Chapter 8 Theories of Systems

8-1 Laplace Transform Solutions of Linear Systems

Linear Systems $\dot{X} = AX + F(t)$ **:** Consider $y^{(n)}+a_{n-1}y^{(n-1)}+a_{n-2}y^{(n-2)}+\cdots+a_1y'+a_0y=f(t)$.

Let $x_1(t)=y(t)$, $x_2(t)=y'(t)$, …, $x_n(t)=y^{(n-1)}(t)$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

Eg. $y''-3xy'+2y=\sin(x)$ **can be transformed into** $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$ **by** $t=x$, $x_1(t)=y(x)$ **and** $x_2(t)=y'(x)$.

Eg. Solve $\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = -2x_1 + x_2 \end{cases}$, $x_1(0)=8$, $x_2(0)=3$ **by Laplace transform.**

(Sol.) $L[f'(t)]=sF(s)-f(0) \Rightarrow \begin{cases} sX_1(s)-8=2X_1(s)-3X_2(s) \\ sX_2(s)-3=-2X_1(s)+X_2(s) \end{cases} \Rightarrow \begin{cases} (s-2)X_1(s)+3X_2(s)=8 \\ 2X_1(s)+(s-1)X_2(s)=3 \end{cases}$

$$\Rightarrow \begin{cases} X_1(s) = \dfrac{8s-17}{s^2-3s-4} = \dfrac{5}{s+1} + \dfrac{3}{s-4} \\ X_2(s) = \dfrac{3s-22}{s^2-3s-4} = \dfrac{5}{s+1} - \dfrac{2}{s-4} \end{cases} \Rightarrow \begin{matrix} x_1(t) = 5e^{-t} + 3e^{4t} \\ x_2(t) = 5e^{-t} - 2e^{4t} \end{matrix}.$$

Eg. Solve $\begin{cases} x'' - 2x' + 3y' + 2y = 4 \\ 2y' - x' + 3y = 0 \end{cases}$, $x(0)=x'(0)=y(0)=y'(0)=0$ **by Laplace transform.**

(Sol.)

$L[f''(t)]=s^2F(s)-sf(0)-f'(0) \Rightarrow \begin{matrix} s^2X(s) - s0 - 0 - 2[sX(s)-0] + 3[sY(s)-0] + 2Y(s) = \dfrac{4}{s} \\ 2[sY(s)-0] - [sX(s)-0] + 3Y(s) = 0 \end{matrix}$

$$\Rightarrow \begin{matrix} (s^2-2s)X(s) + (3s+2)Y(s) = \dfrac{4}{s} \\ -sX(s) + (2s+3)Y(s) = 0 \end{matrix} \Rightarrow \begin{matrix} (s^2-2s)X(s) + (3s+2)Y(s) = \dfrac{4}{s} \\ (s^2-2s)X(s) + (-2s^2+s+6)Y(s) = 0 \end{matrix}$$

$$\Rightarrow \begin{matrix} X(s) = \dfrac{4s+6}{s^2(s+2)(s-1)} = -\dfrac{7}{2s} - \dfrac{3}{s^2} + \dfrac{\frac{1}{6}}{s+2} + \dfrac{\frac{10}{3}}{s-1} \\ Y(s) = \dfrac{4}{2s(s+2)(s-1)} = -\dfrac{1}{s} + \dfrac{\frac{1}{3}}{s+2} + \dfrac{\frac{2}{3}}{s-1} \end{matrix} \Rightarrow \begin{matrix} x(t) = -\dfrac{7}{2} - 3t + \dfrac{1}{6}e^{-2t} + \dfrac{10}{3}e^{t} \\ y(t) = -1 + \dfrac{1}{3}e^{-2t} + \dfrac{2}{3}e^{t} \end{matrix}$$

Eg. According to optical waveguide theory, the *E*-fields of two identical waveguides *A* and *B* fulfill the coupled mode equations $\begin{cases} \dfrac{dE_A}{dz} = -j\beta E_A - j\kappa E_B \\ \dfrac{dE_B}{dz} = -j\beta E_B - j\kappa E_A \end{cases}$, $\begin{cases} E_A(0) = 1 \\ E_B(0) = 0 \end{cases}$, **where κ is the coupling coefficient. Find $E_A(z)$ and $E_B(z)$.**

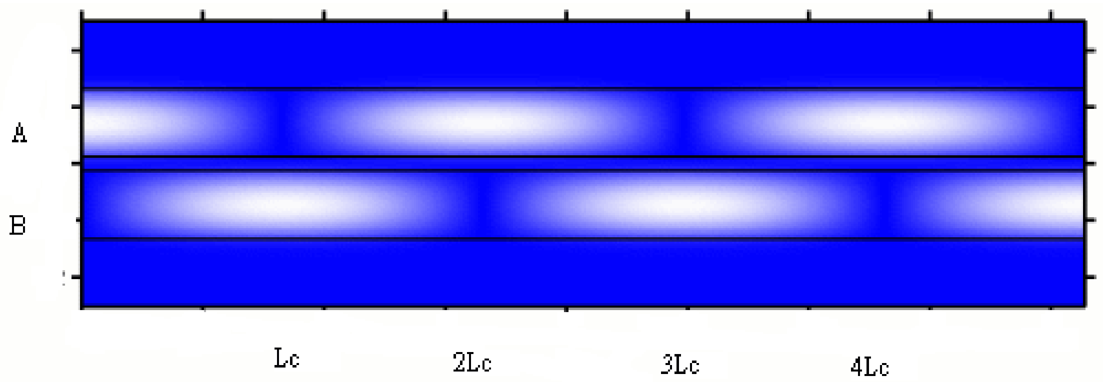
(Sol.) $L(E_A)=\Sigma_A(s)$ and $L(dE_A/dz)=s\Sigma_A(s)-E_A(0)=s\Sigma_A(s)-1$.

Similarly, $L(E_B)=\Sigma_B(s)$ and $L(dE_B/dz)=s\Sigma_B(s)-E_B(0)=s\Sigma_B(s)$

$$\Rightarrow \begin{cases} (s+j\beta)\Sigma_A(s) + j\kappa\Sigma_B(s) = 1 \\ j\kappa\Sigma_A(s) + (s+j\beta)\Sigma_B(s) = 0 \end{cases} \Rightarrow \begin{cases} \Sigma_A(s) = \dfrac{s+j\beta}{(s+j\beta)^2+\kappa^2} \\ \Sigma_B(s) = \dfrac{-j\kappa}{(s+j\beta)^2+\kappa^2} \end{cases},$$

$\because L^{-1}[F(s+a)] = f(t)e^{-at}$, $\cos(at) = L^{-1}[\dfrac{s}{s^2+a^2}]$ and $\sin(at) = L^{-1}[\dfrac{a}{s^2+a^2}]$

$\Rightarrow E_A(z) = L^{-1}[\Sigma_A(s)] = \cos(\kappa z)e^{-j\beta z}$ and $E_B(z) = L^{-1}[\Sigma_B(s)] = -j\sin(\kappa z)e^{-j\beta z}$.



