Chapter 7 Vector Analysis

7-1 Vector Functions

One-variable vector function: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Multi-variable vector function: $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$

The *derivatives* of vector functions: $\dfrac{d\vec{R}(t)}{df(t)} = \dfrac{d\vec{R}(t)}{dt} \cdot \dfrac{dt}{df(t)} = \dfrac{d\vec{R}(t)}{dt} \Big/ \dfrac{df(t)}{dt}$

$$\frac{\partial \vec{F}(x, y, z)}{\partial g(x, y, z)} = \frac{\partial \vec{F}(x, y, z)}{\partial x} \cdot \frac{\partial x}{\partial g(x, y, z)} + \frac{\partial \vec{F}(x, y, z)}{\partial y} \cdot \frac{\partial y}{\partial g(x, y, z)} + \frac{\partial \vec{F}(x, y, z)}{\partial z} \cdot \frac{\partial z}{\partial g(x, y, z)}$$

Eg. For $\vec{R}(t) = 2t\hat{i} - \cos(3t)\hat{j} + t^3\hat{k}$, $0 \le t \le 1$, **find** $\vec{R}'(t) = d\vec{R}(t)/dt$. **Let** $s(t) = \int_0^1 \sqrt{4 + 9\sin^2(3t) + 9t^4} \cdot dt$, **then find** $d\vec{R}(t)/ds(t)$.

(Sol) $d\vec{R}(t)/dt = 2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}$

$$\therefore \quad \frac{d\vec{R}(t)}{ds(t)} = \frac{d\vec{R}(t)}{dt} \cdot \frac{dt}{ds(t)} = \frac{2\hat{i} + 3\sin(3t)\hat{j} + 3t^2\hat{k}}{\sqrt{4 + 9\sin^2(3t) + 9t^4}}$$

Some theorems of derivatives of vector functions:

1. $(\vec{F} \cdot \vec{G})' = \vec{F}' \cdot \vec{G} + \vec{F} \cdot \vec{G}'$

2. $(\vec{F} \times \vec{G})' = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$

3. $(\vec{F} \times \vec{F}')' = \vec{F} \times \vec{F}''$

(Proof) $(\vec{F} \times \vec{F}')' = \vec{F}' \times \vec{F}' + \vec{F} \times \vec{F}'' = \vec{F} \times \vec{F}''$

4. For $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, **if** $\vec{R}(t)$ **does not change direction, then** $\vec{R}(t) \times \vec{R}'(t) = 0$, **and vice versa.**

5. Let $\vec{R}(t)$ **denote the position of a particle at time** *t***. If the particle moves so that equal areas are swept out in equal times, then we have** $\vec{R}(t) \times \vec{R}''(t) = 0$, **and vice versa. (Kepler's law)**

(Proof)

$$\text{Area} = \frac{1}{2}R^2\theta \quad \text{and} \quad |\vec{R}(t + \Delta t) - \vec{R}(t)| \approx R\theta$$

$$\therefore \ 2\,\text{area} = R^2\theta = \vec{R}(t) \times [\vec{R}(t + \Delta t) - \vec{R}(t)]$$
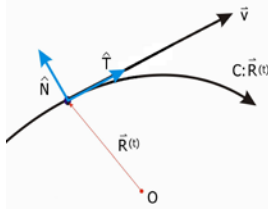
If area $= 0 \Leftrightarrow \vec{R}(t) \times [\vec{R}(t + \Delta t) - \vec{R}(t)] = 0 \Leftrightarrow \vec{R}(t) \times \vec{R}'(t) \cdot \Delta t = 0$

$$\Leftrightarrow \vec{R}(t) \times \vec{R}'(t) = 0$$

Equal area in equal time

$$\Leftrightarrow \vec{R}(t) \times \vec{R}'(t) = \text{constant} \Leftrightarrow [\vec{R}(t) \times \vec{R}'(t)]' = 0 \Leftrightarrow \vec{R}(t) \times \vec{R}''(t) = 0$$

7-2 Differential Geometry

Position vector: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Velocity: $\vec{v}(t) = \vec{R}'(t) = \dfrac{dx(t)}{dt}\hat{i} + \dfrac{dy(t)}{dt}\hat{j} + \dfrac{dz(t)}{dt}\hat{k}$

Arc length: $s(t) = \displaystyle\int_{t_1}^{t_2} |\vec{R}'(t)|\, dt, \quad \dfrac{ds}{dt} = |\vec{R}'(t)| = |\vec{v}(t)|$

Acceleration: $\vec{a}(t) = \vec{v}'(t) = \dfrac{d^2 x(t)}{dt^2}\hat{i} + \dfrac{d^2 y(t)}{dt^2}\hat{j} + \dfrac{d^2 z(t)}{dt^2}\hat{k}$

Curvature: $\kappa = \left|\dfrac{d\hat{T}}{ds}\right|$, where $\hat{T} = \dfrac{\vec{R}'(t)}{|\vec{R}'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{d\vec{R}(t)}{ds(t)}$

Eg. C: $\vec{R}(t) = t\hat{i} + (t-2)\hat{j} + (3t-1)\hat{k}$ **is a straight line.**

$$\hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}}, \quad \kappa = \left|\dfrac{d\hat{T}}{ds}\right| = \left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{dt}{ds}\right| = 0.$$

Eg. C: $\vec{R}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + 4\hat{k}$ **is a circle of radius 2 at z=4.**

$$\hat{T} = \dfrac{-2\sin(t)\hat{i} + 2\cos(t)\hat{j}}{2}, \quad \kappa = \left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{dt}{ds}\right| = \dfrac{1}{2} = \dfrac{1}{r}.$$

Theorem $\vec{a} = \dfrac{d|\vec{v}|}{dt}\hat{T} + \dfrac{|\vec{v}|^2}{\rho}\hat{N} = $ **tangential acceleration+centripetal acceleration**

