Chapter 4 Laplace Transforms

4-1 Laplace Transform $F(s)=L[f(t)]=\int_0^\infty e^{-st}\cdot f(t)dt$

Eg. Evaluate $L[\cos(at)]$ and $L[\sin(at)]$.

(Sol.) $e^{iat}=\cos(at)+i\sin(at)$

$$L[e^{iat}]=\int_0^\infty e^{-st}\cdot e^{iat}dt=\int_0^\infty e^{-(s-ia)t}dt=\frac{-1}{s-ia}\cdot e^{-(s-ia)t}\Big|_0^\infty=\frac{1}{s-ia}$$

$$=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}=L[\cos(at)]+iL[\sin(at)]$$

$$\therefore\ L[\cos(at)]=\frac{s}{s^2+a^2}\ \text{ and }\ L[\sin(at)]=\frac{a}{s^2+a^2}$$

Eg Find $\int_0^\infty e^{-ax}\cdot\cos(bx)dx$, with $a>0$. [台大電研] (Ans.) $a/(a^2+b^2)$

Basic theorems of Laplace transforms $F(s)=L[f(t)]$ and $G(s)=L[g(t)]$:

1. $L[c_1f(t)+c_2g(t)]=c_1F(s)+c_2G(s)$

2. $L[f'(t)]=sF(s)-f(0)$, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, $L[f'''(t)]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$, and $L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

(Proof) $L[f'(t)]=\int_0^\infty e^{-st}\cdot f'(t)dt=\int_0^\infty e^{-st}df(t)=e^{-st}f(t)\|_0^\infty-\int_0^\infty(-s)e^{-st}\cdot f(t)dt$

$$=-f(0)+s\int_0^\infty e^{-st}\cdot f(t)dt=sF(s)-f(0)$$

Similarly, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, and by mathematical induction, we have $L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

3. $L\left[\int_0^t f(u)du\right]=\frac{F(s)}{s}$

(Proof) $L\left[\int_0^t f(u)du\right]=\int_0^\infty e^{-st}\int_0^t f(u)dudt=\int_0^\infty\int_0^t e^{-st}f(u)dudt=\int_0^\infty\int_u^\infty e^{-st}f(u)dtdu$

$$=\int_0^\infty f(u)du\int_u^\infty e^{-st}dt=-\frac{1}{s}\int_0^\infty f(u)[0-e^{-su}]du=\frac{1}{s}\int_0^\infty f(u)e^{-su}du=\frac{F(s)}{s}$$

4. $L[t^nf(t)]=(-1)^nF^{(n)}(s)$

(Proof) $L[tf(t)]=\int_0^\infty e^{-st}\cdot tf(t)dt=\int_0^\infty -\frac{de^{-st}}{ds}\cdot f(t)dt=-\frac{d}{ds}\int_0^\infty e^{-st}\cdot f(t)dt=-F'(s)$

By mathematical induction, we have $L[t^nf(t)]=(-1)^nF^{(n)}(s)$

5. $L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(u)du$

6. $L[f(at)]=[F(s/a)]/a$ and $L[f(t/a)]=aF(as)$, $a>0$

(Proof) For $a>0$, let $at=u$

$$L[f(at)]=\int_0^\infty e^{-st}\cdot f(at)dt=\frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})at}\cdot f(at)d(at)=\frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})u}\cdot f(u)du=\frac{1}{a}F[(\frac{s}{a})]$$

Let $b=\frac{1}{a}\Rightarrow L[f(\frac{t}{a})]=L[f(bt)]=\frac{1}{b}F[(\frac{s}{b})]=aF(as)$ and then $L^{-1}[F(as)]=\frac{1}{a}[f(\frac{t}{a})]$

7. $L[f(t)]=\frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}\cdot f(t)dt$ if $f(t+T)=f(t)$

