Chapter 9 Complex Numbers, Functions, and Operators

9-1 Complex Numbers and Complex Functions

Complex number z: $z=a+ib=r\angle\theta=r\cos\theta+ir\sin\theta$

Conjugate complex of z: $\bar{z}=a-ib=r\angle-\theta=r\cos\theta-ir\sin\theta$

Theorem $z\in R \Leftrightarrow z=\bar{z}$, **z is pure imaginary** $\Leftrightarrow z=-\bar{z}$

Theorem $R_e(z)=\dfrac{z+\bar{z}}{2}=a$, $I_m(z)=\dfrac{z-\bar{z}}{2i}=b$

Theorem $\overline{zw}=\bar{z}\,\bar{w}$, $\overline{z+w}=\bar{z}+\bar{w}$, $|z|^2=z\cdot\bar{z}$

Theorem **$|zw|=|z||w|$, arg(zw)=arg(z)+arg(w), $|z/w|=|z|/|w|$ if $w\neq 0$, arg(z/w)=arg(z)-arg(w)**

Theorem

$$w^2=z=a+ib=r\angle\theta \Leftrightarrow w=\sqrt{r}\angle(\frac{\theta}{2}+n\pi)=\sqrt{r}\cos(\frac{\theta}{2}+n\pi)+i\sqrt{r}\sin(\frac{\theta}{2}+n\pi)$$

Theorem $z\cdot w=R_e(\bar{z}w)=\dfrac{1}{2}(\bar{z}w+z\bar{w})=|z||w|\cos\phi$

$$z\times w=I_m(\bar{z}w)=\frac{1}{2i}(\bar{z}w-z\bar{w})=|z||w|\sin\phi$$

Analytic function $f(z)$: $f(z)$ and $f'(z)$ are continuous and bounded at a certain point.

Theorem For an analytic complex function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$, we have $f'(z)=\dfrac{df(z)}{dz}=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$.

Cauchy-Riemann Equations to test whether $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic:

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic $\Leftrightarrow \begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}$

Eg. Is $f(z)=1/z$ is an analytic function?

(Sol.) $f(z)=\dfrac{1}{z}\to\infty$ as $z\to 0$.

If $z\neq 0$, $f(z)=\dfrac{1}{z}=\dfrac{1}{x+iy}=\dfrac{x}{x^2+y^2}-\dfrac{iy}{x^2+y^2}=u(x,y)+iv(x,y)$

$$\Rightarrow\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{-x^2+y^2}{(x^2+y^2)^2}=\dfrac{\partial v}{\partial y}\\ \dfrac{\partial u}{\partial y}=\dfrac{-2xy}{(x^2+y^2)^2}=-\dfrac{\partial v}{\partial x}\end{cases}$$, Cauchy-Riemann Equations are fulfilled.

∴ If $z=0$, $f(z)=1/z$ is not analytic. But it is analytic except $z=0$.

Eg. Is $f(x,y)=x+y+i(-x+y)$ an analytic function?

(Sol.) $u(x,y)=x+y$, $v(x,y)=-x+y \Rightarrow \frac{\partial u}{\partial x}=1=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=1=-\frac{\partial v}{\partial x}$, $\therefore$ $f(x,y)$ is analytic.

Eg. Is $f(x,y)=x^2+iy^2$ an analytic function?

(Sol.) $\begin{matrix} u(x,y)=x^2 \\ v(x,y)=y^2 \end{matrix} \Rightarrow \begin{cases} \frac{\partial u}{\partial x}=2x \neq \frac{\partial v}{\partial y}=2y \\ \frac{\partial u}{\partial y}=0=-\frac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is not analytic.

Eg. Is $f(x,y)=x^2-y^2-2y+i(2xy+2x)$ an analytic function?

(Sol.) $\begin{cases} u(x,y)=x^2-y^2-2y \\ v(x,y)=2xy+2x \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x}=2x=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-2y-2=-\frac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is analytic.

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=x^2-y^2$. Find out its imaginary part $v(x,y)$.

(Sol.) $u(x,y)=x^2-y^2 \Rightarrow \begin{cases} \frac{\partial u}{\partial x}=2x=\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}=-2y=-\frac{\partial v}{\partial x} \end{cases} \Rightarrow v(x,y)=2xy+c$.

Eg. Show that $f(z)=z^3$ is an analytic function.

(Proof) $f(z)=f(x+iy)=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)=u(x,y)+iv(x,y)$

$\frac{\partial u}{\partial x}=3x^2-3y^2=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-6xy=-\frac{\partial v}{\partial x}$, Cauchy-Riemann Equations are fulfilled.

Eg. Show that $f(z)=|z|^2=z\bar{z}$ is not analytic at any point $z\neq 0$.