(Proof) $\vec{a}(t) = \dfrac{d\vec{v}(t)}{dt} = \dfrac{d}{dt}[|\vec{v}(t)|\hat{T}] = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\dfrac{d\hat{T}}{dt}$

$= \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|\cdot\left(\dfrac{ds}{dt}\cdot\dfrac{d\hat{T}}{ds}\right) = \dfrac{d|\vec{v}(t)|}{dt}\cdot\hat{T} + |\vec{v}(t)|^2\dfrac{d\hat{T}}{ds}$. Define $\hat{N} = \rho\dfrac{d\hat{T}}{ds}$

$\Rightarrow \vec{a}(t) = \dfrac{d|\vec{v}(t)|}{dt}\hat{T} + \dfrac{|\vec{v}(t)|^2}{\rho}\hat{N} \quad (\hat{N}\perp\hat{T} \Leftarrow \hat{T}\cdot\hat{T} = 1), \quad \dfrac{d(\hat{T}\cdot\hat{T})}{ds} = 0$

Eg. For $\vec{R}(t) = [\cos(t) + t\sin(t)]\hat{i} + [\sin(t) - t\cos(t)]\hat{j} + t^2\hat{k}$ **, t>0, we have**

$\vec{v}(t) = t\cos(t)\hat{i} + t\sin(t)\hat{j} + 2t\hat{k}$

$\vec{a}(t) = [\cos(t) - t\sin(t)]\hat{i} + [\sin(t) + t\cos(t)]\hat{j} + 2\hat{k}$

$\hat{T} = \dfrac{\vec{v}(t)}{|v(t)|} = \dfrac{1}{\sqrt{5}}\cos(t)\hat{i} + \dfrac{1}{\sqrt{5}}\sin(t)\hat{j} + \dfrac{2}{\sqrt{5}}\hat{k}$

$\rho = \dfrac{1}{\kappa} = \dfrac{1}{\left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{dt}{ds}\right|} = \dfrac{1}{\left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{1}{|\vec{v}(t)|}\right|} = 5t$

$\hat{N} = \rho\dfrac{d\hat{T}}{ds} = \rho\dfrac{dt}{ds}\cdot\dfrac{d\hat{T}}{dt} = \dfrac{\rho}{|\vec{v}(t)|}\cdot\dfrac{d\hat{T}}{dt} = -\sin(t)\hat{i} + \cos(t)\hat{j}$

$\therefore \ \vec{a}(t) = \sqrt{5}\hat{T} + t\hat{N}$ **,** $a_t = \sqrt{5}$ **,** $a_n = t$

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Binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

Fernet formulae:
$$\begin{cases} \dfrac{d\hat{T}}{ds} = \kappa\hat{N} = \dfrac{\hat{N}}{\rho} & (1) \\[3mm] \dfrac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B} & (2) \\[3mm] \dfrac{d\hat{B}}{ds} = -\tau\hat{N} & (3) \end{cases}$$

Torsion **of a curve: τ**

(Proof of (3)) $\dfrac{d\hat{B}}{ds} = \dfrac{d}{ds}(\hat{T} \times \hat{N}) = \dfrac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \dfrac{d\hat{N}}{ds} = \hat{T} \times (-\kappa\hat{T}) + \hat{T} \times (\tau\hat{B}) = -\tau\hat{N}$

Note: $\hat{T}, \ \hat{N}, \ and \ \hat{B}$ are unit vectors.

Basic theorems of curvature and torsion:

$\kappa = \dfrac{|\vec{R}' \times \vec{R}''|}{|\vec{R}'|^3}$, $\tau = [\hat{T}, \hat{N}, \hat{N}'] = \dfrac{1}{\kappa^2}[\vec{R}', \vec{R}'', \vec{R}''']$, **where** $[\vec{A}, \vec{B}, \vec{C}] \equiv \vec{A} \cdot (\vec{B} \times \vec{C})$

Eg. For $C : \vec{R}(t) = 3\cos(t)\hat{i} + 3\sin(t)\hat{j} + 4t\hat{k}$**, find** $\hat{T}, \ \hat{N}, \ \hat{B}$**, κ, τ, and ρ.**

(Sol.) $\hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = -\dfrac{3}{5}\sin(t)\hat{i} + \dfrac{3}{5}\cos(t)\hat{j} + \dfrac{4}{5}\hat{k}$, $\kappa = \dfrac{|\vec{R}' \times \vec{R}''|}{|R'|^3} = \dfrac{3}{25}$, $\rho = \dfrac{1}{\kappa} = \dfrac{25}{3}$

$\hat{N} = \rho\dfrac{d\hat{T}}{ds} = \dfrac{\rho}{|\vec{v}(t)|} \cdot \dfrac{d\hat{T}}{dt} = -\cos(t)\hat{i} - \sin(t)\hat{j}$, $\hat{B} = \hat{T} \times \hat{N} = \dfrac{4}{5}\sin(t)\hat{i} - \dfrac{4}{5}\cos(t)\hat{j} + \dfrac{3}{5}\hat{k}$

$\dfrac{d\hat{B}}{ds} = \dfrac{1}{|\vec{v}(t)|} \dfrac{d\hat{B}}{dt} = \dfrac{4}{25}\cos(t)\hat{i} + \dfrac{4}{25}\sin(t)\hat{j} = -\tau\hat{N} = -\tau(-\cos(t)\hat{i} - \sin(t)\hat{j})$

$$\Rightarrow \tau = \dfrac{4}{25}$$

Eg. Consider the curve: $\hat{r} = a\cos(t)\hat{i} + a\sin(t)\hat{j} + bt\hat{k}$**, $0 \le t \le 2\pi$. What is the equation of tangential vector at $t=\pi/2$. 【1991 中山機研】**

(Sol.) $\vec{v} = \dfrac{d\vec{r}}{dt} = -a\sin(t)\hat{i} + a\cos(t)\hat{j} + b\hat{k}$. At $t = \dfrac{\pi}{2} \Rightarrow \vec{v} = -a\hat{i} + b\hat{k}$

$\hat{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{-a\hat{i} + b\hat{k}}{\sqrt{a^2 + b^2}}$

7-3 Gradient, Divergence, and Curl in the Rectangular Coordinate System

Gradient $\nabla\phi(x,y,z)$: $\nabla\phi(x,y,z) = \frac{\partial\phi(x,y,z)}{\partial x}\hat{i} + \frac{\partial\phi(x,y,z)}{\partial y}\hat{j} + \frac{\partial\phi(x,y,z)}{\partial z}\hat{k}$ is a vector function if $\phi(x,y,z)$ is a scalar function.