(Proof) Let $t+T=u$, $t+2T=v$, …

$$L[f(t)]=\int_0^\infty e^{-st}\cdot f(t)dt=\int_0^T e^{-st}\cdot f(t)dt+\int_T^{2T} e^{-st}\cdot f(t)dt+\int_{2T}^{3T} e^{-st}\cdot f(t)dt+\dots$$

$$=\int_0^T e^{-st}\cdot f(t)dt+\int_T^{2T} e^{-st}\cdot f(t+T)dt+\int_{2T}^{3T} e^{-st}\cdot f(t+2T)dt+\dots$$

$$=\int_0^T e^{-st}\cdot f(t)dt+e^{-sT}\int_T^{2T} e^{-s(t+T)}\cdot f(t+T)dt+e^{-2sT}\int_{2T}^{3T} e^{-s(t+2T)}\cdot f(t+2T)dt+\dots$$

$$=\int_0^T e^{-st}\cdot f(t)dt+e^{-sT}\int_0^T e^{-su}\cdot f(u)du+e^{-2sT}\int_0^T e^{-sv}\cdot f(v)dv+\dots$$

$$=(1+e^{-sT}+e^{-2sT}+\dots)\int_0^T e^{-st}\cdot f(t)dt=\frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}\cdot f(t)dt$$

8. $L[f(t-a)u(t-a)]=e^{-as}F(s)$

(Proof) For $t>a$, let $t-a=u$

$$L[f(t-a)u(t-a)]=\int_0^\infty e^{-st}\cdot f(t-a)u(t-a)dt=\int_a^\infty e^{-st}\cdot f(t-a)d(t-a)$$

$=e^{-as}\int_a^\infty e^{-s(t-a)}\cdot f(t-a)d(t-a)=e^{-as}\int_0^\infty e^{-su}\cdot f(u)du=e^{-as}F(s)$, and then we have $L^{-1}[F(s)e^{-as}]=f(t-a)u(t-a)$

9. $L[f(t)e^{at}]=F(s-a)$, $s>a$

(Proof) For $s>a$, $L[f(t)e^{at}]=\int_0^\infty e^{-st}\cdot f(t)e^{at}dt=\int_0^\infty e^{-(s-a)t}\cdot f(t)dt=F(s-a)$, and then $L^{-1}[F(s+a)]=f(t)e^{-at}$

10. $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

11. $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

Eg. Find $L[t^n]$.

(Sol.) According to $L[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}F(s)$ and $L[1]=\int_0^\infty e^{-st}dt=\frac{1}{s}$

$$L[t^n]=L[t^n\cdot 1]=(-1)^n\frac{d^n}{ds^n}(s^{-1})=n!/s^{n+1}=\frac{\Gamma(n+1)}{s^{n+1}}$$

Eg. Find $L[e^{at}\cos(kt)]$ and $L[e^{at}\sin(kt)]$.

(Sol.) $L[\cos(kt)]=\dfrac{s}{s^2+k^2}$ and $L[\sin(kt)]=\dfrac{k}{s^2+k^2}$

According to $L[f(t)\cdot e^{at}]=F(s-a)$,

$$\therefore\ L[e^{at}\cdot\cos(kt)]=\frac{s-a}{(s-a)^2+k^2}\ \text{ and }\ L[e^{at}\cdot\sin(kt)]=\frac{k}{(s-a)^2+k^2}$$

Eg. Find $L[3t\text{-}5\sin(2t)]$. [台大電研]

(Sol.) $L[3t\text{-}5\sin(2t)]=3L[t]\text{-}5L[\sin(2t)]=\dfrac{3}{s^2}-\dfrac{10}{s^2+4}$

Eg. Find $L[e^{-t}f(3t)]$ in case of $L[f(t)]=e^{-1/s}$.

(Sol.) 1. $L[f(3t)]=\dfrac{1}{3}e^{-1/(s/3)}=\dfrac{1}{3}e^{-3/s}$, $L[e^{-t}\cdot f(3t)]=\dfrac{1}{3}e^{\frac{-3}{s+1}}$. The result is correct!

2. $L[e^{-t}\cdot f(t)]=e^{-\frac{1}{s+1}}$, $L[e^{-t}\cdot f(3t)]=\dfrac{1}{3}e^{\frac{-1}{(s/3)+1}}=\dfrac{1}{3}e^{\frac{-3}{s+3}}$. The result is wrong!