In this case, the coupling length is $L_c=\pi/2\kappa$. While the waveguiding mode traverses a distance of odd multiple of the coupling length (L_c, $3L_c$, $5L_c$, …, etc), the optical power is completely transferred into the other waveguide. But it is back after a distance of even multiple of the coupling lengths ($2L_c$, $4L_c$, $6L_c$, …, etc). If the waveguiding mode traverses a distance of odd multiple of the half coupling length ($L_c/2$, $3L_c/2$, $5L_c/2$, …, etc), the optical power is equally distributed in the two guides.

Eg. Solve $\begin{cases} y'(t) + 6y(t) = x'(t) \\ 3x(t) - x'(t) = 2y'(t) \end{cases}$, ***x*(0)=2, *y*(0)=3. 【1991 交大電信】**

(Ans.) $\begin{cases} x(t) = 4e^{2t} - 2e^{-3t} \\ y(t) = e^{2t} + 2e^{-3t} \end{cases}$

8-2 Matrix Solutions of Linear Systems

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(Sol.) $det\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = 0$, λ=2, -1 $\Rightarrow$ diagonalizable

λ_1=2, $\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\exp[\int 2dt] = e^{2t}$

λ_2=-1, $\begin{bmatrix} 3-(-1) & -2 \\ 2 & -2-(-1) \end{bmatrix}\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\exp[\int(-1)dt] = e^{-t}$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-t}$$

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(Sol.) $det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = (\lambda-2)^2 = 0$, λ=2, 2

$dim(V)$-$Rank(A-2I)$=1 $\neq$ 2=Multiplicity of $(\lambda-2)^2 \Rightarrow$ not diagonalizable

λ=2, $\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix}\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{2t} + c_2\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{2t}\right\}$$

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$.

(Sol.) $f(\lambda)=-\lambda^3+3\lambda+2=0$, λ=2, -1, -1.

For λ=-1, $dim(V)$-$Rank(A+I)$=2=Multiplicity of $(\lambda+1)^2 \Rightarrow$ diagonalizable

λ=-1, $\begin{bmatrix} 0+1 & 1 & 1 \\ 1 & 0+1 & 1 \\ 1 & 1 & 0+1 \end{bmatrix}\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

λ=2, $\begin{bmatrix} 0-2 & 1 & 1 \\ 1 & 0-2 & 1 \\ 1 & 1 & 0-2 \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-t}$$

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(Sol.) $f(\lambda) = \lambda^2 + 2\lambda + 2 = 0$, $\lambda = -1 \pm i$

$\lambda=-1+i$, $\begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}, \quad e^{\int [-1+i]dt} = e^{-t}\cdot[\cos(t) + i\sin(t)]$$

$\lambda=-1-i$, $\begin{bmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{bmatrix}\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1 = \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 2+i \end{bmatrix}, \quad e^{\int [-1-i]dt} = e^{-t}\cdot[\cos(t) - i\sin(t)]$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2-i \end{bmatrix}e^{-t}\cdot[\cos(t) + i\sin(t)] + c_2\begin{bmatrix} 1 \\ 2+i \end{bmatrix}e^{-t}\cdot[\cos(t) - i\sin(t)]$$

$$= c_1'\begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix}e^{-t} + c_2'\begin{bmatrix} \sin(t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}e^{-t}$$

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/t \\ 2/t+4 \end{bmatrix}$.

(Sol.) $f(\lambda) = \lambda^2 + 5\lambda = 0$, $\lambda = 0, -5$: diagonalizable

$$\lambda_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \lambda_2 = -5 \Rightarrow x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Homogeneous solutions: $X_h = \begin{bmatrix} x_{1c}(t) \\ x_{2c}(t) \end{bmatrix} = \tilde{c}_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \tilde{c}_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-5t}$

Particular solution: $X_p = TY_p$, $X_p' = AX_p + g(t) \Rightarrow TY_p' = ATY_p + g(t) \Rightarrow Y_p' = T^{-1}ATY_p + T^{-1}g(t)$

Choose T such that $T^{-1}AT = D$, $T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $T^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1/t \\ 2/t+4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5\ln(t) + 8t + c_1' \\ \frac{4}{5} + c_2'e^{-5t} \end{bmatrix} \Rightarrow X_p = TY_p = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\cdot\frac{1}{5}\cdot\begin{bmatrix} 5\ln(t) + 8t + c_1' \\ \frac{4}{5} + c_2'e^{-5t} \end{bmatrix}$$

$$\Rightarrow X_p = \begin{bmatrix} x_{1p}(t) \\ x_{2p}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln(t) + \frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}t + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \frac{c_1'}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{c_2'}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-5t}$$

$$\Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\ln(t) + \frac{8}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}t + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-5t}$$

Eg. There are two species of animals: wolf and fox, interacting within the same forest ecosystem. Let $w(t)$ and $f(t)$ denote the wolf and fox population, respectively, at time t. Suppose further that wolves might eat foxes as food and foxes might also eat wolves as food, but only wolves are hunted by human. If there are no foxes, then one might expect that the wolves, lacking an adequate food supply, would decline in number at a rate of $-11w(t)$. When foxes are present, there is a supply of food, and so wolves are added to the forest at a rate of $3f(t)$. Furthermore the change rate of the wolf population also positively depends on a seasonal hunting factor, denoted as $100\sin(t)$. On the other hand, if there is no wolves, then the foxes, lacking an adequate food supply, would decline in number at a rate of $-3f(t)$. But when wolves are present, the fox population is increased by a rate of $3w(t)$. Please answer the following questions: (a) Formulate the above system by a set of differential equations. (b) Use variation of parameters to solve the system. (c) What are the steady-state populations of the wolf and fox, respectively? [2006台大電研]

(Sol.) (a) $\frac{d}{dt}\begin{bmatrix} w(t) \\ f(t) \end{bmatrix} = \begin{bmatrix} -11 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} w(t) \\ f(t) \end{bmatrix} + \begin{bmatrix} 100\sin(t) \\ 0 \end{bmatrix}$.

