Chapter 5 Fourier Analysis

5-1 Fourier Series of a Periodical Function

Fourier series: $f(x)$ is a periodical function with period=$2L$ and defined on an interval: $-L \leqq x \leqq L$. $f(x+2L)=f(x)$, and then $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$, where

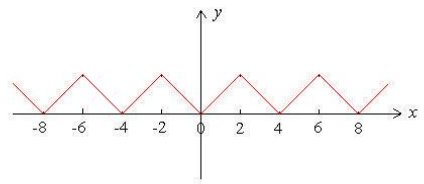
$$\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\text{ and }\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

In case $f(x)$ is $\begin{matrix} odd \\ even\end{matrix} \Rightarrow \begin{matrix} a_n=0 \\ b_n=0\end{matrix}$

Parseval's Identity for Fourier series: $\frac{1}{L}\int_{-L}^{L}|f(x)|^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$

Orthogonalities:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=\begin{cases}0, n\neq m\\ L, n=m\end{cases}\ \text{ and }\ \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\begin{cases}0, n\neq m\\ L, n=m\end{cases}$$



Eg. Expand $f(x)=\begin{cases}x, 0\le x\le 2\\ -x, -2\le x\le 0\end{cases}$, $f(x+4)=f(x)$ into Fourier series and $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=?$ [交大資科所]

(Sol.) $f(x)=\begin{cases}x, 0\le x\le 2\\ -x, -2\le x\le 0\end{cases}$, $f(x+4)=f(x)$, $2L=4$, $L=2$,

$\because$ Even function, $\therefore$ $b_n=0$, $\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}f(x)dx=\frac{1}{4}\left[\int_{-2}^{0}-x\,dx+\int_{0}^{2}x\,dx\right]=1$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}-x\cos\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{2}{2}\cdot\int_{0}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx=\left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)-\frac{4}{n^2\pi^2}\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\right]\Bigg|_0^2$$

$$=\frac{4}{n^2\pi^2}\left[\cos(n\pi)-1\right]=\begin{cases}\frac{-8}{n^2\pi^2}, n: odd\\ 0, n: even\end{cases}=\frac{-8}{(2m-1)^2\pi^2}, m=1,2,\cdots\cdots$$

$$\therefore\ f(x)=1+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos\left(\frac{n\pi x}{2}\right)=1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\cdots\right)$$

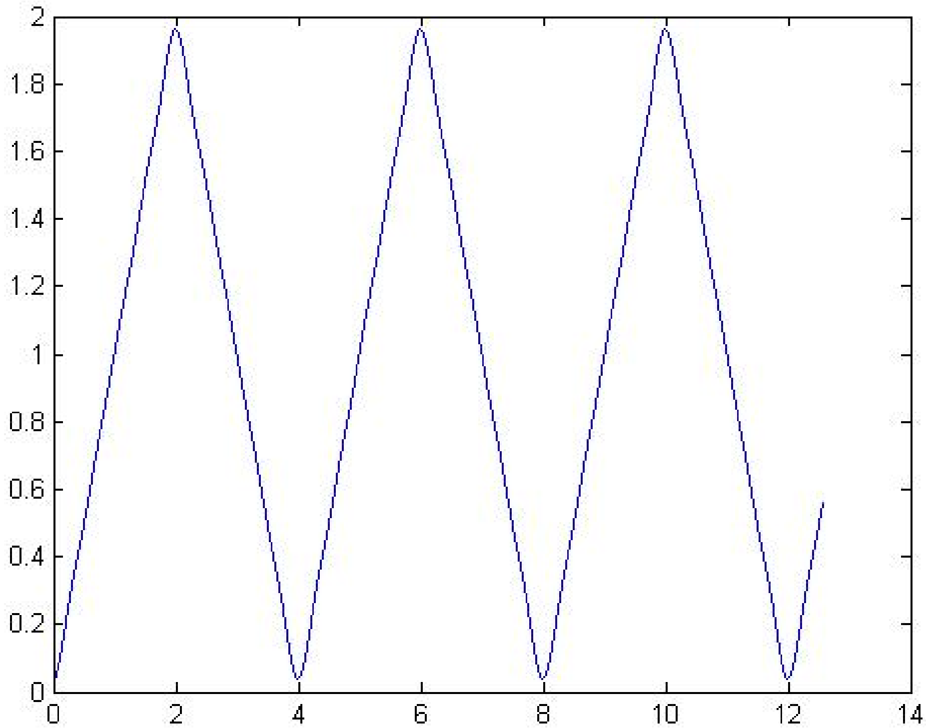
$$f(0)=0=1-\frac{8}{\pi^2}\left(\cos 0+\frac{1}{3^2}\cos 0+\frac{1}{5^2}\cos 0+\cdots\cdots\right)$$