Another method:
$$L[e^{-t}\cdot f(3t)]=\int_0^\infty e^{-st}\cdot e^{-t}\cdot f(3t)dt=\int_0^\infty e^{-(s+1)t}\cdot f(3t)dt$$
$$=\frac{1}{3}\int_0^\infty e^{-\left(\frac{s+1}{3}\right)\cdot(3t)}f(3t)d(3t)=\frac{1}{3}\int_0^\infty e^{-\left(\frac{s+1}{3}\right)u}f(u)du$$
$$=\frac{1}{3}F\left(\frac{s+1}{3}\right)=\frac{1}{3}e^{\frac{-3}{s+1}}$$

Eg. Find $L[f(t)]$ if $f(t+2)=f(t)$ and $f(t)=\begin{cases}1\ , & 0<t\le 1\\ -1, & 1<t\le 2\end{cases}$.

(Sol.) According to $L[f(t)]=\dfrac{1}{1-e^{-sT}}\cdot\displaystyle\int_0^T e^{-st}\cdot f(t)dt$ if $f(t+T)=f(t)$ and T=2,

$$L[f(t)]=\frac{1}{1-e^{-2s}}\cdot\left[\int_0^1 e^{-st}dt+\int_1^2(-1)e^{-st}dt\right]=\frac{1}{1-e^{-2s}}\cdot\left[\frac{e^{-st}}{-s}\bigg|_0^1+\frac{e^{-st}}{s}\bigg|_1^2\right]$$
$$=\frac{1}{1-e^{-2s}}\cdot\left[\frac{1-e^{-s}+e^{-2s}-e^{-s}}{s}\right]=\frac{1}{s}\cdot\frac{(1-e^{-s})^2}{(1+e^{-s})(1-e^{-s})}=\frac{1}{s}\cdot\frac{1-e^{-s}}{1+e^{-s}}$$

Eg. Find $L[\int_0^t \frac{1-e^{-u}}{u}du]$.

(Sol.) According to $L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(u)du$ and $L[1]=\frac{1}{s}$, $L\left[\frac{1}{t}\right]=\int_s^\infty \frac{du}{u}=\ell n(s)\Big|_s^\infty$

$$L[e^{-t}]=\frac{1}{s+1},\quad L\left[\frac{e^{-t}}{t}\right]=\int_s^\infty \frac{ds}{s+1}=\ell n(s+1)\Big|_s^\infty$$

$$L\left[\frac{1-e^{-t}}{t}\right]=\ell n(s)\Big|_s^\infty-\ell n(s+1)\Big|_s^\infty=-\ell n\left(\frac{s}{s+1}\right)$$

According to $L\left[\int_0^t f(u)du\right]=\frac{F(s)}{s}$, $\therefore$ $L\left[\int_0^t \frac{1-e^{-u}}{u}du\right]=\frac{-1}{s}\ell n\left(\frac{s}{s+1}\right)$

Eg. Find $\int_0^\infty \int_0^t \frac{e^{-t}\sin(u)}{u}dudt$.

(Sol.) $\int_0^\infty e^{-t}\cdot\left[\int_0^t \frac{\sin(u)}{u}du\right]dt=L\left[\int_0^t \frac{\sin(u)}{u}du\right]_{s=1}$

According to $L\left[\int_0^t f(u)du\right]=\frac{F(s)}{s}$ and $L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$

$$L\left[\int_0^t \frac{\sin(u)}{u}du\right]=\frac{1}{s}L\left[\frac{\sin(t)}{t}\right]=\frac{1}{s}\int_s^\infty L[\sin(t)]ds=\frac{1}{s}\int_s^\infty\left(\frac{1}{s^2+1}\right)ds=\frac{1}{s}[\frac{\pi}{2}-\tan^{-1}(s)]$$

Set s=1, $\int_0^\infty \int_0^t \frac{e^{-t}\cdot\sin(u)}{u}dudt=\frac{\pi}{4}$

Eg. Find $\int_0^\infty \frac{\sin(x)}{x}dx$. **[中央光電所、交大應數研]**

(Sol.)

$$L\left[\frac{\sin(t)}{t}\right]=\int_0^\infty e^{-st}\cdot\frac{\sin(t)}{t}dt=\int_s^\infty L[\sin(t)]ds=\int_s^\infty \frac{1}{s^2+1}ds=\tan^{-1}(s)\Big|_s^\infty=\frac{\pi}{2}-\tan^{-1}(s)$$