(b) $A=\begin{bmatrix} -11 & 3 \\ 3 & -3 \end{bmatrix}$, $f(\lambda)=\lambda^2+14\lambda+24=0$, $\lambda_1=-2 \Rightarrow x_1=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\lambda_2=-12 \Rightarrow x_2=\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Homogeneous solution X_h: $X_h=\begin{bmatrix} w_h(t) \\ f_h(t) \end{bmatrix} = \tilde{c}_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t} + \tilde{c}_2\begin{bmatrix} 3 \\ -1 \end{bmatrix}e^{-12t}$

Particular solution: $X_p = TY_p$, $X_p' = AX_p + g(t) \Rightarrow TY_p' = ATY_p + g(t) \Rightarrow Y_p' = T^{-1}ATY_p + T^{-1}g(t)$

Let T fulfill $T^{-1}AT = D$, $T=\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$, $T^{-1}=\begin{bmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{bmatrix}$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{bmatrix}\begin{bmatrix} 100\sin(t) \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 4\sin(t) - 2\cos(t) + c_1'e^{-2t} \\ \frac{-72\sin(t)}{29} + \frac{6\cos(t)}{29} + c_2'e^{-12t} \end{bmatrix} = Y_p,\ X_p = TY_p$$

$$\Rightarrow \begin{bmatrix} w(t) \\ f(t) \end{bmatrix} = \tilde{c}_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t} + \tilde{c}_2\begin{bmatrix} 3 \\ -1 \end{bmatrix}e^{-12t} + \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4\sin(t) - 2\cos(t) + c_1'e^{-2t} \\ \frac{-72\sin(t)}{29} + \frac{6\cos(t)}{29} + c_2'e^{-12t} \end{bmatrix}$$

$$= c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 3 \\ -1 \end{bmatrix}e^{-12t} + \frac{1}{29}\begin{bmatrix} 332 \\ 276 \end{bmatrix}\sin(t) - \frac{1}{29}\begin{bmatrix} 76 \\ 168 \end{bmatrix}\cos(t)$$

(c) As $t \to \infty$, $\begin{bmatrix} w(t) \\ f(t) \end{bmatrix} = \frac{1}{29}\begin{bmatrix} 332 \\ 276 \end{bmatrix}\sin(t) - \frac{1}{29}\begin{bmatrix} 76 \\ 168 \end{bmatrix}\cos(t)$: steady-state solution

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 5/t & -1/t \\ 3/t & 1/t \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(Sol.) $det\begin{bmatrix} 5/t-\lambda & -1/t \\ 3/t & 1/t-\lambda \end{bmatrix} = 0 \Rightarrow \lambda = \frac{2}{t},\ \frac{4}{t}$

$$\lambda_1 = \frac{2}{t},\ (A-\lambda_1 I)x_1 = \begin{bmatrix} 3/t & -1/t \\ 3/t & -1/t \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = 0 \Rightarrow x_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda_2 = \frac{4}{t},\ (A-\lambda_2 I)x_2 = \begin{bmatrix} 1/t & -1/t \\ 3/t & -3/t \end{bmatrix}\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = 0 \Rightarrow x_2 = \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2,\ e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = t^4$$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}t^2 + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}t^4 = \begin{bmatrix} t^2 & t^4 \\ 3t^2 & t^4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Phi(t)\cdot C$$

Eg. Solve $\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 5/t & -1/t \\ 3/t & 1/t \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$.

(Sol.) Homogeneous solutions: $\Phi(t)\cdot C$. Note: $\Phi'(t) = \begin{bmatrix} 5/t & -1/t \\ 3/t & 1/t \end{bmatrix}\Phi(t)$

Particular solution: $X_p(t) = \Phi(t)\cdot u(t)$

$$X_p'(t) = \Phi'(t)\cdot u(t) + \Phi(t)\cdot u'(t) = \begin{bmatrix} 5/t & -1/t \\ 3/t & 1/t \end{bmatrix}\Phi\cdot u(t) + \Phi(t)\cdot u'(t)$$

$$= \begin{bmatrix} 5/t & -1/t \\ 3/t & 1/t \end{bmatrix}X_p(t) + \begin{bmatrix} 1 \\ t \end{bmatrix} \Rightarrow \Phi(t)u'(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$\Rightarrow u'(t) = \frac{d}{dt}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \Phi^{-1}(t)\cdot\begin{bmatrix} 1 \\ t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1/t^2 & 1/t^2 \\ 3/t^4 & -1/t^4 \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2t^2} + \frac{1}{2t} \\ \frac{3}{2t^4} - \frac{1}{2t^3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2t} + \frac{1}{2}\ln|t| \\ -\frac{1}{2t^3} + \frac{1}{4t^2} \end{bmatrix}$$

$$\Rightarrow X_p(t) = \begin{bmatrix} x_{1p}(t) \\ x_{2p}(t) \end{bmatrix} = \begin{bmatrix} t^2 & t^4 \\ 3t^2 & t^4 \end{bmatrix}\begin{bmatrix} \frac{1}{2t} + \frac{1}{2}\ln|t| \\ -\frac{1}{2t^3} + \frac{1}{4t^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t^2\ln|t| + \frac{t^2}{4} \\ t + \frac{3t^2}{2}\ln(t) + \frac{t^2}{4} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t^2 & t^4 \\ 3t^2 & t^4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t^2\ln|t| + \frac{t^2}{4} \\ t + \frac{3t^2}{2}\ln(t) + \frac{t^2}{4} \end{bmatrix}$$

8-3 Nonlinear Systems and Phase Planes

Autonomous system: $\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$, where $F(x,y)$ and $G(x,y)$ are both independent of t.

Phase plane: The xy-plane for the analysis of the autonomous systems.

Theorem $\begin{cases} \dfrac{dx}{dt} = F(x,y) = ax + by + P(x,y) \\ \dfrac{dy}{dt} = G(x,y) = cx + dy + Q(x,y) \end{cases}$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{P(x,y)}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{Q(x,y)}{\sqrt{x^2+y^2}} = 0$.

We have $\lambda^2-(a+d)\lambda+ad-bc=0 \Rightarrow \lambda=\lambda_1, \lambda_2$

1. $\lambda_1 \neq \lambda_2$, λ_1, λ_2: real, $\lambda_1\lambda_2>0$, then (0,0) is a node.