$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\cdots=\frac{\pi^2}{8}$$

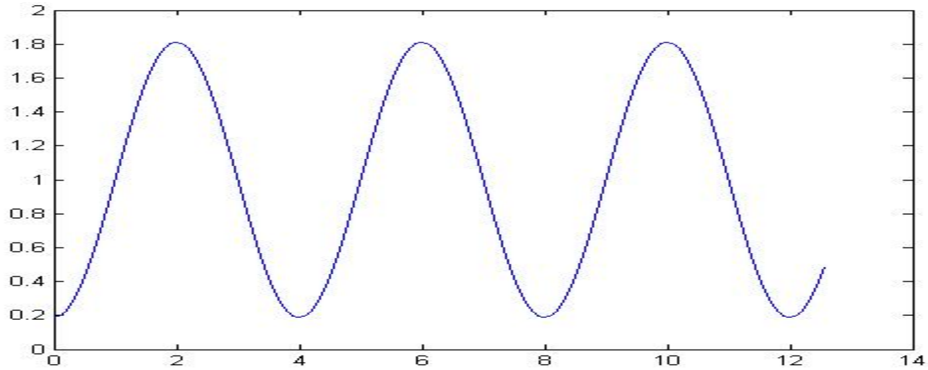
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$1+\sum_{n=1}^{10}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos(\frac{n\pi x}{2})=1-\sum_{i=1}^{5}\frac{8}{(2i-1)^2\pi^2}\cos(\frac{(2i-1)\pi x}{2}).$$

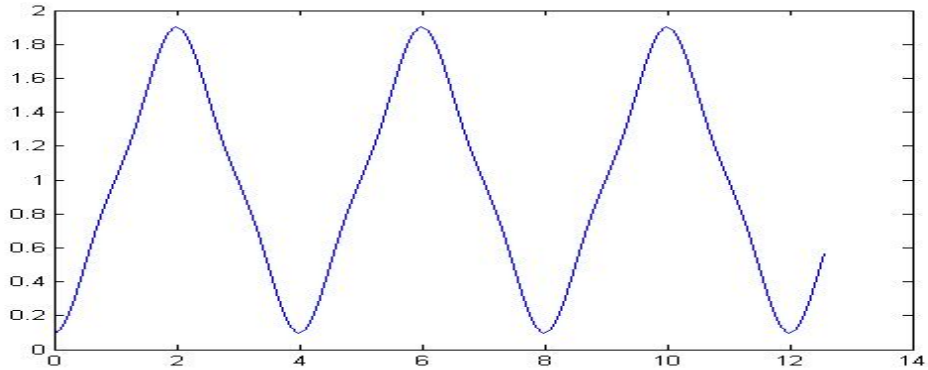
```
>>x = 0:0.001:4*pi; y=1;
>>for i=1:5
y=y-8*cos((2*i-1)*pi*x/2)/(2*i-1)^2/pi^2
end
>>plot (x,y)
```



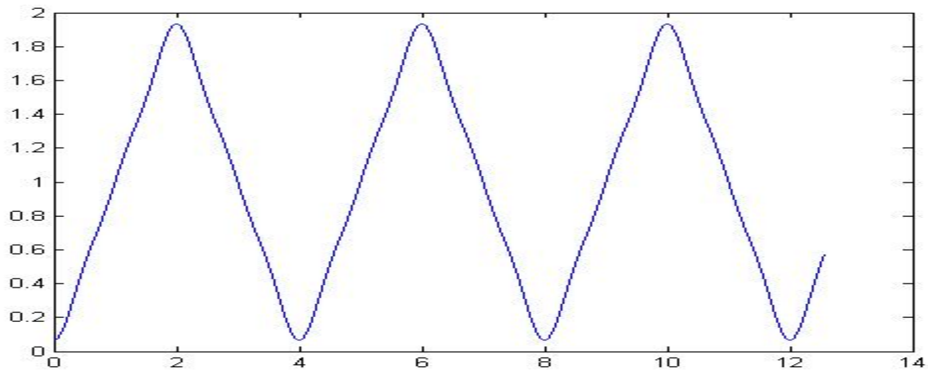
$$1-\frac{8}{\pi^2}\cos(\frac{\pi x}{2})$$