Set s=0, $\int_0^\infty \frac{\sin(x)}{x}dx=\frac{\pi}{2}$

4-2 Inverse Laplace Transform $L^{-1}[F(s)]=f(t)$

Basic theorems of the inverse Laplace Transforms:

1. $L^{-1}[F(s+a)]=f(t)e^{-at}$
2. $L^{-1}[F(s)\cdot e^{-as}]=f(t-a)\cdot u(t-a)=\begin{cases}f(t-a), t\geq a\\ 0, t<a\end{cases}$
3. $L^{-1}[F(as)]=[f(t/a)]/a$
4. $L^{-1}[F^{(n)}(s)]=(-1)^n t^n f(t)$

5. $L^{-1}[\int_s^{\infty} F(u)du] = \dfrac{f(t)}{t}$

6. $L^{-1}[sF(s)]=f'(t)+f(0)\delta(t)$

7. $L^{-1}\left[\dfrac{F(s)}{s}\right] = \int_0^t f(u)du$

8. $L^{-1}[1/s^n]=t^{n-1}/\Gamma(n)$

9. $L^{-1}[c_1F(s)+c_2G(s)]=c_1f(t)+c_2g(t)$

Heaviside's formula:

$$F(s) = \frac{A_1}{(s-a)} + \cdots + \frac{A_m}{(s-a)^m} + \frac{B}{s-b} + \frac{C}{s-c} \cdots + \frac{\alpha(s-r)}{(s-r)^2+\omega^2} + \frac{\beta\omega}{(s-r)^2+\omega^2}$$

$$\Rightarrow L^{-1}[F(s)] = e^{at} \cdot \left\{ A_1 + A_2 t + \cdots + \frac{A_m t^{m-1}}{(m-1)!} \right\} + Be^{bt} + Ce^{ct} + \cdots$$

$$+ \alpha\cos(\omega t) \cdot e^{rt} + \beta\sin(\omega t) \cdot e^{rt}$$

Eg. Find $L^{-1}\left[\dfrac{2s^2-9s+19}{(s-1)^2(s+3)}\right]$.

(Sol.) $\dfrac{2s^2-9s+19}{(s-1)^2(s+3)} = \dfrac{A_1}{(s-1)} + \dfrac{A_2}{(s-1)^2} + \dfrac{B}{s+3}$

$$\Rightarrow \begin{cases} A_1 + B = 2 \\ 2A_1 + A_2 - 2B = -9 \\ -3A_1 + 3A_2 + B = 19 \end{cases} \Rightarrow \begin{cases} A_1 = -2 \\ A_2 = 3 \\ B = 4 \end{cases},$$

$$L^{-1}\left[\frac{2s^2-9s+19}{(s-1)^2(s+3)}\right] = e^t(-2+3t) + 4e^{-3t}$$

Eg. Find $L^{-1}\left[\dfrac{1}{s(s-4)^2}\right]$. [清大電研]

(Sol.) $\dfrac{1}{s(s-4)^2} = \dfrac{a}{s-4} + \dfrac{b}{(s-4)^2} + \dfrac{c}{s} \Rightarrow \begin{matrix} a+c=0 \\ -4a+b-8c=0 \\ 16c=1 \end{matrix} \Rightarrow a = \dfrac{-1}{16}, b = \dfrac{1}{4}, c = \dfrac{1}{16}$

$$L^{-1}[\frac{1}{s(s-4)^2}] = \frac{-e^{4t}+4te^{4t}+1}{16}$$

Eg. Find $L^{-1}\left[\dfrac{2s-1}{s(s-1)}\right]$ **and** $L^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right]$. [交大材研]

(Sol.)

$$\frac{2s-1}{s(s-1)}=\frac{A}{s}+\frac{B}{s-1}\Rightarrow A(s-1)+Bs=2s-1\Rightarrow A=1, B=1$$

$$\Rightarrow L^{-1}\left[\frac{2s-1}{s(s-1)}\right]=e^t+1$$

$$\frac{1}{s^2(s^2+1)}=\frac{A_1}{s}+\frac{A_2}{s^2}+\frac{\alpha s+\beta}{s^2+1}$$

$$\Rightarrow\begin{cases}A_1+\alpha=0\\A_2+\beta=0\\A_1=0\\A_2=1\end{cases}\Rightarrow A_1=0, A_2=1, \alpha=0, \beta=-1$$

$$L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]=e^{0t}(0+t)+0+(-1)\sin(t)e^{0t}=t-\sin t$$

Eg. Find $L^{-1}\left[\dfrac{1}{s(s+1)^3}\right]$.