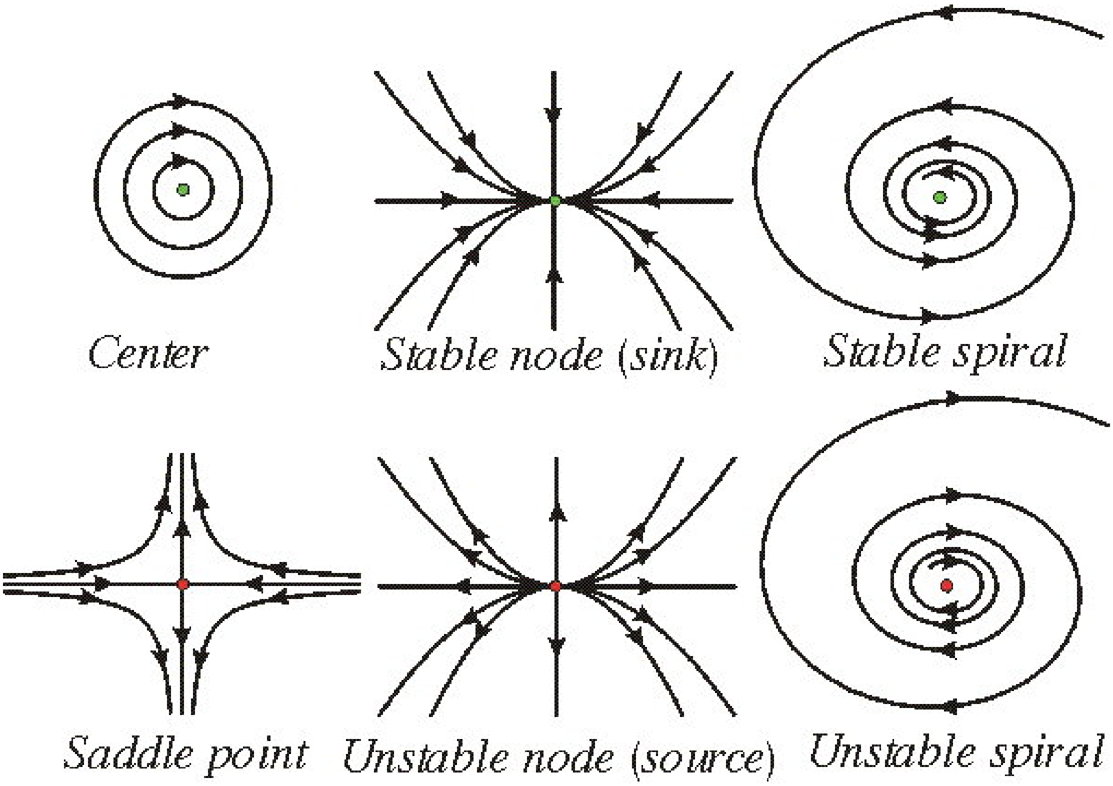
Stable node in case of $\lambda_1<0$ and $\lambda_2<0$. Unstable node if $\lambda_1>0$, $\lambda_2>0$.

2. $\lambda_1 \neq \lambda_2$, λ_1, λ_2: real, $\lambda_1\lambda_2<0$, then (0,0) is a saddle point.

3. λ_1, λ_2: complex with nonzero real part, then (0,0) is the point, which a spiral approaches it.

Stable spiral in case of $Re(\lambda)<0$. Unstable spiral if $Re(\lambda)>0$.

4. λ_1, λ_2: pure imaginary, then (0,0) is a center of a closed curve.



Eg. $\begin{cases} \dfrac{dx}{dt} = 4x - 2y - 4xy \\ \dfrac{dy}{dt} = x + 6y - 8x^2y \end{cases}$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{-4xy}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{-8x^2y}{\sqrt{x^2+y^2}} = 0$

$\lambda^2-10\lambda+26=0 \Rightarrow \lambda=5\pm i$, $Re(\lambda)=5>0$, ∴ (0,0) is an unstable spiral point.

Eg. $\begin{cases} \dfrac{dx}{dt} = -3x + y \\ \dfrac{dy}{dt} = x - 3y \end{cases}$ **$\Rightarrow \lambda^2+6\lambda+8=0$, $\lambda=-4$, -2, and $(-4)(-2)=8>0$: negative and distinct,**

$\therefore$ (0,0) is a stable node. Check: $\begin{cases} x(t) = c_1 e^{-4t} + c_2 e^{-2t} \to 0 \\ y(t) = -c_1 e^{-4t} + c_2 e^{-2t} \to 0 \end{cases}$

Eg. $\begin{cases} \dfrac{dx}{dt} = -x + 3y \\ \dfrac{dy}{dt} = 2x - 2y \end{cases}$ **$\Rightarrow \lambda^2+3\lambda-4=0$, $\lambda=1$, -4, $1 \neq -4$, and $1(-4)<0$, $\therefore$ (0,0) is a saddle point. Check:** $\begin{cases} x(t) = c_1 e^{t} + c_2 e^{-4t} \\ y(t) = \dfrac{2}{3} c_1 e^{t} - c_2 e^{-4t} \end{cases}$. **If $c_1=0$, $(x(t),y(t)) \to (0,0)$. But in case of** $c_1 \neq 0$, $(x(t), y(t)) \to (\infty, \infty)$

Eg. $\begin{cases} \dfrac{dx}{dt} = 3x + y \\ \dfrac{dy}{dt} = -13x - 3y \end{cases}$ **$\Rightarrow \lambda^2+4=0$, $\lambda=\pm 2i$, $\therefore$ (0,0) is a center of a closed curve.**

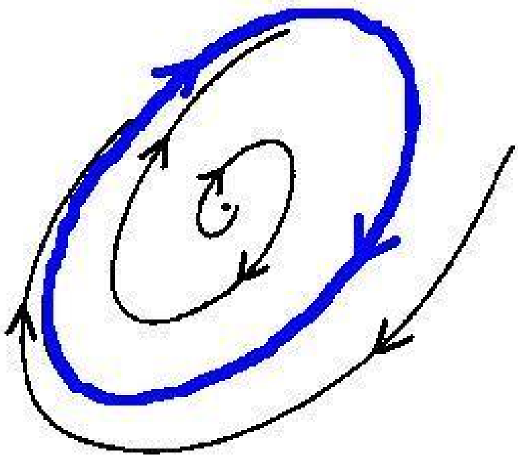
Check: $\begin{cases} x(t) = c_1 \cos(2t) + c_2 \sin(2t) \\ y(t) = (2c_2 - 3c_1)\cos(2t) + (-2c_1 - 3c_2)\sin(2t) \end{cases}$

Critical point: (x_c, y_c) fulfills both $F(x_c, y_c)=0$ and $G(x_c, y_c)=0$.

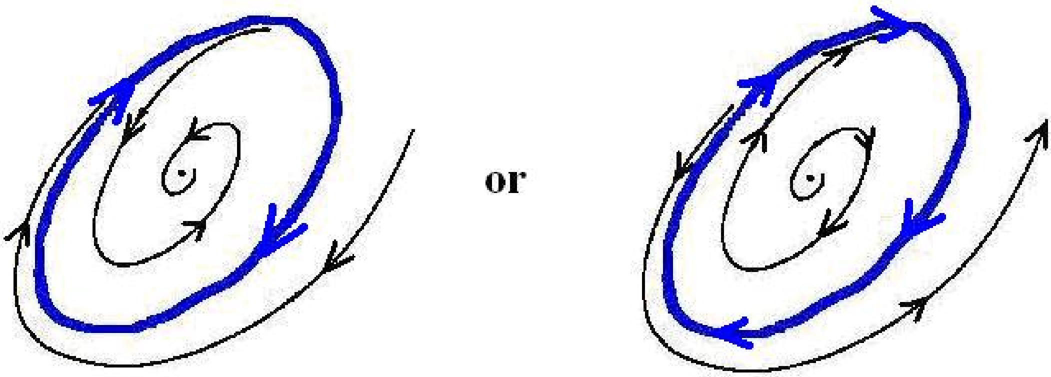
Theorem ***C* is the closed trajectory of the autonomous system** $\begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$, **where *F* and $G \in C^1(x,y)$, then there exists at least one critical point of the system enclosed by *C*.**

Three types of limit cycles: Stable, unstable, and semi-stable limit cycles.