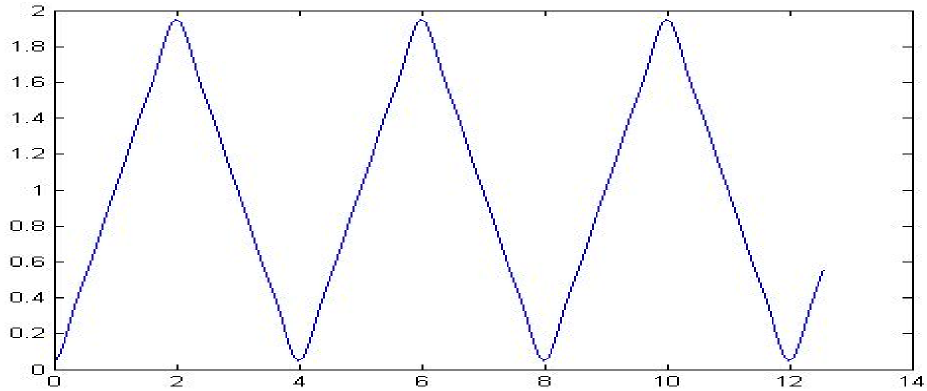
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}\right)$$



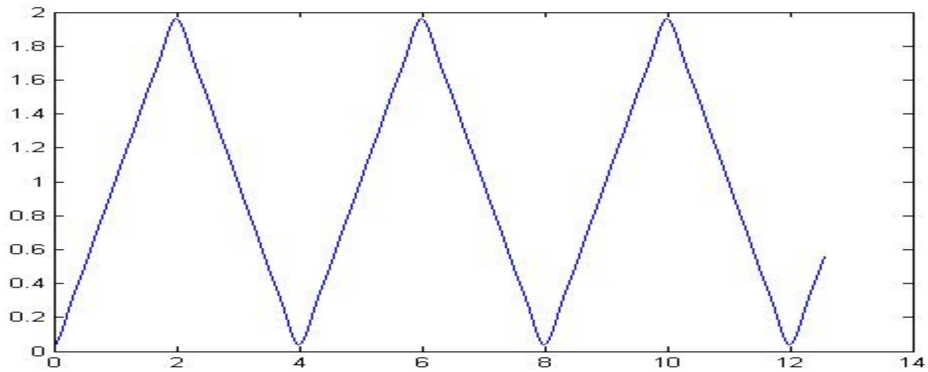
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}\right)$$



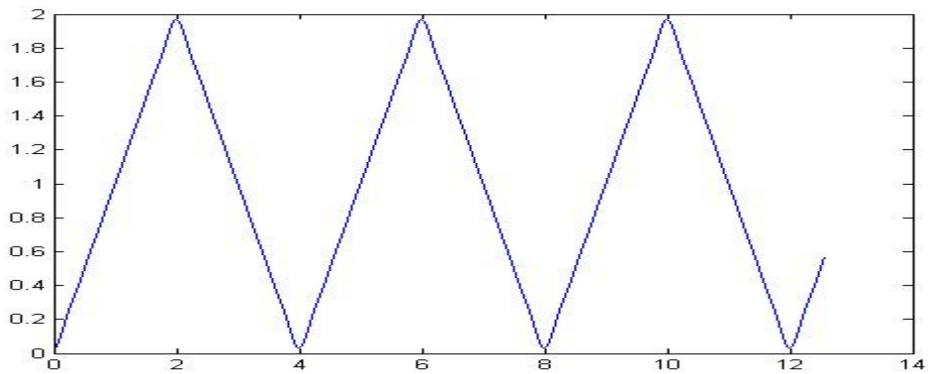
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\frac{1}{7^2}\cos\frac{7\pi x}{2}\right)$$