Eg. For $\phi(x,y,z) = e^{xyz}$, $\nabla\phi(x,y,z) = yze^{xyz}\hat{i} + xze^{xyz}\hat{j} + xye^{xyz}\hat{k}$.

Divergence $\nabla\cdot\vec{F}(x,y,z)$: $\nabla\cdot\vec{F}(x,y,z) = \frac{\partial F_1(x,y,z)}{\partial x} + \frac{\partial F_2(x,y,z)}{\partial y} + \frac{\partial F_3(x,y,z)}{\partial z}$

is a scalar function if $\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$ is a vector function.

Curl $\nabla\times\vec{F}(x,y,z)$:

$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$ is a vector

function if $\vec{F}(x,y,z) = F_1(x,y,z,)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$ is a vector function.

Eg. $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, **compute** $\nabla\cdot\vec{F}$ **and** $\nabla\times\vec{F}$.

(Sol.) $\nabla\cdot\vec{F} = 1+1+1 = 3$, $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$

Eg. $\vec{F}(x,y,z) = x^2\hat{i} - 2x^2y\hat{j} + 2yz^4\hat{k}$, **find** $\nabla\times\vec{F}$ **and** $\nabla\cdot\vec{F}$ **at (1,-1,1).** 【1991 中山電研】

(Sol.) $\nabla\cdot\vec{F} = 2x - 2x^2 + 8yz^3$, at $(1,-1,1) \Rightarrow \nabla\cdot\vec{F} = -8$

$\nabla\times\vec{F} = 2z^4\hat{i} - 4xy\hat{k}$, at $(1,-1,1) \Rightarrow \nabla\times\vec{F} = 2\hat{i} + 4\hat{k}$

Laplacian operator: $\nabla^2\phi = \nabla\cdot\nabla\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$

Eg. For $\phi(x,y,z) = e^{xyz}$, $\nabla^2\phi(x,y,z) = y^2z^2e^{xyz} + x^2z^2e^{xyz} + x^2y^2e^{xyz}$.

Theorems (*a*) $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$

(*b*) $\nabla\cdot(\nabla\times\vec{F}) = 0$, $\forall\vec{F}\in C^2$

(*c*) $\nabla\times(\nabla\phi) = 0$, $\forall\phi\in C^2$

(*d*) $\nabla\times(\vec{F}\times\vec{G}) = (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G} + (\nabla\cdot\vec{G})\vec{F} - (\nabla\cdot\vec{F})\vec{G}$

(*e*) $\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F} + \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F})$

(*f*) $\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$

(*g*) $\nabla\cdot(\phi\vec{F}) = (\nabla\phi)\cdot\vec{F} + \phi(\nabla\cdot\vec{F})$

Theorem $\nabla\phi(x,y,z) \perp$ **the surface of** $\phi(x,y,z)$**=constant.**

(Proof) $\because$ $\phi(x,y,z)$=constant

$$\therefore \ d\phi(x,y,z) = 0 = \frac{\partial\phi(x,y,z)}{\partial x}dx + \frac{\partial\phi(x,y,z)}{\partial y}dy + \frac{\partial\phi(x,y,z)}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k}) = \nabla\phi\cdot d\vec{R}$$

$\therefore$ $\nabla\phi(x,y,z) \perp d\vec{R}$, and $d\vec{R}$ is the tangential increment on the surface $\phi(x,y,z)$

Eg. Find the tangential plane and normal line to $z=x^2+y^2$ at (2,-2,8).

(Sol.) Let $\phi(x,y,z)=z-x^2-y^2$, $\nabla\phi = -2x\hat{i} - 2y\hat{j} + \hat{k}$, and (2,-2,8) is on the surface.

For $z-x^2-y^2=0$, the normal vector at (2,-2,8) is $-4\hat{i} + 4\hat{j} + \hat{k}$.

The tangential plane at (2,-2,8) is $-4(x-2) + 4(y+2) + (z-8) = 0 \Rightarrow -4x + 4y + z = -8$

The normal line is $\dfrac{x-2}{-4} = \dfrac{y+2}{4} = \dfrac{z-8}{1}$.

Eg. Find a unit normal vector of $z^2=4(x^2+y^2)$ at (1,0,2). 【1993 中山電研、2007 師大光電所】

Theorem $\nabla\cdot\vec{F}(x,y,z)$ **is the outward flux per unit volume of the flow at point (x,y,z) and time t.**

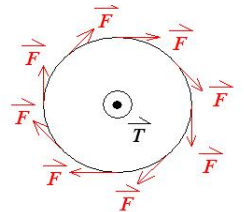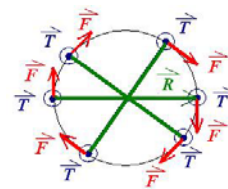(Proof) $\vec{F} \ at \ P = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

The x-direction flux $= \left(F_x + \dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\Delta z - \left(F_x - \dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\Delta z = \dfrac{\partial F_x}{\partial x}\Delta x\Delta y\Delta z$

The y-direction flux $= \dfrac{\partial F_y}{\partial y}\Delta x\Delta y\Delta z$, and the z- direction flux $= \dfrac{\partial F_z}{\partial z}\Delta x\Delta y\Delta z$

$$\therefore \text{Total flux} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\cdot\Delta x\Delta y\Delta z / \Delta x\Delta y\Delta z = \nabla\cdot\vec{F}$$