(Sol.) $\dfrac{1}{s(s+1)^3}=\dfrac{a}{s+1}+\dfrac{b}{(s+1)^2}+\dfrac{c}{(s+1)^3}+\dfrac{d}{s}$

$$\begin{cases}a+d=0\\2a+b+3d=0\\a+b+c+3d=0\\d=1\end{cases}\Rightarrow a=-1, b=-1, c=-1, d=1\Rightarrow$$

$$L^{-1}\left[\frac{1}{s(s+1)^3}\right]=e^{-t}\left[-1-t+\frac{-1\times t^{3-1}}{(3-1)!}\right]+e^{0t}=e^{-t}\left[-1-t-\frac{t^2}{2}\right]+1$$

Eg. Find $L^{-1}\left[\dfrac{e^{-2s}}{(s-2)^4}\right]$. [中山電研]

(Sol.) According to $L^{-1}[F(s)\cdot e^{-as}]=f(t-a)\cdot u(t-a)$, $L^{-1}\left[\dfrac{1}{(s-2)^4}\right]=\dfrac{e^{2t}\cdot t^3}{3!}$

$$L^{-1}\left[\frac{e^{-2s}}{(s-2)^4}\right]=\frac{e^{2(t-2)}\cdot(t-2)^3}{3!}\cdot u(t-2)$$

Eg. Find $L^{-1}(\frac{s+\pi}{s^2+\pi^2}e^{-s})$. **[成大電研]**

(Sol.) $L^{-1}(\frac{s+\pi}{s^2+\pi^2})=\cos(\pi t)+\sin(\pi t)$, $\because$ $L^{-1}[F(s)\cdot e^{-as}]=f(t-a)\cdot u(t-a)$,

$\therefore$ $L^{-1}(\frac{s+\pi}{s^2+\pi^2}e^{-s})=\{\cos[\pi(t-1)]+\sin[\pi(t-1)]\}\cdot u(t-1)$

Eg. Find $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$.

(Sol.) According to $L^{-1}[F(s+a)]=e^{-at}\cdot f(t)$ and $L^{-1}[F^{(n)}(s)]=(-1)^n t^n f(t)$

$$L^{-1}\left[\frac{s+1}{\left[(s+1)^2+1\right]^2}\right]=e^{-t}\cdot L^{-1}\left[\frac{s}{(s^2+1)^2}\right]=e^{-t}\cdot L^{-1}\left[\frac{1}{2}\cdot(-1)\cdot\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]=\frac{e^{-t}}{2}\cdot t\cdot\sin(t)$$

Eg. Find $L^{-1}\left[\ell n\left(\frac{s+2}{s+1}\right)\right]$.

(Sol.) According to $L^{-1}[\int_s^{\infty}F(u)du]=\frac{f(t)}{t}$ and $L^{-1}[F(s+a)]=e^{-at}\cdot f(t)$

$$L^{-1}\left[\ell n\left(\frac{s+2}{s+1}\right)\right]=L^{-1}\left[\int_s^{\infty}\left(\frac{1}{s+1}-\frac{1}{s+2}\right)ds\right]=\frac{1}{t}\cdot[e^{-t}-e^{-2t}]$$

4-3 Laplace Transform Solutions of Differential Equations with Polynomial Coefficients

Eg. Solve $xy''-xy'-y=0$, $y(0)=0$ and $y'(0)=3$. [成大電研]

(Sol.) $L[y(x)]=\int_0^\infty y(x)e^{-sx}dx = Y(s)$, $L[x^n y(x)] = (-1)^n \frac{d^n}{ds^n}Y(s)$, and

$$L[y^{(n)}(x)] = s^n Y(s) - s^{n-1}\cdot y(0) - s^{n-2}\cdot y'(0) - \cdots\cdots - s\cdot y^{(n-2)}(0) - y^{(n-1)}(0)$$