$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\frac{1}{7^2}\cos\frac{7\pi x}{2}+\frac{1}{9^2}\cos\frac{9\pi x}{2}\right)$$



$$1+\sum_{n=1}^{22}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos(\frac{n\pi x}{2})$$



Eg. Find the Fourier series of *f*(*x*)=|*x*| for –π<*x*<π. [台大電研]

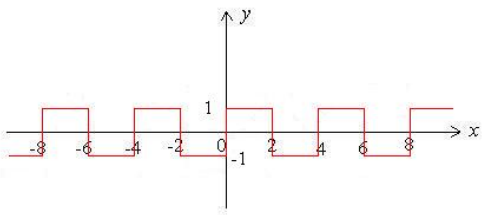
(Sol.) $2L=2\pi$, $L=\pi$, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$

$\because$ Even function, $\therefore$ $b_n=0$, $\forall n$.

$$\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}-x\,dx+\int_{0}^{\pi}x\,dx\right]=\frac{\pi}{2},$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{2}{n^2\pi}[\cos(n\pi)-1],$$

$$\therefore\ f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}(\cos n\pi-1)\cdot\cos(nx)$$



Eg. Expand *f*(*x*)= $\begin{cases}1, & 0\le x\le 2\\ -1, & -2\le x\le 0\end{cases}$ **and *f*(*x*+4)=*f*(*x*) into Fourier series. Find (a)** $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}$ **and (b)** $\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$**. [文化電機轉學考]**

(Sol.) $2L=4$, $L=2$, $f(x)=\begin{cases}1, & 0\le x\le 2\\ -1, & -2\le x\le 0\end{cases}$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$. $\because$ Odd function, $\therefore$ $a_n=0$, $\forall n$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}-\sin\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{2}{n\pi}-\frac{2\cos(n\pi)}{n\pi}=\frac{2}{n\pi}[1-\cos(n\pi)]\Rightarrow f(x)=\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$

(a) Set $x=1$,

$$f(1)=1=\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi}{2}\right)\right\}=\frac{4}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\cdots\cdots\right\}$$

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\cdots\cdots=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}=\frac{\pi}{4}$$

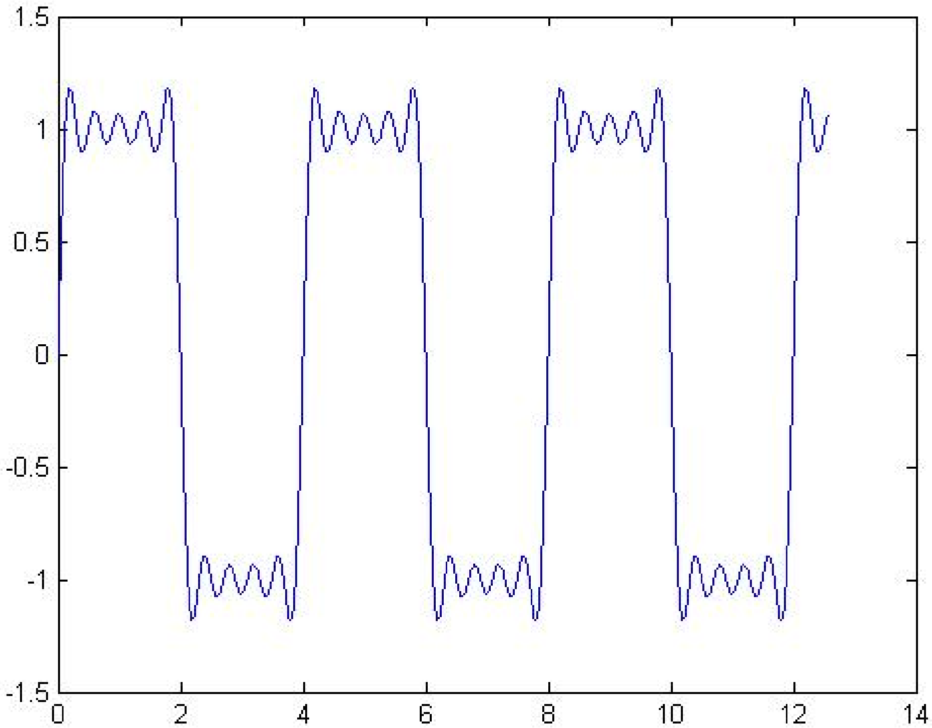
(b) $\frac{1}{L}\int_{-L}^{L}|f(x)|^2\,dx=\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left[a_n^{\,2}+b_n^{\,2}\right]$,

$$\frac{1}{2}\int_{-2}^{2}1^2dx=2=\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[1-\cos(n\pi)]^2=\frac{16}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\cdots\right]$$