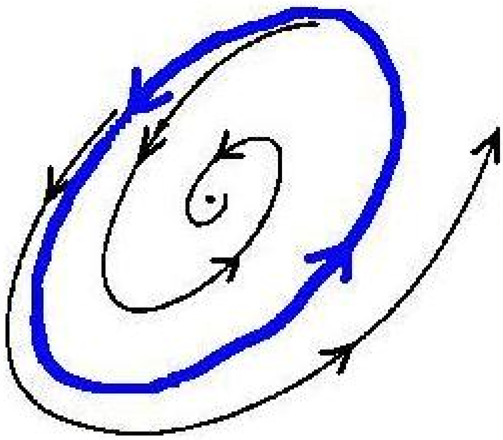
Stable limit cycle



Semi-stable limit cycle



Unstable limit cycle



Eg. Find the trajectory of the following system: $\begin{cases} x' = x + y - x\sqrt{x^2+y^2} \\ y' = -x + y - y\sqrt{x^2+y^2} \end{cases}$.

(Sol.) Let $x=r\cos\theta$, $y=r\sin\theta$, $x'=-r\sin\theta\cdot\theta'$, $y'=r\cos\theta\cdot\theta'$

$\because r^2=x^2+y^2$,

$\therefore rr'=xx'+yy'=x^2+y^2-(x^2+y^2)\sqrt{x^2+y^2}=r^2-r^3 \Rightarrow r'=r(1-r)\cdots(1)$

$\because yx'-xy'=r\sin\theta(-r\sin\theta\cdot\theta')-r\cos\theta(r\cos\theta\cdot\theta')=-r^2\theta'$,

$\therefore -r^2\theta'=yx'-xy'=x^2+y^2=r^2 \Rightarrow -r^2\theta'=r^2 \Rightarrow \theta'=-1\cdots(2)$

$$(1)\text{ and }(2) \Rightarrow \begin{cases} \theta=\theta_0-t \\ r=\dfrac{1}{1-\dfrac{r_0-1}{r_0}\cdot e^{-t}} \end{cases}$$

If $r_0=1 \Rightarrow r=1$; else if $r_0<1 \Rightarrow r\to 1^-$; else, $r_0>1 \Rightarrow r\to 1^+$.

In this example, the unit circle $r=1$ is a stable limit cycle.

8-4 Some Approximate Solutions of Nonlinear Ordinary Differential Equations

Eg. For a simple pendulum of mass *m* with a thread of length *l*, show that $\frac{d^2\theta}{dt^2}+\frac{g}{\ell}\sin\theta=0$**. Solve it and obtain its period. [1991 台大電研、2009 交大電控所]**

(Sol.) Total energy is conservative: $mg(l-l\cos\theta)+\frac{1}{2}m(l\frac{d\theta}{dt})^2$=Constant

$$\Rightarrow \frac{d^2\theta}{dt^2}+\frac{g}{\ell}\sin\theta=0$$

Let $u=\frac{d\theta}{dt}$, $u'=\frac{du}{dt}=\frac{d^2\theta}{dt^2} \Rightarrow u'+\frac{g}{l}\sin\theta=0 \Rightarrow u\frac{du}{dt}+\frac{g}{\ell}\sin\theta\cdot\frac{d\theta}{dt}=0$

$\Rightarrow udu=-\frac{g}{\ell}\sin\theta d\theta \Rightarrow \frac{u^2}{2}=\frac{g}{\ell}(\cos\theta-\cos\theta_0)$, where θ_0 is the initial angle

$$\Rightarrow u=\frac{d\theta}{dt}=\left[\frac{2g}{\ell}(\cos\theta-\cos\theta_0)\right]^{1/2} \Rightarrow dt=\frac{d\theta}{\left[\frac{2g}{\ell}(\cos\theta-\cos\theta_0)\right]^{1/2}}$$

$$\text{Period: } T=4\int_0^{T_0}dt=4\int_0^{\theta_0}\frac{d\theta}{\sqrt{\frac{2g}{\ell}(\cos\theta-\cos\theta_0)}}$$

If θ is small, $\sin\theta\approx\theta \Rightarrow \frac{d^2\theta}{dt^2}+\frac{g}{\ell}\theta=0 \Rightarrow \theta=c_1\cos\left(\sqrt{\frac{g}{\ell}}t\right)+c_2\sin\left(\sqrt{\frac{g}{\ell}}t\right)$

Eg. Solve *Van der Pol's* equation $y''(t)+\varepsilon[y^2(t)-1]y'(t)+y(t)=0$.

(Sol.) For $\varepsilon=0$, $y(t)=A\sin(t+\phi)$, $y'(t)=\underline{A}\cos(t+\underline{\phi})$

If ε is small, $y(t)=A(t)\sin[t+\phi(t)]\cdots(1)$, $y'(t)=A(t)\cos[t+\phi(t)]\cdots(2)$

$$y'(t)=A'(t)\sin[t+\phi(t)]+A(t)\cos[t+\phi(t)]\cdot[1+\phi'(t)]\cdots(3)$$

$$(2),(3)\Rightarrow A'(t)\sin[t+\phi(t)]+A(t)\phi'(t)\cos[t+\phi(t)]=0\cdots(4)$$

$$(2)\Rightarrow y''(t)=A'(t)\cos[t+\phi(t)]-A(t)\cdot\sin[t+\phi(t)]\cdot[1+\phi'(t)]\cdots(5)$$

$$(2),(5)\Rightarrow A'(t)\cos[t+\phi(t)]-A(t)\phi'(t)\cdot\sin[t+\phi(t)]$$

$$=\varepsilon[1-A^2(t)\sin^2[t+\phi(t)]\cdot A(t)\cos[t+\phi(t)]\quad(\text{we set }\ \theta=t+\phi(t))$$

$$=\varepsilon\left\{\left[A(t)-\frac{A^3(t)}{4}\right]\cos(\theta)+\frac{A^3(t)}{4}\cos(3\theta)\right\}\cdots(6)$$

$$\begin{cases}\int_0^{2\pi}\cos(\theta)\times(6)\,d\theta\Rightarrow A'(t)=A(t)-\dfrac{A^3(t)}{4}:\ \textit{Bernoulli's equation}\\ \int_0^{2\pi}\sin(\theta)\times(6)\,d\theta\Rightarrow\phi'(t)=0\end{cases}\Rightarrow\begin{cases}A(t)=2\left[\dfrac{1}{1+ce^{-2t}}\right]^{1/2}\\ \phi(t)=\phi_0\end{cases}$$

$$\Rightarrow y(t)=2\left[\frac{1}{1+ce^{-2t}}\right]^{1/2}\cdot\sin(t+\phi_0)$$

Eg. Obtain the particular solution of $\dfrac{d^2y}{dx^2}+\omega^2y+\varepsilon y^3=\Gamma\cos(x)$.

