

Chapter 3 Series Solutions of Differential Equations

3-1 Simple Power Series Solutions of Ordinary Differential Equations

For $y'+g(x)y=r(x)$ or $y''+P(x)y'+Q(x)y=F(x)$, if $g(x)$, $r(x)$, $P(x)$, $Q(x)$, and $F(x)$ are

analytical at zero, then $y(x)=\sum_{n=0}^{\infty} a_n x^n$,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Eg. Solve $(1+x^2)y''+2xy'=0$, $y(0)=0$, $y'(0)=1$. [2004 台大電研]

(Sol.) Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$$(1+x^2)y'' + 2xy' = y'' + x^2 y'' + 2xy' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n$$

$$= \sum_{n=0}^{\infty} [n(n+1) a_n + (n+2)(n+1) a_{n+2}] x^n = 0 \Rightarrow a_{n+2} = -\frac{n}{(n+2)} a_n$$

$$\Rightarrow a_2 = a_4 = a_6 = \dots = 0 \quad \text{and} \quad a_{2n+1} = \frac{(-1)^n}{2n+1} a_1$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) = a_0 + a_1 \tan^{-1}(x)$$

$$y(0)=0, y'(0)=1 \Rightarrow a_0=0 \text{ and } a_1=1 \Rightarrow y(x)=\tan^{-1}(x)$$

Eg. Solve $(1+x^2)y''-2xy'+2y=0$. [1990 台大土木所]

(Sol.) Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(1+x^2)y'' - 2xy' + 2y = y'' + x^2 y'' - 2xy' + 2y$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n-2)(n-1) a_n] x^n = 0 \Rightarrow a_{n+2} = -\frac{(n-1)(n-2)}{(n+2)(n+1)} a_n$$

$$a_2 = -\frac{(-1)(-2)}{1 \cdot 2} a_0 = -a_0, a_4=0, a_6=0, \dots, \text{ and } a_3=0, a_5=0, a_7=0, \dots$$

$$\therefore y(x) = a_0 + a_1 x - a_0 x^2 = a_0(1-x^2) + a_1 x$$

Eg. Solve $y''-xy=0$ by series expansions. [2013 中正電研]

$$(\text{Sol.}) y = \sum_{n=0}^{\infty} a_n x^n, y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n, xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1}x^m = \sum_{n=1}^{\infty} a_{n-1}x^n,$$

$$y'' - xy = \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n + 2a_2 = 0$$

$$\Rightarrow a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, n=1, 2, 3, \dots, a_2=0, \Rightarrow a_3 = \frac{a_0}{3 \cdot 2}, a_4 = \frac{a_1}{4 \cdot 3}, a_5=0,$$

$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, a_8=0, a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2},$$

$$a_{10} = \frac{a_7}{10 \cdot 9} = \frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}, \dots,$$

$$\therefore y = a_0 \left(1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \frac{x^9}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} + \dots \right) + a_1 \left(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \frac{x^{10}}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} + \dots \right)$$

Eg. Solve $y''+x^2y=0$ by series expansions.

$$(\text{Sol.}) y = \sum_{n=0}^{\infty} a_n x^n, y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n, x^2y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{m=2}^{\infty} a_{m-2}x^m = \sum_{n=2}^{\infty} a_{n-2}x^n$$

$$y'' + x^2y = \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-2}]x^n + 2a_2 + 6a_3x = 0$$

$$\Rightarrow a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)}, n = 2, 3, \dots, a_2 = a_3 = 0$$

$$\Rightarrow a_4 = \frac{-a_0}{4 \cdot 3}, a_8 = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3} \quad a_5 = \frac{-a_1}{5 \cdot 4}, a_9 = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4}$$

\vdots

\vdots

$$\therefore y = a_0 \left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \dots \right) + a_1 \left(x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \dots \right)$$

Eg. Solve $y'+ky=0$ by series expansions.