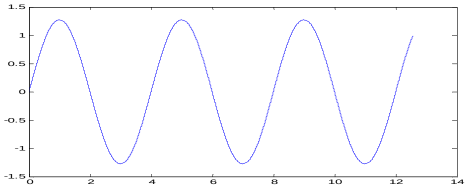
$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\cdots=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}$$

In **Matlab** language, we can use the following instructions to obtain the finite sum of $\sum_{n=1}^{10}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}=\sum_{i=1}^{5}\left\{\frac{4}{(2i-1)\pi}\cdot\sin\left(\frac{(2i-1)\pi x}{2}\right)\right\}$.

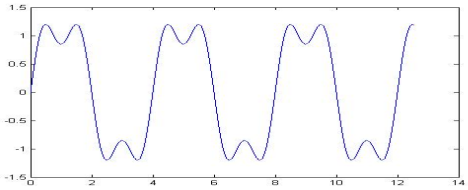
```
>>x = 0:0.001:4*pi;
>>y=0;
>>for i=1:5
y=y+4*sin((2*i-1)*pi*x/2)/(2*i-1)/pi
end
>>plot (x,y)
```



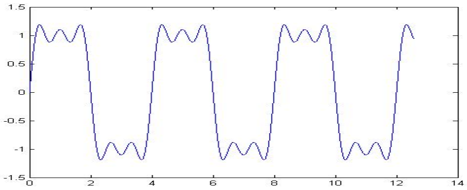
$$\frac{4}{\pi}\sin(\frac{\pi x}{2})$$



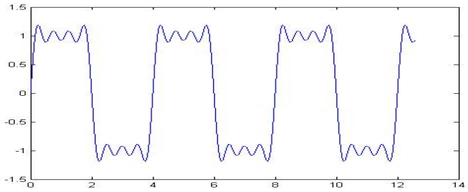
$$\sum_{n=1}^{4}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



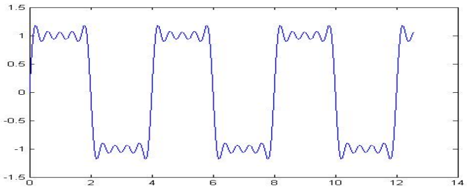
$$\sum_{n=1}^{6}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



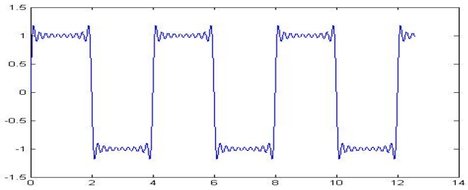
$$\sum_{n=1}^{8}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



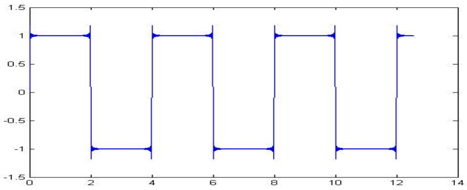
$$\sum_{n=1}^{10}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



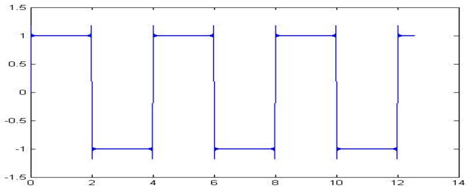
$$\sum_{n=1}^{20}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



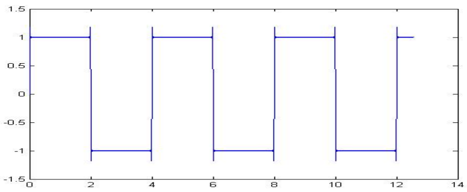
$$\sum_{n=1}^{200}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



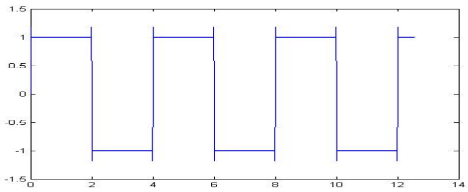
$$\sum_{n=1}^{400}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$