(Sol.) Suppose $y(x,\varepsilon)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\cdots$

$$[y_0''(x)+\varepsilon y_1''(x)+\varepsilon^2y_2''(x)+\cdots]+\omega^2[y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\cdots]$$

$$\varepsilon[y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\cdots]^3=\Gamma\cos(x)$$

$$\Rightarrow[y_0''(x)+\omega^2y_0(x)]+\varepsilon[y_1''(x)+\omega^2y_1(x)+y_0^3(x)]$$

$$+\varepsilon^2[y_2''(x)+\omega^2y_2(x)+3y_0{}^2(x)y_1(x)]+O(\varepsilon^3)=\Gamma\cos(x)$$

$$\Rightarrow y_0''(x)+\omega^2y_0(x)=\Gamma\cos(x),\quad y_1''(x)+\omega^2y_1(x)=-y_0^3(x)$$

$$y_2''(x)+\omega^2y_2(x)=-3y_0^2(x)y_1(x)$$

$$\vdots$$

$\Rightarrow$ Solve $y_0(x)$, $y_1(x)$, …

Eg. Obtain the approximate particular solution of $y''+\dfrac{1}{4}y+0.1y^3=\cos(x)$.

(Sol.) $y(x;\varepsilon)=y_0(x)+\varepsilon y_1(x)+\varepsilon^2y_2(x)+\varepsilon^3y_3(x)+\cdots\approx y_0(x)+\varepsilon y_1(x)$

$$\Rightarrow y_0''+\frac{1}{4}y_0(x)=\cos(x)\Rightarrow y_0=-\frac{4}{3}\cos(x)$$

$$y_1''(x)+\frac{1}{4}y_1(x)=-\left[-\frac{4}{3}\cos(x)\right]^3\Rightarrow y_1=-\frac{64}{27}\cos(x)-\frac{64}{945}\cos(3x)$$

$$\Rightarrow y(x)\approx y_0(x)+0.1y_1(x)\approx-1.57\cos(x)-0.006\cos(3x)$$

8-5 Sturm-Liouville Theory

Consider $y''+R(x)y'+[Q(x)+\lambda P(x)]y=F(x)$, multiply it by $e^{\int R(x)dx}$, and then let $r(x)=e^{\int R(x)dx}$, $q(x)=Q(x)e^{\int R(x)dx}$, $p(x)=P(x)e^{\int R(x)dx}$, and $f(x)=F(x)e^{\int R(x)dx}$

$\Rightarrow$ Sturm-Liouville form: $[r(x)y'(x)]'+[q(x)+\lambda p(x)]y(x)=f(x)$

Eg. Find the eigenvalues and eigenfunctions of $y''-2y'+2(1+\lambda)y=0$ with boundary conditions $y(0)=y(1)=0$, and transform it into the Sturm-Liouville form. [1990 台大造船所]

(Sol.) $r^2-2r+2(1+\lambda)=0$, $r=1\pm\sqrt{1-2(1+\lambda)}=1\pm\sqrt{-(2\lambda+1)}$

If $\lambda=-\frac{1}{2}$, $r=1, 1\Rightarrow y=Ae^x+Bxe^x$

$y(0)=y(1)=0\Rightarrow A=B=0$, $\therefore$ trivial solutions

If $\lambda<-\frac{1}{2}$, $-(2\lambda+1)>0$, $r=1\pm k\Rightarrow y=Ae^{(1+k)x}+Be^{(1-k)x}$

$y(0)=y(1)=0\Rightarrow A=B=0$, $\therefore$ trivial solutions

If $\lambda>-\frac{1}{2}$, $-(2\lambda+1)<0$, $r=1\pm ki\Rightarrow y=e^x(A\cos kx+B\sin kx)$

$$\begin{cases} y(0)=0\Rightarrow A=0 \\ y(1)=0\Rightarrow k=n\pi\Rightarrow \lambda=\dfrac{n^2\pi^2-1}{2} \end{cases}$$, $\therefore$ the corresponding eigenfunction is $e^x\sin(n\pi x)$

$e^{\int -2dx}=e^{-2x}$, $y''e^{-2x}-2e^{-2x}y'+2(1+\lambda)e^{-2x}y=0\Rightarrow[e^{-2x}y'(x)]'+[2e^{-2x}+\lambda 2e^{-2x}]y(x)=0$

Eg. Find the eigenvalues and eigenfunctions of $x^2y''+xy'-\lambda y=0$ with boundary conditions $y(1)=y(a)=0$, $1<x<a$. 【1991 台大機研】

(Sol.) $r^2+(1-1)r-\lambda=0$, $r=\pm\sqrt{\lambda}$

If $\lambda=0$, $y(x)=c_1x^0+c_2x^0\ell n(x)=c_1+c_2\ell n(x)$, $\begin{cases} c_1+c_2\ell n(1)=0 \\ c_1+c_2\ell n(a)=0 \end{cases}\Rightarrow c_1=c_2=0$.

If $\lambda>0$, set $\lambda=k^2$, $y(x)=d_1x^k+d_2x^{-k}$, $\begin{cases} d_1\cdot 1+d_2\cdot 1=0 \\ d_1a^k+d_2a^{-k}=0 \end{cases}\Rightarrow d_1=d_2=0$

If $\lambda<0$, set $\lambda=-k^2$,

$$y(x)=e_1x^{ki}+e_2x^{-ki}=e_1\cdot e^{ik\ln(x)}+e_2\cdot e^{-ik\ln(x)}=h_1\cos[k\ln(x)]+h_2\sin[k\ln(x)]$$

$y(1)=0\Rightarrow h_1=0$, $y(a)=0\Rightarrow k=\dfrac{n\pi}{\ln(a)}\Rightarrow\lambda=-[\dfrac{n\pi}{\ln(a)}]^2\Rightarrow y(x)=\sin[\dfrac{n\pi\ln(x)}{\ln(a)}]$

Sturm-Liouville theorems

1. For the two distinct λ_n and λ_m of the Sturm-Liouville problem, with corresponding functions φ_n and φ_m, then $\exists p(x)$ fulfills $\int_a^b p(x)\phi_n(x)\phi_m(x)dx = 0$ if $n \neq m$.