$$(\text{Sol.}) y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + k \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} [(n+1)a_{n+1} + k a_n]x^n = 0$$

$$\Rightarrow a_{n+1} = -\frac{k a_n}{n+1}, n=0, 1, 2, 3, \dots \Rightarrow a_n = \frac{(-1)^n k^n a_0}{n!}, n = 1, 2, 3, \dots$$

$$\therefore y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0 (-1)^n (kx)^n}{n!} = a_0 \cdot \sum_{n=0}^{\infty} \frac{(-kx)^n}{n!} = a_0 e^{-kx}$$

Eg. Solve $(x-1)y''-xy'+y=0$. [1990 台大應力所] (Ans.) $y(x)=c_1 e^x + c_2 x$

Eg. Solve $(x+1)y''-(x+2)y'+y=0$. [1991 台大土木所] (Ans.) $y(x)=c_1 e^x + c_2 (x+2)$

Eg. Solve $y''+k^2y=0$ by series expansions.

$$\text{(Sol.) } y = \sum_{n=0}^{\infty} a_n x^n, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

$$y'' + k^2 y = \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + k^2 a_n] x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{-k^2 a_n}{(n+2)(n+1)}; \quad n = 0, 1, 2, \dots$$

$$\Rightarrow a_2 = \frac{-k^2 a_0}{2 \cdot 1} \quad a_3 = \frac{-k^2 a_1}{3 \cdot 2}$$

$$a_4 = \frac{k^4 a_0}{4 \cdot 3 \cdot 2 \cdot 1} \quad a_5 = \frac{k^4 a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{2n} = \frac{(-1)^n k^{2n} \cdot a_0}{(2n)!} \quad a_{2n+1} = \frac{(-1)^n k^{2n} a_1}{(2n+1)!}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m (kx)^{2m}}{(2m)!} + \frac{a_1}{k} \sum_{m=0}^{\infty} \frac{(-1)^m (kx)^{2m+1}}{(2m+1)!}$$

$$= a_0 \cos(kx) + \frac{a_1}{k} \cdot \sin(kx)$$

Eg. Solve $y''-e^x y=0$ by series expansions.

$$\text{(Sol.) } y = \sum_{n=0}^{\infty} a_n x^n, \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)$$

$$y'' - e^x y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \cdot (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots)$$

$$- \left[a_0 + (a_0 + a_1)x + \left(\frac{a_0}{2} + a_1 + a_2 \right) x^2 + \left(\frac{a_0}{6} + \frac{a_1}{2} + a_2 + a_3 \right) x^3 + \dots \right] = 0$$

$$\Rightarrow 2a_2 - a_0 = 0, \quad 6a_3 - a_0 - a_1 = 0, \quad 12a_4 - \frac{a_0}{2} - a_1 - a_2 = 0$$

$$\Rightarrow a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_0 + a_1}{6}, \quad a_4 = \frac{a_0 + a_1}{12}, \dots$$

$$\Rightarrow y = a_0 + a_1 x + \frac{a_0}{2} x^2 + \left(\frac{a_0 + a_1}{6} \right) x^3 + \left(\frac{a_0 + a_1}{12} \right) x^4 + \dots$$

$$= a_0 \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \dots \right) + a_1 \left(x + \frac{x^3}{6} + \frac{x^4}{12} + \dots \right)$$

$$= a_0 y_0(x) + a_1 y_1(x)$$

Eg. Solve $y''-e^x y=\sin(x)+1$. [1990 台大化工所]

3-2 Method of Frobenius

For $P(x)y''+Q(x)y'+R(x)y=0$. If $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ have a regular singular point at

$x_0=0$, and $\begin{cases} x\frac{Q(x)}{P(x)} \\ x^2\frac{R(x)}{P(x)} \end{cases}$ are analytical at $x_0=0$, then $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ is a

solution $\Rightarrow r^2+Ar+B=0 \Rightarrow r=r_1, r_2$

Case 1 $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer, then

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Case 2 $r_1 - r_2$ is a positive integer, then $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ and

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2} + Ay_1 \ln(x), \text{ where } A \text{ may be } 0.$$

Case 3 $r_1 = r_2$, then $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ and $y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2}$

Eg. Solve $3xy'' + y' - y = 0$.