Theorem If $\vec{T} = \vec{F}\times\vec{R}$ **and** $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$**, then** $\vec{F} = \dfrac{1}{2}\nabla\times\vec{T}$**.**

(Proof) $\nabla\times\vec{T} = \nabla\times(\vec{F}\times\vec{R}) = \nabla\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \\ x & y & z \end{vmatrix}$
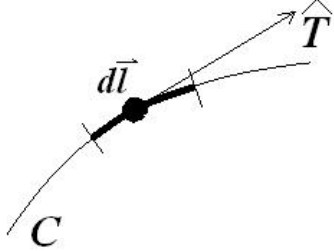

$$= \nabla\times[(F_2z - F_3y)\hat{i} + (F_3x - F_1z)\hat{j} + (F_1y - F_2x)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2z - F_3y & F_3x - F_1z & F_1y - F_2x \end{vmatrix} = 2(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) = 2\vec{F} \Rightarrow \vec{F} = \frac{1}{2}\nabla\times\vec{T}$$

7-4 Line Integrals & Surface Integrals

Line integral: $\exists$ Curve C: $\vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \ \ a \le t \le b$

$\exists$ Vector $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$

(a) $\displaystyle\int_C \vec{F} \cdot d\vec{l} = \int_a^b \vec{F}[x(t), y(t), z(t)] \cdot \vec{R}'(t)dt$,

(b) $\displaystyle\int_C \vec{F} \times d\vec{l} = \int_a^b \vec{F}[x(t), y(t), z(t)] \times \vec{R}'(t)dt$,

(c) $\displaystyle\int_C f(x, y, z)dl = \int_a^b f[x(t), y(t), z(t)] \cdot | \vec{R}'(t) | \, dt$

Eg. For $C : \vec{R}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}$, $\ \mathbf{0 \le t \le 2\pi},$ **and** $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ **, determine** $\displaystyle\int_C \vec{F} \cdot d\vec{l}$ **and** $\displaystyle\int_C \vec{F} \times d\vec{l}$ **.**

(Sol.) $C : \vec{R}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}$, $\vec{R}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j}$

$\vec{F}(x(t), y(t), z(t)) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}$

$\vec{F} \cdot d\vec{l} = \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t)dt = (\cos(t)\hat{i} + \sin(t)\hat{j} + \hat{k}) \cdot (-\sin(t)\hat{i} + \cos(t)\hat{j})dt$

$= (-\cos(t)\sin(t) + \sin(t)\cos(t))dt = 0$

$$\vec{F} \times d\vec{l} = \vec{F}(x(t), y(t), z(t)) \times \vec{R}'(t)dt = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(t) & \sin(t) & 1 \\ -\sin(t) & \cos(t) & 0 \end{vmatrix} dt$$
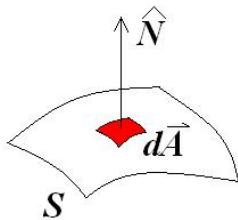
$$= [-\cos(t)\hat{i} - \sin(t)\hat{j} + \hat{k}]\,dt$$

$\therefore \ \displaystyle\int_C \vec{F} \cdot d\vec{l} = 0$, $\displaystyle\int_C \vec{F} \times d\vec{l} = \int_0^{2\pi} [-\cos(t)\hat{i} - \sin(t)\hat{j} + \hat{k}]dt = 2\pi\hat{k}$

Surface integral: $\exists$ Surface $S : \ \phi(x, y, z) = z - \varphi(x, y) = 0$. $\vec{N} = -\dfrac{\partial \varphi}{\partial x}\hat{i} - \dfrac{\partial \varphi}{\partial y}\hat{j} + \hat{k}$

is a normal vector on S, and $\hat{N} = \vec{N} / | \vec{N} |$.

$\exists$ Vector $\vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$

(a) $\displaystyle\iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y, z) \cdot \hat{N} \sqrt{1 + \left(\dfrac{\partial \varphi}{\partial x}\right)^2 + \left(\dfrac{\partial \varphi}{\partial y}\right)^2} \, dxdy$,

(b) $\displaystyle\iint_S \vec{F} \times d\vec{A} = \iint_D \vec{F}(x, y, z) \times \hat{N} \sqrt{1 + \left(\dfrac{\partial \varphi}{\partial x}\right)^2 + \left(\dfrac{\partial \varphi}{\partial y}\right)^2} \, dxdy$,

(c) $\displaystyle\iint_S f(x, y, z)dA = \iint_D f(x, y, z) \sqrt{1 + \left(\dfrac{\partial \varphi}{\partial x}\right)^2 + \left(\dfrac{\partial \varphi}{\partial y}\right)^2} \, dxdy$,

Note: $d\vec{l} = \hat{T}dl$ is parallel to the tangential direction of the curve, but $d\vec{A} = \hat{N}dA$ is normal to the surface.

Eg. For $S: x^2 + y^2 + z^2 = 1, z \geq 0$ **, and** $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$ **, determine** $\iint\limits_S \vec{F} \cdot d\vec{A}$ **and** $\iint\limits_S \vec{F} \times d\vec{A}$ **.**

(Sol.) $S: x^2 + y^2 + z^2 = 1 \wedge z \geq 0 \Rightarrow z = \sqrt{1 - x^2 - y^2}$, $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1-x^2-y^2}} = -\dfrac{x}{z}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{1-x^2-y^2}} = -\dfrac{y}{z} \Rightarrow \vec{N} = -\dfrac{\partial z}{\partial x}\hat{i} - \dfrac{\partial z}{\partial y}\hat{j} + \hat{k} = \dfrac{x}{z}\hat{i} + \dfrac{y}{z}\hat{j} + \hat{k}$,

$\hat{N} = \dfrac{\vec{N}}{|\vec{N}|} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$, and $x^2 + y^2 + z^2 = 1$