2. For the regular Sturm-Liouville problem, and two eigenfunctions corresponding to a given eigenvalue are linearly dependent.

Eg. For $y''+\lambda y=0$, $y(0)=y(\pi/2)=0$.

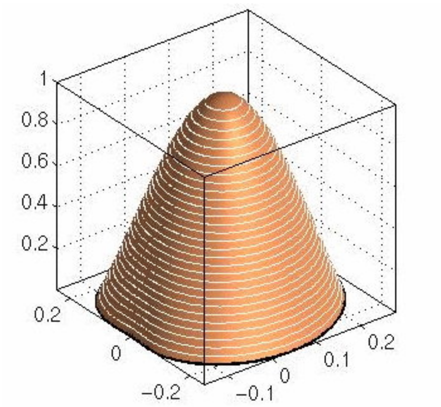
1. If $\lambda=0$, $y=ax+b \Rightarrow a=b=0$: trivial solution

2. If $\lambda=k^2>0$, $y=a\cos(kx)+b\sin(kx) \Rightarrow a=0$, $k=2n \Rightarrow \lambda=k^2=4n^2$

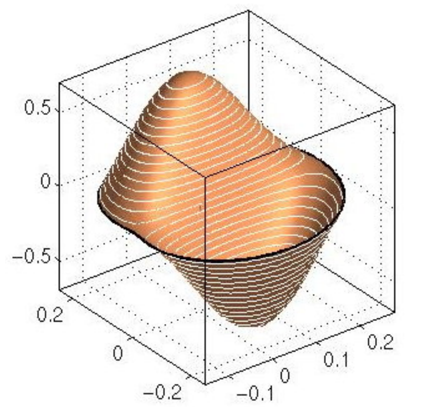
3. If $\lambda=-k^2<0$, $y=ae^{kx}+be^{-kx} \Rightarrow$ no solutions

And $p(x)=1$, $\int_0^{\pi/2} p(x)\sin(2nx)\sin(2mx)dx = 0$ if $n \neq m$.

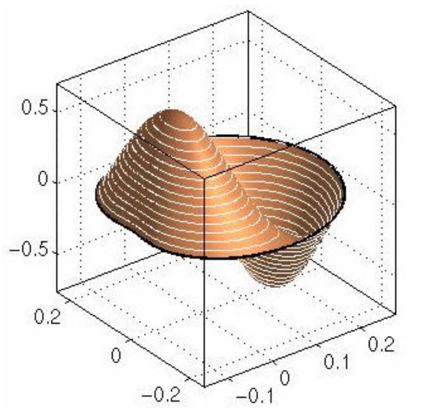
Eg. The eigenfunctions and their corresponding eigenvalues of the stationary Helmholtz equation $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-k^2\psi$ are presented as follows.



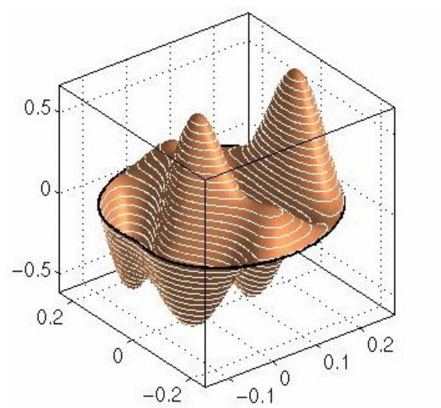
The first eigenfunction, $k^2 = 106.6774$



The second eigenfunction, $k^2 = 254.2339$

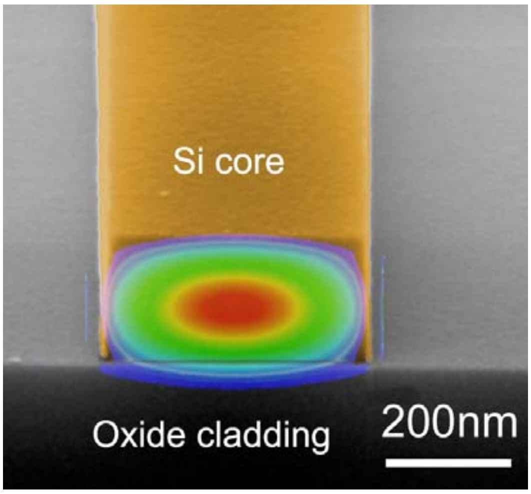


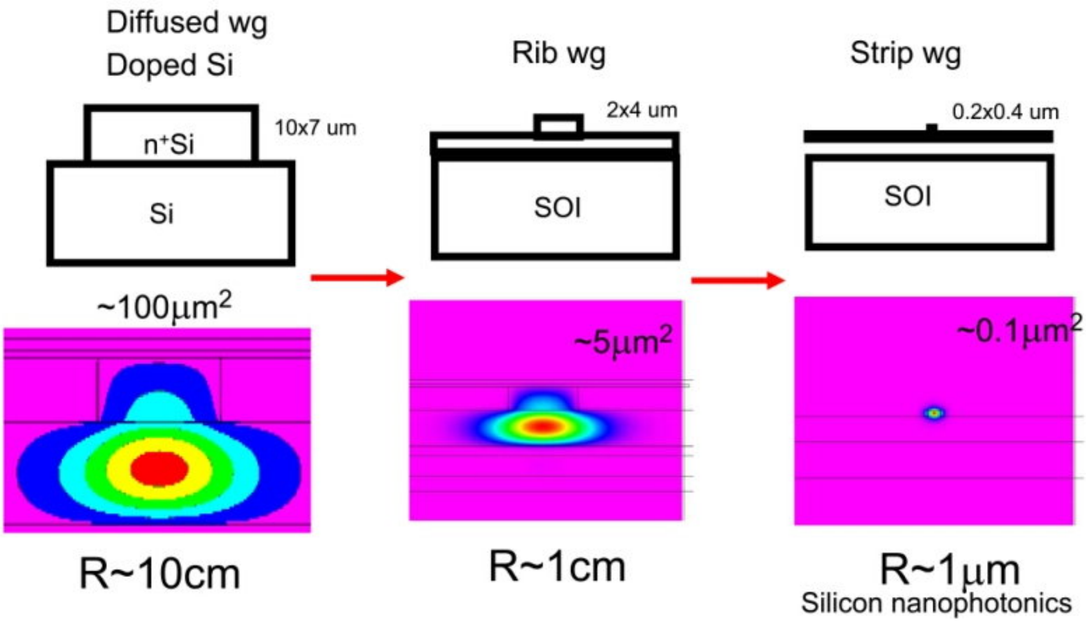
The third eigenfunction, $k^2 = 286.0975$

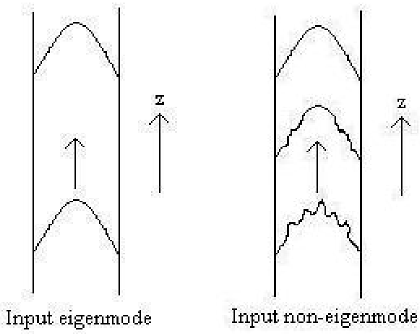


The tenth eigenfunction, $k^2 = 960.0726$

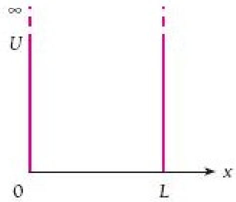
Eg. Given a refractive index distribution *n*(*x*,*y*), the eigenmodal function *Φ*(*x*,*y*) of an optical waveguide fulfills $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2n(x,y)^2\Phi=\beta^2\Phi$, where β^2 is the eigenvalue and β represents the phase constant of the lossy waveguide or the propagation constant of the lossless waveguide. The eigenmodes of some optical waveguides are presented as follows.