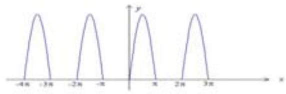


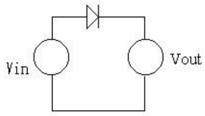
$$\sum_{n=1}^{1000}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



$$\sum_{n=1}^{2000}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$







Eg. Find the Fourier series of $f(x)=\begin{cases}0, & -\pi<x<0\\ \sin x, & 0\le x<\pi\end{cases}$ **and use the results to show that** $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\times 3}-\frac{1}{3\times 5}+\frac{1}{5\times 7}-\frac{1}{7\times 9}+\cdots$. **[台大電研]**

(Sol.)

$(a)\ 2L=2\pi,\ L=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\frac{a_0}{2}=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_0^{\pi}\sin xdx\right]=\frac{1}{2\pi}[1-\cos\pi]=\frac{1}{\pi}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi x}{L})dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot\cos(nx)dx+\int_0^{\pi}\sin(x)\cdot\cos(nx)dx\right]=\frac{1}{\pi}\int_0^{\pi}\frac{1}{2}[\sin(x+nx)+\sin(x-nx)]dx$$

$$=\frac{1}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n}+\frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{1+n}+\frac{1}{1-n}\right)-\frac{\cos(1+n)\pi}{1+n}-\frac{\cos(1-n)\pi}{1-n}\right]=\frac{1}{2\pi}\left[\frac{2}{1-n^2}+\frac{\cos n\pi}{1+n}+\frac{\cos n\pi}{1-n}\right]$$

$$=\frac{1}{2\pi}\left[\frac{2}{1-n^2}+\frac{2\cos n\pi}{1-n^2}\right]=\frac{1}{2\pi}\cdot\frac{2}{1-n^2}(1+\cos n\pi)$$

$$=\frac{1+\cos n\pi}{\pi(1-n^2)}(n\ne 1)=\begin{cases}0, \forall n=3,5,7,\cdots\\ \dfrac{2}{\pi(1-n^2)}, \forall n=2,4,6,\cdots\end{cases}$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot\sin(nx)dx+\int_0^{\pi}\sin(x)\cdot\sin(nx)dx\right]$$

$$=\frac{1}{\pi}\left\{\int_0^{\pi}\frac{1}{2}[\cos(x-nx)-\cos(x+nx)]dx\right\}=\frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{1-n}-\frac{\sin(1+n)\pi}{1+n}\right]=\begin{cases}1/2, n=1\\ 0,\ \ n>1\end{cases}$$

$$\therefore\ f(x)=\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}+\frac{\cos(6x)}{35}+\frac{\cos(8x)}{63}+\cdots\right]+\frac{1}{2}\sin(x)$$

$$(b)\ f(-\frac{\pi}{2})=0=\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(-\pi)}{3}+\frac{\cos(-2\pi)}{15}+\frac{\cos(-3\pi)}{35}+\frac{\cos(-4\pi)}{63}+\cdots\right]-\frac{1}{2}$$