$\Rightarrow \vec{F} \cdot d\vec{A} = \vec{F} \cdot \hat{N}\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\,dxdy = \dfrac{x^2 + y^2 + z^2}{z}dxdy = \dfrac{dxdy}{\sqrt{1-x^2-y^2}}$

$\Rightarrow \vec{F} \times d\vec{A} = \vec{F} \times \hat{N}\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\,dxdy = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ x & y & z \end{vmatrix}\dfrac{dxdy}{z} = 0$

$\therefore \iint\limits_S \vec{F} \cdot d\vec{A} = \iint\limits_{x^2+y^2\leq1} \dfrac{dxdy}{\sqrt{1-x^2-y^2}} = \int_0^{2\pi}\int_0^1 \dfrac{rdrd\theta}{\sqrt{1-r^2}} = 2\pi$, $\iint\limits_S \vec{F} \times d\vec{A} = 0$

Eg. $f = \dfrac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$ **, find** $\iint\limits_S \nabla f \cdot \hat{n}\,dA$ **for S: $x^2+y^2+z^2=a^2$.** 【1990 中山電研】

(Sol.) $S: x^2 + y^2 + z^2 = a^2 \Rightarrow z = \pm\sqrt{a^2 - x^2 - y^2}$,

For $z = \sqrt{a^2 - x^2 - y^2} \Rightarrow \vec{N} = -\dfrac{\partial z}{\partial x}\hat{i} - \dfrac{\partial z}{\partial y}\hat{j} + \hat{k} = \dfrac{x}{z}\hat{i} + \dfrac{y}{z}\hat{j} + \hat{k}$,

$\hat{N} = \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$, $\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2} = \dfrac{a}{z}$

$\nabla f = \nabla(\dfrac{1}{r}) = \nabla(\dfrac{1}{\sqrt{x^2+y^2+z^2}}) = \dfrac{-1}{(\sqrt{x^2+y^2+z^2})^3}(x\hat{i} + y\hat{j} + z\hat{k}) = \dfrac{-1}{a^3}(x\hat{i} + y\hat{j} + z\hat{k})$

$\Rightarrow \nabla(\dfrac{1}{r}) \cdot \hat{n}\,dA = \nabla(\dfrac{1}{r}) \cdot \hat{N}\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\,dxdy = -\dfrac{x^2+y^2+z^2}{a^3 z}dxdy = \dfrac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}}$

$\iint\limits_{z=\sqrt{a^2-x^2-y^2}}\nabla f \cdot \hat{n}\,dA = \iint\limits_{x^2+y^2\leq a^2} \dfrac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}} = \dfrac{-1}{a}\int_0^{2\pi}\int_0^a \dfrac{rdrd\theta}{\sqrt{a^2-r^2}} = \dfrac{-2\pi}{a}\int_0^a \dfrac{rdr}{\sqrt{a^2-r^2}}$

=-2π. Similarly, for $z = -\sqrt{a^2 - x^2 - y^2}$, $\iint\limits_{z=-\sqrt{a^2-x^2-y^2}}\nabla f \cdot \hat{n}\,dA = -2\pi$

$\therefore \iint\limits_S \nabla f \cdot \hat{n}\,dA = (-2\pi)+(-2\pi)=-4\pi$

Eg. Compute the surface integral $\iint\limits_S (x+y+z)dA$ **, where** $S : z = x+y, 0 \le y \le x, 0 \le x \le 1$. 【1990 台大農工研】

(Sol.) $\sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2} = \sqrt{3}$, $\iint\limits_S (x+y+z)dA = \int_0^1 \int_0^x 2(x+y)\cdot\sqrt{3}dydx = \sqrt{3}$

Eg. Find $\int\limits_C \vec{F}\cdot d\vec{r}$ **, where** $\vec{F} = e^x\sin y\hat{i} + e^x\cos y\hat{j} + z^2\hat{k}$

and $C : \ \vec{r}(t) = t^{1/2}\hat{i} + t^3\hat{j} + e^{\sqrt{t}}\hat{k}, 0 \le t \le 1$. 【1990 清大核工】

(Ans.) $e\cdot\sin(1) + \dfrac{e^3}{3} + \dfrac{2}{3}$

Eg. $\vec{F} = \hat{i}xy + \hat{j}yz + \hat{k}xz$ **, evaluate** $\iint\limits_S \vec{F}\cdot\hat{n}dA$ **for** $S : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

Green's theorem Let C be a regular, closed, positively-oriented curve enclosing a region D, $\vec{F}(x,y) = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$.

$$\oint\limits_C F_1(x,y)dx + F_2(x,y)dy = \iint\limits_D \left[\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right]dxdy$$

Eg. $\hat{F} = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ **, find** $\oint\limits_C \vec{F}\cdot d\vec{r}$ **, where C is any closed curve.**

(Sol.) $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0, \forall(x,y) \ne 0$, $\oint\limits_C \vec{F}\cdot d\vec{r} = 0$ if C does not enclose 0.

Else, $\oint \vec{F}\cdot d\vec{r} = 2\pi$

Stokes's theorem Let S be a regular surface with coherently oriented boundary C,
$\iint\limits_S (\nabla\times\vec{F})\cdot d\vec{A} = \oint\limits_C \vec{F}\cdot d\vec{\ell}$.

Divergence theorem Let S be a regular, positive-oriented closed surface, enclosing a region V, $\oiint\limits_S \vec{F}\cdot d\vec{A} = \iiint\limits_V (\nabla\cdot\vec{F})dxdydz$.