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] - (-\frac{d}{ds})[sY(s) - y(0)] - Y(s) = 0$$

$$-2sY(s) - s^2Y'(s) + Y(s) + sY'(s) - Y(s) = 0$$

$$(-s^2+s)Y'(s) - 2sY(s) = 0,\quad Y'(s) + \frac{2}{s-1}Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{A}{(s-1)^2} \Rightarrow y(x) = Axe^x,\ y'(0)=3 \Rightarrow A=3,\ \therefore\ y(x)=3xe^x$$

Eg. Solve $y''+2xy'-4y=1$, $y(0)=y'(0)=0$. [清大電研]

(Sol.) $L[y''+2xy'-4y]=L(1)=\frac{1}{s}$

$$s^2Y(s) - sy(0) - y'(0) + 2\cdot(-1)\frac{d}{ds}[sY(s) - y(0)] - 4Y(s) = \frac{1}{s}$$

$$s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s},\quad Y'(s) + \left(\frac{3}{s} - \frac{s}{2}\right)Y(s) = -\frac{1}{2s^2}$$

$$Y(s) = \frac{1}{s^3} + c\frac{e^{s^2/4}}{s^3},\quad \lim_{s\to\infty} sY(s) = \lim_{x\to 0} y(x) \neq \infty,\ \therefore\ c=0 \Rightarrow y(x) = L^{-1}\left(\frac{1}{s^3}\right) = \frac{x^2}{2}$$

Eg. Solve $y''+9y=\cos(2t)$, $y(0)=1$ and $y\left(\frac{\pi}{2}\right) = -1$.

(Sol). $L(y'') + 9L(y) = \frac{s}{s^2+4}$, we set $y'(0) = c$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2+4},\quad s^2Y(s) - s - c + 9Y(s) = \frac{s}{s^2+4}$$

$$Y(s) = \frac{s+c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)} = \frac{4}{5}\cdot\frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4}$$

$$\Rightarrow y(t) = \frac{4}{5}\cos(3t) + \frac{c}{3}\sin(3t) + \frac{1}{5}\cos(2t)$$

$$y\left(\frac{\pi}{2}\right) = -1 \Rightarrow c = \frac{12}{5},\ \therefore\quad y(t) = \frac{4}{5}\cos(3t) + \frac{4}{5}\sin(3t) + \frac{1}{5}\cos(2t)$$

4-4 Convolution and Dirac Delta Function

Convolution in Laplace transform: $f(t)*g(t)=\int_0^t f(t-\alpha)g(\alpha)d\alpha$

Theorem $L[f(t)*g(t)]=L[\int_0^t f(t-\alpha)g(\alpha)d\alpha]=F(s)G(s)$

Eg. Solve $y''+y-4\int_0^t y(\tau)\sin(t-\tau)d\tau=e^{-2t}$ **,** $y(0)=1$ **and** $y'(0)=0$**. [交大電信所]**

(Sol.) $s^2Y(s)-sy(0)-y'(0)+Y(s)-4Y(s)\dfrac{1}{s^2+1}=\dfrac{1}{s+2}$

$$\Rightarrow Y(s)=\frac{(s^2+1)(s+1)}{(s+2)(s^2+3)(s-1)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{Cs+D}{s^2+3}$$

$$\Rightarrow A=\frac{1}{3},\ B=\frac{5}{21},\ C=\frac{3}{7},\ D=\frac{1}{7}$$

$$\Rightarrow y(t)=\frac{1}{3}e^t+\frac{5}{21}e^{-2t}+\frac{3}{7}\cos\left(\sqrt{3}t\right)+\frac{1}{7\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

Eg. Solve $\int_0^t f(\tau)f(t-\tau)d\tau=6t^3$ **. [台大電研]**

(Sol.) According to $L[t^n]=\dfrac{\Gamma(n+1)}{s^{n+1}}$ and $L^{-1}[1/s^n]=t^{n-1}/\Gamma(n)$, $[F(s)]^2=6L[t^3]=\dfrac{36}{s^4}$,