(Proof) $f(z)=f(x+iy)=x^2+y^2=u(x,y)+iv(x,y) \Rightarrow v(x,y)=0$

$\frac{\partial u}{\partial x}=2x\neq\frac{\partial v}{\partial y}$ if $x\neq 0$, $\frac{\partial u}{\partial y}=2y\neq-\frac{\partial v}{\partial x}$ if $y\neq 0 \Rightarrow f(z)$ is not analytic at $z\neq 0$

Eg. Is $f(z)=\bar{z}$ an analytic function?

(Sol.) $f(z)=f(x+iy)=x-iy=u(x,y)+iv(x,y)\ \begin{cases}\dfrac{\partial u}{\partial x}=1\neq\dfrac{\partial v}{\partial y}=-1\\ \dfrac{\partial u}{\partial y}=0=-\dfrac{\partial v}{\partial x}\end{cases}$,

$\therefore$ $f(z)$ is not analytic.

Check $f'(i)=\lim\limits_{z\to i}\dfrac{f(z)-f(i)}{z-i}=\lim\limits_{z\to i}\dfrac{\bar{z}-(-i)}{z-i}=\lim\limits_{z\to i}\dfrac{\bar{z}+i}{z-i}$

(a) if $z=\alpha i$, α is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{-\alpha i+i}{\alpha i-i}=\dfrac{1-\alpha}{\alpha-1}=-1$, (b) if $z=\beta+i$, β is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{\beta-i+i}{\beta+i-i}=1\Rightarrow f'(i)$ does not exist! $\therefore$ $f(z)=\bar{z}$ is not analytic

9-2 Elementary Complex Functions

For $z=x+iy$, where $x,y\in R$

Exponential function: $e^z=e^x[\cos(y)+i\sin(y)]$

Theorem $|e^z|=e^x$.

Theorem $e^z=e^w \Leftrightarrow z=w+2n\pi i$, where n is an integer.

Eg. Find e^z for (a) $z=-\dfrac{i\pi}{4}$, (b) $z=3+\dfrac{\pi}{2}i$. **【1990 清大材料所】**

(Sol.) (a) $e^{\frac{-i\pi}{4}}=\cos(\frac{\pi}{4})-i\sin(\frac{\pi}{4})=\dfrac{1-i}{\sqrt{2}}$

(b) $e^{3+\frac{i\pi}{2}}=e^3\cdot[\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2})]=ie^3$

Eg. Solve $e^{4z}=1-i\sqrt{3}$. **【1991 台大機研】**

(Sol.) $e^{4z}=1-i\sqrt{3}=2e^{-i\frac{\pi}{3}}=e^{\ln 2-i\frac{\pi}{3}+i2n\pi}\Rightarrow 4z=\ln 2+i\left(2n\pi-\dfrac{\pi}{3}\right)$

$\Rightarrow z=\dfrac{1}{4}\left[\ln 2+i\left(2n\pi-\dfrac{\pi}{3}\right)\right]$

Eg. $(-i)^i$=? and $(-1)^{2i}$=?

(Sol.) $-1=e^{i\pi}$, $ln(-1)=i\pi$ and $-i=e^{-i\pi/2}$, $ln(-i)=-i\pi/2$,

$(-i)^i=e^{i\ell n(-i)}=e^{i(-i\pi/2)}=e^{\frac{\pi}{2}}$

$(-1)^{2i}=e^{2i\ell n(-1)}=e^{2i(i\pi)}=e^{-2\pi}$

Trigonometric functions: $\sin(z)=\dfrac{e^{iz}-e^{-iz}}{2i}=\dfrac{1}{2i}[e^{i(x+iy)}-e^{-i(x+iy)}]$

$=\sin(x)\cosh(y)+i\cos(x)\sinh(y)$

$$\cos(z)=\frac{e^{iz}+e^{-iz}}{2}=\frac{1}{2}[e^{i(x+iy)}+e^{-i(x+iy)}]=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$$

Theorem sin(iy)=$i$sinh(y) and cos(iy)=cosh(y).