Eg. Compute $\oiint\limits_S \vec{F}\cdot d\vec{A}$ **, where** $\vec{F} = (y^2+z^2)^{\frac{2}{3}}\hat{i} + \sin(x^2+z)\hat{j} + e^{x^2-y^2}\hat{k}$ **,**

$S : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. 【1990 成大工程科學所】

(Sol.) $\nabla\cdot\vec{F} = 0 \Rightarrow \oiint\limits_S \vec{F}\cdot d\vec{A} = \iiint\limits_V (\nabla\cdot\vec{F})dxdydz = 0$

Eg. $f = \dfrac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, **find** $\displaystyle\iint_S \nabla f \cdot \hat{n}\,dA$ **for S: $x^2+y^2+z^2=a^2$.** 【1990 中山 電研】

(Sol.)

$$\vec{F} = \nabla f = \nabla\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{-1}{(\sqrt{x^2 + y^2 + z^2})^3}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{-1}{a^3}(x\hat{i} + y\hat{j} + z\hat{k}),$$

$$\nabla \cdot \vec{F} = \frac{-3}{a^3}, \quad \iint_S \nabla f \cdot \hat{n}\,dA = \oiint_S \vec{F} \cdot d\vec{A} = \iiint_V (\nabla \cdot \vec{F})\,dxdydz = \frac{-3}{a^3} \cdot \frac{4a^3\pi}{3} = -4\pi$$

Green's identities

ϕ, ψ are scalars,
$$\begin{cases}(a)\displaystyle\iiint_V [\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi]\,dxdydz = \iint_S (\phi\nabla\psi) \cdot d\vec{A} \\[4mm] (b)\displaystyle\iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dxdydz = \iint_S [\phi\nabla\psi - \psi\nabla\phi]\,d\vec{A}\end{cases}.$$

7-5 Potential Theory

Potential: ϕ is called a potential for the vector field $\vec{F}$ if $\vec{F} = \nabla\phi$ or $\vec{F} = -\nabla\phi$

Test for a potential: If $\vec{F}$ and $\nabla \cdot \vec{F}$ are continuous in a simply-connected domain Ω, then $\vec{F}$ has a potential function. $\Leftrightarrow \nabla \times \vec{F} = 0$

(Proof) ＂$\Rightarrow$＂ : $\vec{F} = \pm\nabla\phi \Rightarrow \nabla \times \vec{F} = 0$.

＂$\Leftarrow$＂ : $\nabla \times \vec{F} = 0$, $\displaystyle\oint_C \vec{F} \cdot d\vec{\ell} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = 0$

Let C be the boundary of $\phi(x,y,z)$=constant, then we choose $\vec{F} = \pm\nabla\phi$, then $\displaystyle\int_C \vec{F} \cdot d\vec{\ell} = 0$ is always valid. $\Rightarrow \vec{F}$ has a potential function.

Eg. Check (a) $\vec{F} = 2xy\hat{i} + z^2\hat{j} + (x - y + z)\hat{k}$,

and (b) $\vec{F} = (yze^{xyz} - 4x)\hat{i} + (xze^{xyz} + z)\hat{j} + (xye^{xyz} + y)\hat{k}$, which does have a potential? 【1991 成大機研】

(Sol.) (a) $\nabla \times \vec{F} = (-2z - 1)\hat{i} - \hat{j} - 2x\hat{k} \neq 0$, $\therefore$ no potential.

(b) $\nabla \times \vec{F} = 0$, $\therefore$ there exists a potential.

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (yze^{xyz} - 4x)\hat{i} + (xze^{xyz} + z)\hat{j} + (xye^{xyz} + y)\hat{k}$$

$$\Rightarrow \phi(x, y, z) = e^{xyz} - 2x^2 + I(y, z) = e^{xyz} + yz + J(x, z) = e^{xyz} + yz + K(x, y)$$

$$\Rightarrow \phi(x, y, z) = e^{xyz} - 2x^2 + zy + \text{Constant}$$

Eg. $\vec{F} = (x^2 + y^2 + z^2)^n(x\hat{i} + y\hat{j} + z\hat{k})$, **find a scalar potential $\phi(x,y,z)$ so that** $\vec{F} = -\nabla\phi$. 【1990 台大材研】 (Ans.) $\phi(x, y, z) = \dfrac{-(x^2 + y^2 + z^2)^{n+1}}{2(n + 1)} + C$

Theorem If $\vec{F}$ has a potential, then the line integral of $\vec{F}$ is independent of path in Ω. That is, $\int_C \vec{F} \cdot d\vec{\ell} = \int_K \vec{F} \cdot d\vec{\ell}$, whenever C and K are regular curves in Ω with the same initial point and the same terminal point.

(Proof) $\int_C \vec{F} \cdot d\vec{\ell} - \int_K \vec{F} \cdot d\vec{\ell} = \oint_{C'} \vec{F} \cdot d\vec{\ell}$

If $\vec{F}$ has a potential, then $\oint_{C'} \vec{F} \cdot d\vec{\ell} = 0$, $\therefore$ $\int_C \vec{F} \cdot d\vec{\ell} = \int_K \vec{F} \cdot d\vec{\ell}$.

Theorem Let $\vec{F}$ be a 2-D vector field of a simply-connected domain Ω. Then $\vec{F}$ has a potential on Ω. $\Leftrightarrow \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$. (In this case, $\vec{F} = F_1(x, y)\hat{i} + F_2(x, y)\hat{j}$)

(Proof) If $\vec{F} = \nabla\phi$, then $F_1 = \dfrac{\partial \phi(x, y)}{\partial x}$, $F_2 = \dfrac{\partial \phi(x, y)}{\partial y}$, $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial^2 \phi}{\partial y \partial x} = \dfrac{\partial^2 \phi}{\partial x \partial y} = \dfrac{\partial F_2}{\partial x}$

By Green's theorem, $\oint_C F_1 dx + F_2 dy = \iint_D \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \right) dxdy = 0$, $\therefore$ $\vec{F}$ is conservative on Ω.

Eg. (a) Is $\vec{F}(x, y, z) = -\dfrac{y}{z}\hat{i} - \dfrac{x}{z}\hat{j} + \dfrac{xy}{z^2}\hat{k}$ conservative in the region $z>0$? **(b)** Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = -\dfrac{y}{z}\hat{i} - \dfrac{x}{z}\hat{j} + \dfrac{xy}{z^2}\hat{k}$, C is a piecewisely smooth curve from (1,1,1) to (2,-1,3) and not crossing the xy-plane. 【1990 成大化工所】

(Sol.) (a) $\nabla \times \vec{F} = 0$ in the region $z>0$, $\therefore$ $\vec{F}$ is conservative.