(Sol.) $y'' + \frac{1}{3x}y' - \frac{1}{3x}y = 0$, $\frac{1}{3x}$ and $-\frac{1}{3x}$ have a regular singular point at 0, but

$\frac{x}{3x} = \frac{1}{3}$ and $-\frac{x^2}{3x} = -\frac{x}{3}$ are analytic at 0.

$$\therefore y = \sum_{n=0}^{\infty} C_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)C_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)C_n x^{n+r-2},$$

$$\begin{aligned} 3xy'' + y' - y &= 3 \sum_{n=0}^{\infty} (n+r)(n+r-1)C_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} \\ &= \sum_{n=0}^{\infty} (n+r)(3n+3r-2)C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} \\ &= r(3r-2)C_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r)(3n+3r-2)C_n x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r} \\ &= r(3r-2)C_0 x^{r-1} + \sum_{m=0}^{\infty} (m+r+1)(3m+3r+1)C_{m+1} x^{m+r} - \sum_{n=0}^{\infty} C_n x^{n+r} \\ &= r(3r-2)C_0 x^{r-1} + \sum_{n=0}^{\infty} [(n+r+1)(3n+3r+1)C_{n+1} - C_n] x^{n+r} \end{aligned}$$

$$C_{n+1} = \frac{C_n}{(n+r+1)(3n+3r+1)} \quad (\text{or } D_{n+1} = \frac{D_n}{(n+r+1)(3n+3r+1)})$$

$\therefore r = \frac{2}{3}, 0 \Rightarrow r_1 - r_2 = \frac{2}{3}$ is not an integer: *Case 1*

$$r_1 = \frac{2}{3} \Rightarrow C_{n+1} = \frac{C_n}{(3n+5)(n+1)} \Rightarrow C_1 = \frac{C_0}{5}, \quad C_2 = \frac{C_1}{8 \cdot 2} = \frac{C_0}{2! \cdot 5 \cdot 8}, \quad \dots,$$

$$C_n = \frac{C_0}{n! \cdot 5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)}$$

$$r_2 = 0 \Rightarrow D_{n+1} = \frac{D_n}{(n+1)(3n+1)} \Rightarrow D_1 = \frac{D_0}{1 \cdot 1}, \quad D_2 = \frac{D_0}{2! \cdot 4}, \quad \dots,$$

$$D_n = \frac{D_0}{n! \cdot 4 \cdot 7 \cdot \dots \cdot (3n-2)}$$

$$\begin{aligned} \therefore y(x) &= C_0 \sum_{n=0}^{\infty} \frac{1}{n! \cdot 5 \cdot 8 \cdot 11 \cdot \dots \cdot (3n+2)} x^{n+\frac{2}{3}} + D_0 \sum_{n=0}^{\infty} \frac{1}{n! \cdot 4 \cdot 7 \cdot \dots \cdot (3n-2)} x^n \\ &= C_0 y_1(x) + D_0 y_2(x) \end{aligned}$$

Eg. Solve $x^2y'' + x^2y' - 2y = 0$.

(Sol.) $y'' + y' - \frac{2}{x^2}y = 0$, $-\frac{2}{x^2}$ has a regular singular point at 0, but $-\frac{2x^2}{x^2} = -2$ is analytic at 0.

$$\therefore y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r}$$

$$x^2 y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r+1} = \sum_{m=1}^{\infty} (m+r-1)a_{m-1} x^{m+r} = \sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^{n+r}$$

$$x^2 y'' + x^2 y' - 2y$$

$$= [r(r-1)a_0 - 2a_0]x^r + \sum_{n=1}^{\infty} [(n+r)(n+r-1)a_n + (n+r-1)a_{n-1} - 2a_n] \cdot x^{n+r} = 0$$

$$\Rightarrow r = 2, -1 \Rightarrow r_1 - r_2 = 3 \text{ is a positive integer: Case 2}$$

$$[(n+r)(n+r-1) - 2]a_n + (n+r-1)a_{n-1} = 0$$

$$r_1 = 2 \Rightarrow a_n = -\frac{n+1}{n(n+3)}a_{n-1} \Rightarrow a_n = (-1)^n \cdot \frac{6a_0}{n!(n+2)(n+3)}$$