Hyperbolic functions: $\sinh(z)=\dfrac{e^{z}-e^{-z}}{2}$, $\cosh(z)=\dfrac{e^{z}+e^{-z}}{2}$

Logarithm function: $\log(z)=\log|z|+i\arg(z)+2n\pi i=\log|z|+i\arg(z)$

$$=\log\sqrt{x^2+y^2}+i\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right)$$

Inverse trigonometric functions:

$$\sin^{-1}z=k\pi+(-1)^k\cdot\left[\frac{\pi}{2}-iLog(z+\sqrt{1-z^2})\right],\quad k=0,\pm1,\pm2,\cdots$$

$$\cos^{-1}z=2k\pi+i(-1)^k\cdot Log(z+\sqrt{1-z^2}),\quad k=0,\pm1,\pm2\cdots,\quad n=1,2$$

$$\tan^{-1}z=k\pi+\frac{1}{2i}Log\left(\frac{1+iz}{1-iz}\right),\quad k=0,\pm1,\pm2\cdots$$

$$\sinh^{-1}z=k\pi i-(-1)^k\cdot\left\{Log\left[iz+\sqrt{1+z^2}\right]+\frac{\pi}{2}i\right\}$$

$$\cosh^{-1}z=2k\pi i+(-1)^n\cdot Log\left[z+\sqrt{1-z^2}\right]$$

Eg. Show that tan^{-1}z=$\dfrac{i}{2}\ln[\dfrac{i+z}{i-z}]$. 【1991 清大動機研】

(Sol.) Let w= tan^{-1}z,

$$z=\tan w=\frac{\sin w}{\cos w}=\frac{e^{iw}-e^{-iw}}{2i}\cdot\frac{2}{e^{iw}+e^{-iw}}=\frac{1}{i}\cdot\frac{e^{2iw}-1}{e^{2iw}+1}\Rightarrow e^{2iw}=\frac{1+iz}{1-iz}=\frac{i-z}{i+z}$$

$$\Rightarrow 2iw=\ell n\frac{i-z}{i+z}\Rightarrow w=\tan^{-1}z=\frac{1}{2i}\ln[\frac{i-z}{i+z}]=-\frac{i}{2}\ln[\frac{i-z}{i+z}]=\frac{i}{2}\ln[\frac{i+z}{i-z}]$$

9-3 Complex Operators $\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$, $\overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$, **and** $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Theorem **Let** ***F*(*x*,*y*)=** $G(z,\bar{z})$ **be a complex scalar function, and** ***A*(*x*,*y*)=*P*(*x*,*y*)+*iQ*(*x*,*y*)=** $B(z,\bar{z})$ **be a complex vector function. Then we have** $\nabla F = 2\frac{\partial G}{\partial \bar{z}}$, $\nabla \cdot A = 2\operatorname{Re}\left(\frac{\partial B}{\partial z}\right)$, $\nabla \times A = 2I_m\left(\frac{\partial B}{\partial z}\right)$, **and** $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

(Proof) $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$

$$\therefore\ \nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}},\ \overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z} \Rightarrow \nabla F(x,y) = 2\frac{\partial}{\partial \bar{z}}\{G(z,\bar{z})\}$$

$$\nabla \cdot A = R_e(\overline{\nabla}A) = R_e\left(2\frac{\partial}{\partial z}B\right) = 2R_e\left(\frac{\partial B}{\partial z}\right) = R_e\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P+iQ)\right\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\nabla \times A = I_m(\overline{\nabla}A) = I_m\left(2\frac{\partial}{\partial z}B\right) = 2I_m\left(\frac{\partial B}{\partial z}\right) = I_m\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P+iQ)\right\} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\nabla^2 = \nabla \cdot \nabla = R_e(\overline{\nabla}\nabla) = R_e\left(2\frac{\partial}{\partial z}2\frac{\partial}{\partial \bar{z}}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Eg. Solve $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{8}{x^2+y^2}$.

(Sol.) $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\frac{\partial^2 U}{\partial z \partial \bar{z}} = \frac{8}{x^2+y^2} = \frac{8}{z\bar{z}}$

$$\Rightarrow U(z,\bar{z}) = 2\ln|z| \cdot \ln|\bar{z}| + F_1(z) + F_2(\bar{z})$$

$$= 2\left[\ell n\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\right] \cdot \left[\ell n\sqrt{x^2+y^2} - i\tan^{-1}\left(\frac{y}{x}\right)\right] + F_1(x+iy) + F_2(x-iy)$$

$$= 2\left[\ell n\sqrt{x^2+y^2}\right]^2 + 2\left[\tan^{-1}\left(\frac{y}{x}\right)\right]^2 + F_1(x+iy) + F_2(x-iy)$$

Green's theorem $\oint_c P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$

General Green's theorem $\oint_c P(z,\bar{z})dz + Q(z,\bar{z})d\bar{z} = 2i\iint_R \left(\frac{\partial P}{\partial \bar{z}} - \frac{\partial Q}{\partial z}\right)dA$

Eg. Show that $A = \frac{1}{4i}\oint_c \bar{z}dz - zd\bar{z}$.

(Proof) $\frac{1}{4i}\oint_c \bar{z}dz - zd\bar{z} = \frac{1}{4i} \cdot 2i\iint_R [1-(-1)]dxdy = \iint_R dxdy = A$