(b) $\nabla \times \vec{F} = 0$, $\therefore \exists \phi(x, y, z)$ such that $\vec{F} = \nabla\phi$
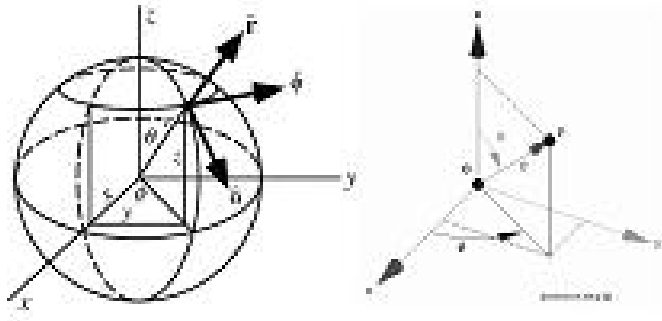
$$\Rightarrow \phi(x, y, z) = -\frac{xy}{z} \Rightarrow \int \vec{F} \cdot d\vec{r} = \phi(x, y, z)\Big|_{(1,1,1)}^{(2,-1,3)} = \frac{5}{3}$$

7-6 Curvilinear Coordinates

Coordinate transformation: $\begin{cases} x = x(q_1, q_2, q_3) \\ y = y(q_1, q_2, q_3) \\ z = z(q_1, q_2, q_3) \end{cases} \Leftrightarrow \begin{cases} q_1 = q_1(x, y, z) \\ q_2 = q_2(x, y, z) \\ q_3 = q_3(x, y, z) \end{cases}$

Scalar factors: If (q_1, q_2, q_3) are orthogonal system, then

$$h_{ij} = \begin{cases} 0, i \neq j \\ \left[\left(\dfrac{\partial x}{\partial q_i} \right)^2 + \left(\dfrac{\partial y}{\partial q_i} \right)^2 + \left(\dfrac{\partial z}{\partial q_i} \right)^2 \right]^{1/2}, i = j \end{cases}$$
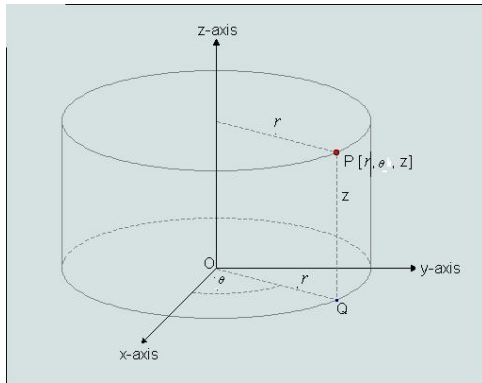
Eg.   Rectangular coordinate

⇔   Spherical coordinate

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \sin^{-1}\left(\sqrt{x^2+y^2}\Big/\sqrt{x^2+y^2+z^2}\right) \\ \phi = \cos^{-1}\left(x\Big/\sqrt{x^2+y^2}\right) \end{cases} \Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\dfrac{\partial x}{\partial r}\right)^2 + \left(\dfrac{\partial y}{\partial r}\right)^2 + \left(\dfrac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\[2mm] h_\theta = h_{22} = \left[\left(\dfrac{\partial x}{\partial \theta}\right)^2 + \left(\dfrac{\partial y}{\partial \theta}\right)^2 + \left(\dfrac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\[2mm] h_\phi = h_{33} = \left[\left(\dfrac{\partial x}{\partial \phi}\right)^2 + \left(\dfrac{\partial y}{\partial \phi}\right)^2 + \left(\dfrac{\partial z}{\partial \phi}\right)^2\right]^{1/2} = r\sin\theta \end{cases}$$

Differential length vector in the spherical coordinate:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r\sin\theta d\phi$$
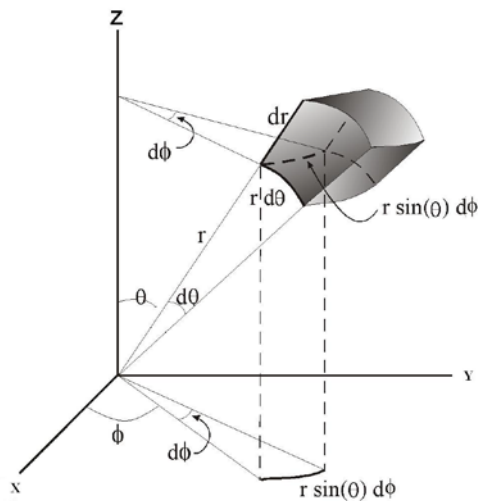
Eg. Rectangular coordinate ⇔ Cylindrical coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases} \Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\dfrac{\partial x}{\partial r}\right)^2 + \left(\dfrac{\partial y}{\partial r}\right)^2 + \left(\dfrac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\[2mm] h_\theta = h_{22} = \left[\left(\dfrac{\partial x}{\partial \theta}\right)^2 + \left(\dfrac{\partial y}{\partial \theta}\right)^2 + \left(\dfrac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\[2mm] h_z = h_{33} = \left[\left(\dfrac{\partial x}{\partial z}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + \left(\dfrac{\partial z}{\partial z}\right)^2\right]^{1/2} = 1 \end{cases}$$

Differential length vector in the cylindrical coordinate: $\quad d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_z dz$

Differential arc length: $\quad ds = [(h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2]^{1/2}$

Differential elements of area on the q_iq_j-plane: $dA_{ij} = ds_i\ ds_j = h_i h_j dq_i dq_j$

Differential element of volume: $dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$

