{5} + \frac{18}{5} c_1 - \frac{18}{5} + \frac{18}{5} c_2 \right) = 0 \\ \delta c_2 \cdot \left( 2c_1 + \frac{8}{3} c_2 - 2c_1 + \frac{13}{4} - 3c_1 - 6c_2 - \frac{1}{5} + \frac{18}{5} c_2 - \frac{18}{5} + \frac{18}{5} c_1 \right) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = \frac{11}{12}, c_3 = 1 - c_1 - c_2 = \frac{1}{12} \Rightarrow y(x) = \frac{x^3}{12} + \frac{11x}{12} \\ c_2 = 0 \end{cases}$$

If we set  $y(x) = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$ , we also obtain  $c_1 = \frac{11}{12}$ ,  $c_2 = 0 = c_4$ ,  $c_3 = \frac{1}{12}$ .  $\therefore$

It has been convergent to an exact solution!

**Eg. Apply the Ritz method to solve  $I = \int_0^1 [(y')^2 + xy] dx$ ,  $y(0)=0$  and  $y(1)=1$ .**

(Sol.) Let  $y(x) \approx c_0x + c_1x(x-1) + \dots + c_nx^n(x-1)$

$$\because y(1)=1, \therefore c_0 = 1$$

1. Assume  $n=1$ ,  $y(x) \approx x + c_1x(x-1)$ ,  $y'(x) = 2c_1x - c_1 + 1$

$$\begin{aligned} \delta I &= \delta \int_0^1 [(y')^2 + xy] dx = \delta \int_0^1 [(1 + 2c_1x - c_1)^2 + x^2 + c_1x^3 - c_1x^2] dx \\ &= \delta \left[ 1 + \frac{1}{3}c_1^2 + \frac{1}{3} - \frac{1}{12}c_1 \right] = \frac{2}{3}c_1\delta c_1 - \frac{1}{12}\delta c_1 \\ &= \left( \frac{2}{3}c_1 - \frac{1}{12} \right) \cdot \delta c_1 = 0 \quad \forall \delta c_1 \neq 0 \end{aligned}$$

$$\Rightarrow c_1 = \frac{1}{8}, \therefore y(x) \approx x + \frac{1}{8}x(x-1)$$

2. Assume  $n=2$ ,  $y(x) \approx x + c_1x(x-1) + c_2x^2(x-1)$

$$\Rightarrow c_1 = \frac{1}{12}, \quad c_2 = \frac{1}{12}$$

$$\Rightarrow y(x) = x + \frac{1}{12}[x(x-1) + x^2(x-1)] = \frac{11}{12}x + \frac{1}{12}x^3$$

3. Assume  $n=3$ ,  $y(x) \approx x + c_1x(x-1) + c_2x^2(x-1) + c_3x^3(x-1)$

$$\Rightarrow c_1 = \frac{1}{12}, \quad c_2 = \frac{1}{12}, \quad c_3 = 0$$

$$\Rightarrow y(x) = x + \frac{1}{12}[x(x-1) + x^2(x-1)] = \frac{11}{12}x + \frac{1}{12}x^3$$

$\therefore y(x) = \frac{11}{12}x + \frac{1}{12}x^3$  is an exact solution!

### 14-3 Relationship between Differential Equations and Calculus of Variations

For  $y''+P(x)y'+Q(x)y=F(x)$ , set  $p(x)=e^{\int P(x)dx}$ ,  $q(x)=-Q(x)e^{\int P(x)dx}=-Q(x)p(x)$ , and  $f(x)=-F(x)e^{\int P(x)dx}=-F(x)p(x)$ . It is equivalent to the Sturm-Liouville form  $-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=f(x)$ . And then it can be transformed into  $\delta I=0$ , where  $I=\int_0^1\{p(x)[y'(x)]^2+q(x)y^2(x)-2f(x)y(x)\}dx$ .