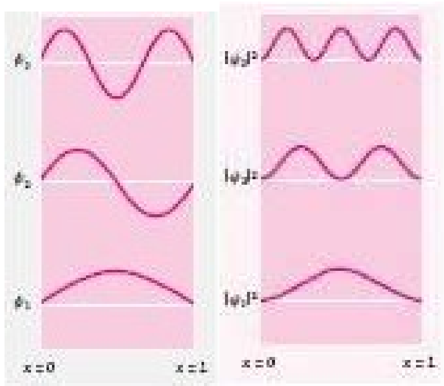


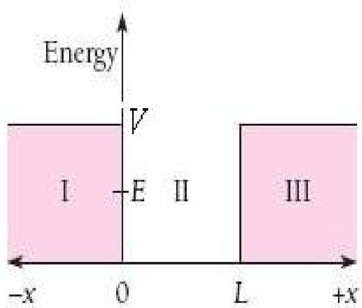


If the eigenmode is injected into an infinitely-long straight waveguide, it can propagate along the waveguide without any deformation. However, in case the input light is not an eigenmode, some optical power loss occurs and then it becomes the eigenmode gradually.



Eg. One-dimensional wave function $\Psi(x)$ in a quantum well with infinitely hard walls, $V=\begin{cases}0, & 0 \le x \le L \\ \infty, & elsewhere\end{cases}$, fulfills $d^2\Psi/dx^2+2mE\Psi/\hbar^2=0$ in $0 \leqq x \leqq L$ with boundary conditions $\Psi(0)=\Psi(L)=0$. It can be transformed into the eigenvalue problem as $-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}=E\Psi$. It is proved that the eigenvalue E is quantized as $E_n=\frac{n^2\pi^2\hbar^2}{2mL^2}$ and the corresponding eigenfunction is $\Psi_n(x)=\sqrt{2/L}\sin(n\pi x/L)$.



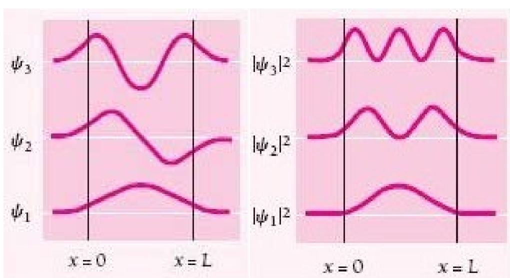


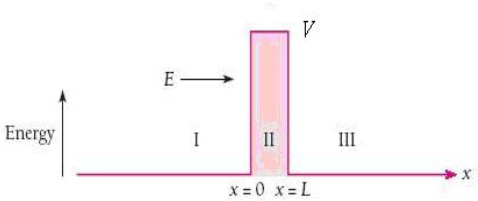
Eg. One-dimensional wave function $\Psi(x)$ in a quantum well with two finite potential walls, $V=\begin{cases}0, & 0 \le x \le L \\ V, & elsewhere\end{cases}$, fulfills

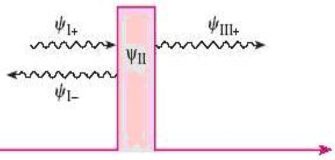
$$\begin{cases}\frac{d^2\Psi_I}{dx^2}+\frac{2m}{\hbar^2}(E-V)\Psi_I=0 \\ \frac{d^2\Psi_{II}}{dx^2}+\frac{2m}{\hbar^2}\Psi_{II}=0 \\ \frac{d^2\Psi_{III}}{dx^2}+\frac{2m}{\hbar^2}(E-V)\Psi_{III}=0\end{cases}$$

with boundary conditions: $\Psi_I(0)=\Psi_{II}(0)$, $\Psi_{II}(L)=\Psi_{III}(L)$, $\Psi_I'(0)=\Psi_{II}'(0)$, $\Psi_{II}'(L)=\Psi_{III}'(L)$. And the eigenfunctions have the forms as

$$\begin{cases}\Psi_I(x)=Ce^{\alpha x} \\ \Psi_{II}(x)=A\sin(\frac{\sqrt{2mE}x}{\hbar})+B\cos(\frac{\sqrt{2mE}x}{\hbar}) \\ \Psi_{III}(x)=De^{-\alpha x}\end{cases}.$$







Eg. Tunnel Effect: One-dimensional wave function $\Psi(x)$ in a quantum barrier, $V(x)$= $\begin{cases} V(>E), & 0 \le x \le L \\ 0, & elsewhere \end{cases}$ **, fulfills**

$$\begin{cases} \frac{d^2\Psi_I}{dx^2} + \frac{2m}{\hbar^2}E\Psi_I = 0 \\ \frac{d^2\Psi_{II}}{dx^2} + \frac{2m}{\hbar^2}(E-V)\Psi_{II} = 0 \\ \frac{d^2\Psi_{III}}{dx^2} + \frac{2m}{\hbar^2}E\Psi_{III} = 0 \end{cases}$$

with boundary conditions: $\Psi_I(0)=\Psi_{II}(0)$, $\Psi_{II}(L)=\Psi_{III}(L)$, $\Psi_I'(0)=\Psi_{II}'(0)$, $\Psi_{II}'(L)=\Psi_{III}'(L)$. And hence the eigenfunctions have the forms as

$$\begin{cases} \Psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x} \\ \Psi_{II}(x) = Ce^{-ik_2x} + De^{-ik_2x} \\ \Psi_{III}(x) = Fe^{ik_3x} \end{cases}$$

, where $k_1=\frac{\sqrt{2mE}}{\hbar}$ **,** $k_2=\frac{\sqrt{2m(V-E)}}{\hbar}$ **,** $k_3=\frac{\sqrt{2mE}}{\hbar}=k_1$.

The quantum mechanics can prove that the transmission probability is $T=|\Psi_{III}^+|^2/|\Psi_I^+|^2=|F|^2/|A|^2 \approx [\frac{16}{4+(K_2/K_1)^2}]\cdot e^{-2k_2L} \approx e^{-2k_2L}$.