Eg. Spherical coordinate:

$$\begin{cases} ds = \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]^{1/2} \\ dA_{\theta,\phi} = r^2 \sin\theta d\theta d\phi, dA_{r,\theta} = rdrd\theta, etc. \\ dV = r^2 \sin\theta drd\theta d\phi \end{cases}$$

Eg. Cylindrical coordinate: $\begin{cases} ds = \left[dr^2 + r^2 d\theta^2 + dz^2\right]^{1/2} \\ dA_{\theta,z} = rd\theta dz, dA_{r,\theta} = rdrd\theta, dA_{r,z} = drdz\ . \\ dV = rdrd\theta dz \end{cases}$

Jacobian determinants: $(p_1, p_2, p_3) \Leftrightarrow (q_1, q_2, q_3)$

$$dp_i dp_j = \left|\det \begin{bmatrix} \dfrac{\partial p_i}{\partial q_i} & \dfrac{\partial p_j}{\partial q_i} \\ \dfrac{\partial p_i}{\partial q_j} & \dfrac{\partial p_j}{\partial q_j} \end{bmatrix}\right| \cdot dq_i dq_j$$

and $dp_1 dp_2 dp_3 = \left|\det \begin{bmatrix} \dfrac{\partial p_1}{\partial q_1} & \dfrac{\partial p_2}{\partial q_1} & \dfrac{\partial p_3}{\partial q_1} \\ \dfrac{\partial p_1}{\partial q_2} & \dfrac{\partial p_2}{\partial q_2} & \dfrac{\partial p_3}{\partial q_2} \\ \dfrac{\partial p_1}{\partial q_3} & \dfrac{\partial p_2}{\partial q_3} & \dfrac{\partial p_3}{\partial q_2} \end{bmatrix}\right| \cdot dq_1 dq_2 dq_3$

Eg. Let $I = \iiint\limits_V f(x, y, z)dxdydz$. Transform the integral from (x,y,z) into (r,θ,φ).

$$|J| = \left|\det \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{bmatrix}\right| = \left|r^2 \sin\theta\cos^2\theta\cos^2\phi\right.$$

$$\left. + r^2 \sin^3\theta\sin^2\phi + r^2 \sin\theta\cos^2\theta\sin^2\phi + r^2 \sin^3\theta\cos^2\phi\right| = \left|r^2 \sin\theta\cos^2\theta + r^2 \sin^3\theta\right| = r^2 \sin\theta = h_1 h_2 h_3$$

$$\therefore\ I = \iiint\limits_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2 \sin\theta drd\theta d\phi$$

Del operators in (q_1,q_2,q_3) coordinate system:

$$\nabla \psi(q_1,q_2,q_3) = \frac{1}{h_1}\frac{\partial \psi}{\partial q_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial q_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial q_3}\hat{u}_3$$

$$\nabla \cdot \vec{F}(q_1,q_2,q_3) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_1 h_3) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2) \right)$$

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$

$$\nabla^2 \psi(q_1,q_2,q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3} \right) \right],$$

where $\vec{F} = F_1\hat{u}_1 + F_2\hat{u}_2 + F_3\hat{u}_3$

Eg. Spherical coordinate system: $h_r=1, h_\theta=r, h_\varphi=r\sin\theta$

$\vec{F} = F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_\phi\hat{u}_\phi$

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial \theta}(\sin\theta \cdot F_\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial \phi}(F_\phi)$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{u}_r & r\hat{u}_\theta & r\sin\theta\hat{u}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta \cdot F_\phi \end{vmatrix}, \quad \nabla\psi = \frac{\partial \psi}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{u}_\phi$$

$$\nabla^2\psi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta}\cdot\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2}$$

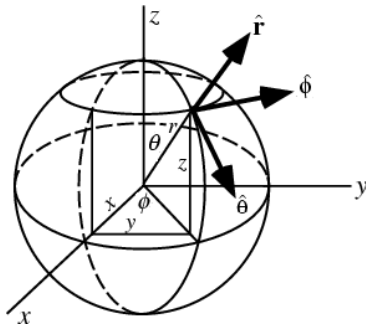**Eg. Cylindrical coordinate system: $h_r=1, h_\theta=r, h_z=1$**

$\vec{F} = F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_z\hat{u}_z$

$$\nabla \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right)\hat{u}_r + \left(\frac{\partial F_\theta}{\partial z} - \frac{\partial F_z}{\partial r} \right)\hat{u}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} \right)\hat{u}_z$$

$$\nabla\psi = \frac{\partial \psi}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{u}_\theta + \frac{\partial \psi}{\partial z}\hat{u}_z, \quad \nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Unit vector conversions between distinct coordinates:

Eg. Rectangular coordinate ⇔ Spherical coordinate

(Proof) $\hat{x} \cdot \hat{r} = \sin\theta\cos\phi$, $\hat{y} \cdot \hat{r} = \sin\theta\sin\phi$,

$\hat{z} \cdot \hat{r} = \cos\theta$, ∴ $\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

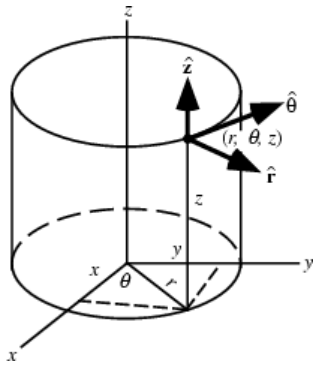Similarly, $\hat{x} \cdot \hat{\theta} = \cos\theta\cos\phi$, $\hat{y} \cdot \hat{\theta} = \cos\theta\sin\phi$,

$\hat{z} \cdot \hat{\theta} = -\sin\theta$, ∴ $\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$,

And $\hat{x} \cdot \hat{\phi} = -\sin\phi$, $\hat{y} \cdot \hat{\phi} = \cos\phi$, $\hat{z} \cdot \hat{\phi} = 0$,

∴ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix},$$

or $$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$$

Eg. Rectangular coordinate ⇔ Cylindrical coordinate

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix}$$ or

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$