**Eg. Transform  $xy''+y'=1/2$  into  $\delta I=0$ .**

(Sol.)  $y''+\frac{y'}{x}+0y=\frac{1}{2x}$ ,  $p(x)=\exp\left[\int\frac{1}{x}dx\right]=x$ ,  $q(x)=-0\cdot x=0$ , and  $f(x)=-\frac{1}{2x}\cdot x=-1/2$ ,

$\therefore xy''+y'=1/2$  is equivalent to  $-\frac{d}{dx}\left[x\frac{dy}{dx}\right]=-\frac{1}{2}$ , so it can be transformed into

$$\delta I=\delta\int_0^1[x(y')^2+y]dx=0.$$

**Eg. Transform  $y''-y=x/2$  into  $\delta I=0$ .**

(Sol.)  $y''+0y'-y=x/2$ ,  $p(x)=\exp\left[\int 0dx\right]=1$ ,  $q(x)=-(-1)\cdot 1=1$ , and  $f(x)=-x/2\cdot 1=-x/2$ ,

$\therefore y''-y=x/2$  is equivalent to  $-\frac{d}{dx}\left[\frac{dy}{dx}\right]+y=-\frac{x}{2}$ , so it can be transformed into

$$\delta I=\delta\int_0^1[(y')^2+y^2+xy]dx=0.$$

**Eg. Transform  $y''=-x^2/2$  into  $\delta I=0$ .**

(Sol.)  $y''+0y'+0y=-x^2/2$ ,  $p(x)=\exp\left[\int 0dx\right]=1$ ,  $q(x)=-0\cdot 1=0$ , and  $f(x)=-\left(-\frac{x^2}{2}\right)\cdot 1=\frac{x^2}{2}$ ,

$\therefore y''=-x^2/2$  is equivalent to  $-\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{x^2}{2}$ , so it can be transformed into

$$\delta I=\int_0^1[(y')^2-x^2y]dx=0.$$

## 14-4 Lagrange's Multiplier Method

To find the relative extrema of  $F(x_1, x_2, \dots, x_n)$  subject to  $k$  conditions  $\varphi_1(x_1, x_2, \dots, x_n)=0$ ,  $\varphi_2(x_1, x_2, \dots, x_n)=0$ ,  $\varphi_3(x_1, x_2, \dots, x_n)=0$ ,  $\varphi_4(x_1, x_2, \dots, x_n)=0$ , ...,  $\varphi_k(x_1, x_2, \dots, x_n)=0$ , we have  $G(x_1, x_2, \dots, x_n; \lambda_1, \lambda_2, \dots, \lambda_k) = F(x_1, x_2, \dots, x_n) + \lambda_1 \varphi_1(x_1, x_2, \dots, x_n) + \dots + \lambda_k \varphi_k(x_1, x_2, \dots, x_n)$

$$\text{subject to the necessary conditions: } \begin{cases} \frac{\partial G}{\partial x_1} = 0 \\ \frac{\partial G}{\partial x_2} = 0 \\ \vdots \\ \frac{\partial G}{\partial x_n} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_1(\lambda_1, \lambda_2, \dots, \lambda_k) \\ x_2 = x_2(\lambda_1, \lambda_2, \dots, \lambda_k) \\ \vdots \\ x_n = x_n(\lambda_1, \lambda_2, \dots, \lambda_k) \end{cases} \Rightarrow \begin{cases} \lambda_1 = ? \\ \lambda_2 = ? \\ \vdots \\ \lambda_k = ? \end{cases}$$

$\Rightarrow F(x_1, x_2, \dots, x_n) = \text{Max or Min.}$

**Eg. Find the extrema of  $F(x, y, z) = x^2 + y^2 + z^2$  subject to the conditions  $x^2/4 + y^2/5 + z^2/25 = 1$  and  $z = x + y$ .**

(Sol.)  $\varphi_1(x, y, z) = x^2/4 + y^2/5 + z^2/25 - 1 = 0$ ,  $\varphi_2(x, y, z) = x + y - z = 0$

$$G(x, y, z; \lambda_1, \lambda_2) = x^2 + y^2 + z^2 + \lambda_1(x^2/4 + y^2/5 + z^2/25 - 1) + \lambda_2(x + y - z)$$

$$\begin{cases} \frac{\partial G}{\partial x} = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0 \\ \frac{\partial G}{\partial y} = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0 \\ \frac{\partial G}{\partial z} = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{-2\lambda_2}{\lambda_1 + 4} \\ y = \frac{-5\lambda_2}{2\lambda_1 + 10} \\ z = \frac{25\lambda_2}{2\lambda_1 + 50} \end{cases}$$