$$\therefore y_1(x) = 6a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!(n+1)(n+3)}$$

$$\text{Let } y_2(x) = Ay_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} = Ay_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-1}$$

$$\Rightarrow Ax^2 y_1'' \cdot \ln(x) + 2Axy_1' - Ay_1 + \sum_{n=0}^{\infty} (n-1)(n-2)b_n x^{n-1} + Ax^2 y_1' \ln(x)$$

$$+ Axy_1 + \sum_{n=1}^{\infty} (n-2)b_{n-1} x^{n-1} - 2 \sum_{n=0}^{\infty} b_n x^{n-1} - 2Ay_1 \ln(x) = 0$$

$$\Rightarrow A(2xy_1' + xy_1 - y_1) + 2b_0 x^{-1} - 2b_0 x^{-1} + \sum_{n=1}^{\infty} [n(n-3)b_n + (n-2)b_{n-1}] x^{n-1} = 0$$

$$\Rightarrow A = 0, \quad b_n = -\frac{n-2}{n(n-3)}b_{n-1} \Rightarrow b_1 = -\frac{1}{2}b_0, \quad b_2 = 0, \quad b_3 = 0, \quad \dots$$

$$\therefore y_2(x) = b_0 \left(\frac{1}{x} - \frac{1}{2} \right)$$

$$\Rightarrow \begin{cases} y_1(x) = 6 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!(n+1)(n+3)} \\ y_2(x) = \frac{1}{x} - \frac{1}{2} \end{cases} \Rightarrow y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Eg. Solve $xy'' + (1-2x)y' + (x-1)y = 0$. [1990 交大資訊所]

(Ans.) $y(x) = c_1 e^x + c_2 e^x \cdot \ln x$

Eg. Solve $x^2y'' + 5xy' + (x+4)y = 0$.

(Sol.) $y'' + \frac{5}{x}y' + \frac{x+4}{x^2}y = 0$, $\frac{5}{x}$ and $\frac{x+4}{x^2}$ have a regular singular point at 0, but $\frac{5x}{x} = 5$ and $x^2 \cdot \frac{x+4}{x^2} = x+4$ are analytic at 0.

$$\therefore y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x^2 y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}$$

$$xy' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r}, \quad xy = \sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{m=1}^{\infty} a_{m-1} x^{m+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$$\therefore x^2 y'' + 5xy' + (x+4)y = \sum_{n=1}^{\infty} [(n+r)(n+r-1)a_n + 5(n+r)a_n + a_{n-1} + 4a_n] x^{n+r} + [r(r-1) + 5r + 4]a_0 x^r = 0$$

$$\Rightarrow r^2 + 4r + 4 = 0, \quad r = -2, -2: \text{Case 3}$$

$$[(n+r)(n+r+4) + 4]a_n + a_{n-1} = 0$$

$$r = -2 \Rightarrow a_n = -\frac{a_{n-1}}{n^2} \Rightarrow a_n = \frac{(-1)^n a_0}{(n!)^2}$$

$$\therefore a_0 y_1(x) = a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \cdot x^{n-2}$$

$$\text{Let } y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n-2}$$

$$\begin{aligned} \Rightarrow 4y_1 + 2xy_1' + \sum_{n=1}^{\infty} (n-2)(n-3)b_n x^{n-2} + \sum_{n=1}^{\infty} 5(n-2)b_n x^{n-2} + \sum_{n=1}^{\infty} b_n x^{n-1} \\ + \sum_{n=1}^{\infty} 4b_n x^{n-2} + \ln(x) \cdot [x^2 y_1'' + 5xy_1' + (x+4)y_1] = 0 \end{aligned}$$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \cdot x^{n-2} \Rightarrow b_1 = 2, \quad b_n = \frac{-b_{n-1}}{n^2} - \frac{2(-1)^n}{n(n!)^2}$$

$$\Rightarrow y_2(x) = y_1 \ln(x) + \frac{2}{x} - \frac{3}{4} + \frac{11}{108}x - \frac{25}{576}x^2 + \dots$$