$F(s)=\dfrac{6}{s^2}\Rightarrow f(t)=6t$

Eg. Solve $f(t)=e^{-t}+2\int_0^t e^{-3\alpha}f(t-\alpha)d\alpha$ **.**

(Sol.) $F(s)=\dfrac{1}{s+1}+\dfrac{2}{s+3}\cdot F(s)$

$$F(s)=\frac{s+3}{(s+1)^2}=\frac{1}{s+1}+\frac{2}{(s+1)^2}\Rightarrow f(t)=e^{-t}+2te^{-t}.$$

Eg. Solve $f(t)=2t^2+\int_0^t \sin(4\alpha)\cdot f(t-\alpha)d\alpha$ **.**

(Sol.) $F(s)=\dfrac{4}{s^3}+L[\sin(4t)*f(t)]=\dfrac{4}{s^3}+\dfrac{4}{s^2+16}\cdot F(s)$

$$F(s)=\frac{4(s^2+16)}{s^3(s^2+12)}=-\frac{1}{9}\cdot\frac{1}{s}+\frac{16}{3}\cdot\frac{1}{s^3}+\frac{1}{9}\cdot\frac{s}{s^2+12}$$

$$f(t)=-\frac{1}{9}+\frac{16}{3}\cdot\frac{t^2}{2}+\frac{1}{9}\cdot\cos\left(\sqrt{12}t\right)=-\frac{1}{9}+\frac{8}{3}t^2+\frac{1}{9}\cos\left(\sqrt{12}t\right)$$

Eg. Solve $\int_0^t x(u)\sin(t-u)du=x(t)+\sin(t)-\cos(t)$**.** **【交大電信所】**

Eg. Solve $y(t)=\sin(2t)+\int_0^t y(\tau)\sin[2(t-\tau)]d\tau$**.** **【淡江機研】**

Dirac delta function:

$$\delta(t-a)=\delta_a(t)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[u(t-a)-u(t-a-\varepsilon)]=\begin{cases}\lim\limits_{\varepsilon\to 0}\frac{1}{\varepsilon}, a\le t\le a+\varepsilon\\ 0, elsewhere\end{cases}$$

Kronecker delta: $\delta_{ab}=\begin{cases}1, a=b\\ 0, a\ne b\end{cases}$

Characteristics of Dirac's delta function:

1. $\int_{-\infty}^{\infty}\delta(t-a)dt=1$

2. $L[\delta(t-a)]=e^{-as}$

3. $\int_0^{\infty}f(t)\delta(t-a)dt=f(a)$

4. $f(t)*\delta(t)=\int_0^{\infty}f(x)\delta(t-x)dx=f(t)$

5. $f(t)\delta(t-a)=f(a)\delta(t-a)$

Eg. Solve $y^{(4)}=\delta(x-a)$, $y(0)=y''(0)=y(1)=0$, $y^{(3)}(0)=1$, $0<a<1$. [交大電子所]

(Sol.) $L(y^{(4)})=s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y^{(3)}(0)=s^4Y(s)-s^2c-1$

$$L[\delta(x-a)]=e^{-as}\Rightarrow Y(s)=\frac{e^{-as}}{s^4}+\frac{c}{s^2}+\frac{1}{s^4}\Rightarrow y(x)=\frac{(x-a)^3}{6}\cdot u(x-a)+cx+\frac{x^3}{6}$$

$$y(1)=0=\frac{(1-a)^3}{6}+c+\frac{1}{6}\Rightarrow c=\frac{-(1-a)^3}{6}-\frac{1}{6}$$

$$\therefore\ y(x)=\frac{(x-a)^3}{6}\cdot u(x-a)-\left[\frac{(1-a)^3}{6}+\frac{1}{6}\right]\cdot x+\frac{x^3}{6}$$

$$=\begin{cases}-\left[\frac{(1-a)^3}{6}+\frac{1}{6}\right]\cdot x+\frac{x^3}{6}\ ,\ 0\le x\le a\\ \frac{(x-a)^3}{6}-\left[\frac{(1-a)^3}{6}+\frac{1}{6}\right]\cdot x+\frac{x^3}{6}\ ,\ a<x\le 1\end{cases}$$

Eg. Solve $y''+5y'+4y=3+2\delta(t)$.【北科大土木所】