$$\phi_2(x, y, z) = x + y - z = 0 \Rightarrow \lambda_1 = -10 \quad \text{or} \quad -\frac{75}{17}$$

$$1. \lambda_1 = -10 \Rightarrow x = \frac{\lambda_2}{3}, \quad y = \frac{\lambda_2}{2}, \quad z = \frac{5}{6}\lambda_2$$

$$\phi_1(x, y, z) = x^2/4 + y^2/5 + z^2/25 - 1 = 0 \Rightarrow \lambda_2 = \pm 6\sqrt{\frac{5}{19}}$$

$$\Rightarrow (x, y, z) = \left( \pm 2\sqrt{\frac{5}{19}}, \pm 3\sqrt{\frac{5}{19}}, \pm 5\sqrt{\frac{5}{19}} \right) \Rightarrow F(x, y, z) = x^2 + y^2 + z^2 = 10$$

$$2. \lambda_1 = -\frac{75}{17} \Rightarrow x = \frac{34}{7}\lambda_2, \quad y = -\frac{17}{4}\lambda_2, \quad z = \frac{17}{28}\lambda_2 \Rightarrow \lambda_2 = \pm \frac{140}{17\sqrt{646}}$$

$$\Rightarrow (x, y, z) = \left( \pm \frac{40}{\sqrt{646}}, \mp \frac{35}{\sqrt{646}}, \pm \frac{5}{\sqrt{646}} \right) \Rightarrow F(x, y, z) = \frac{75}{17}$$

$\therefore \text{Max} = 10, \text{Min} = 75/17$

**Eg. For a simple lens of focal length  $f$ , the object distance  $p$  and the image distance  $q$  are related by  $1/p+1/q=1/f$ . Find the minimum object-image distance  $p+q$ . [2003中央光電所]**

(Sol.) Let  $G(p,q,\lambda)=p+q+\lambda(1/p+1/q-1/f)$

$$\begin{cases} \frac{\partial G}{\partial p} = 0 \Rightarrow 1 - \frac{\lambda}{p^2} = 0 \Rightarrow p = \pm\sqrt{\lambda} \\ \frac{\partial G}{\partial q} = 0 \Rightarrow 1 - \frac{\lambda}{q^2} = 0 \Rightarrow q = \pm\sqrt{\lambda} \end{cases}, 1/p+1/q=2/\sqrt{\lambda}=1/f \Rightarrow \lambda=4f^2 \Rightarrow p=q=2f \Rightarrow p+q=4f$$

**Eg. Apply Lagrange's multiplier method to find the extrema of  $F(x,y)=x^2+y^2$  under the constraint  $x^2/4+y^2/9=1$ .**

(Sol.) Let  $G(x,y,\lambda) = x^2 + y^2 + \lambda\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right)$

$$\begin{cases} \frac{\partial G}{\partial x} = 0 \Rightarrow 2x + \frac{\lambda}{2}x = 0 \Rightarrow x = 0 \text{ or } \lambda = -4 \\ \frac{\partial G}{\partial y} = 0 \Rightarrow 2y + \frac{2\lambda}{9}y = 0 \Rightarrow y = 0 \text{ or } \lambda = -9 \end{cases}$$

$$1. x = 0 \wedge \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow y = \pm 3$$

$$\Rightarrow x^2 + y^2 = 0 + (\pm 3)^2 = 9$$

$$2. y = 0 \wedge \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = \pm 2$$

$$\Rightarrow x^2 + y^2 = (\pm 2)^2 + 0^2 = 4, \therefore \text{Max}=9, \text{Min}=4$$

**Another method:**

$$1. x = 0 \wedge \lambda = -9$$

$$G(x=0, y; \lambda = -9) = 0^2 + y^2 - 9\left(\frac{0^2}{4} + \frac{y^2}{9} - 1\right) = 9$$

$$2. y = 0 \wedge \lambda = -4$$

$$G(x, y=0; \lambda = -4) = x^2 + 0^2 - 4\left(\frac{x^2}{4} + \frac{0^2}{9} - 1\right) = 4, \therefore \text{Max}=9, \text{Min}=4$$