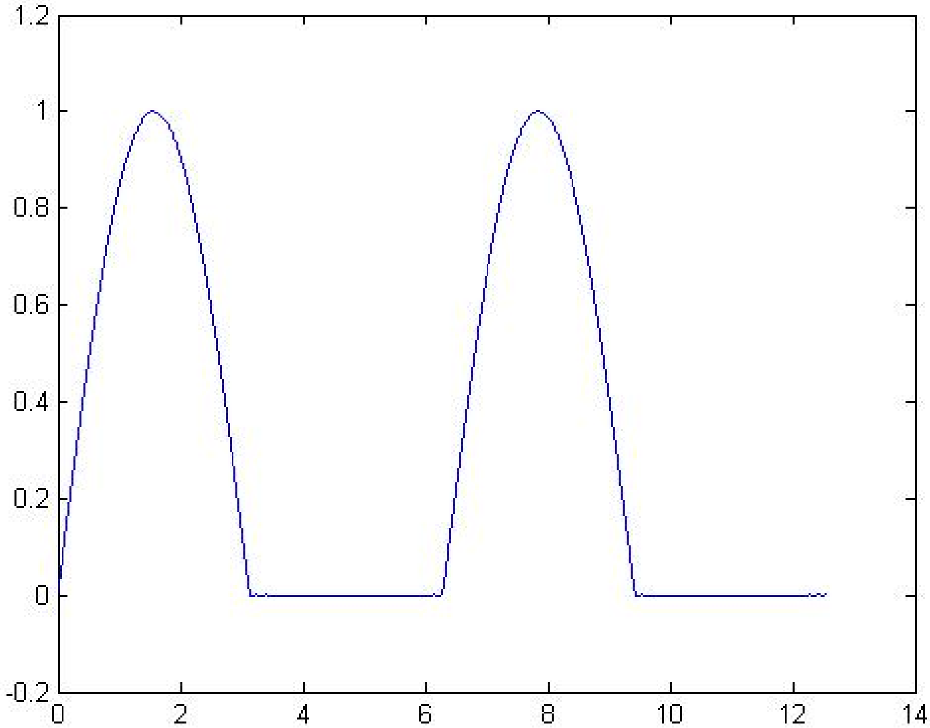
$$\frac{1}{2}=\frac{1}{\pi}+\left[\frac{2}{3\pi}-\frac{2}{15\pi}+\frac{2}{35\pi}-\frac{2}{63\pi}+\cdots\right]=\frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\frac{1}{63}\cdots\right)$$

$$\therefore\ \frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\times 3}-\frac{1}{3\times 5}+\frac{1}{5\times 7}-\frac{1}{7\times 9}+\cdots$$

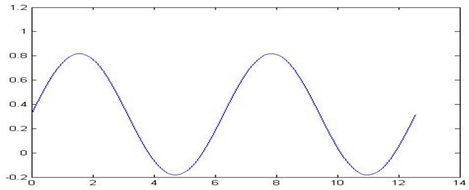
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}+\frac{\cos(6x)}{35}+\frac{\cos(8x)}{63}+\cdots+\frac{\cos(40x)}{1599}\right]+\frac{1}{2}\sin(x).$$

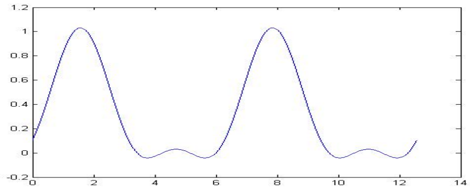
```
>>x = 0:0.001:4*pi; y=1/pi+sin(x)/2;
>>for n=1:20
y=y-2*cos(2*n*x)/pi/(4*n^2-1)
end
>>plot (x,y)
```



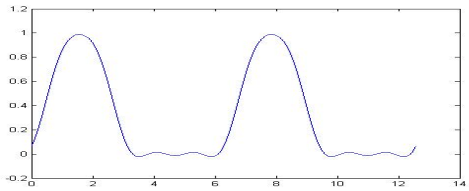
$$\frac{1}{\pi}+\frac{1}{2}\sin(x)$$



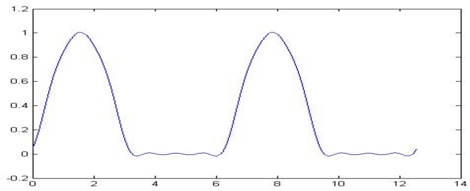
$$\frac{1}{\pi}-\frac{2}{\pi}\cdot\frac{\cos(2x)}{3}+\frac{1}{2}\sin(x)$$



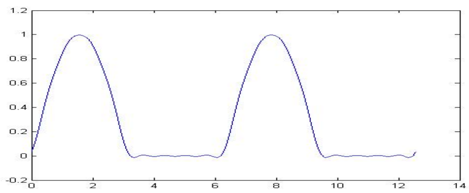
$$\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}\right]+\frac{1}{2}\sin(x)$$



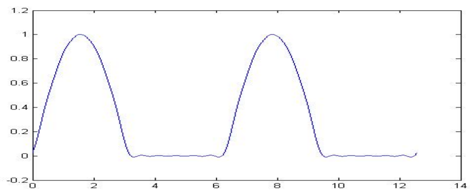
$$\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}+\frac{\cos(6x)}{35}\right]+\frac{1}{2}\sin(x)$$



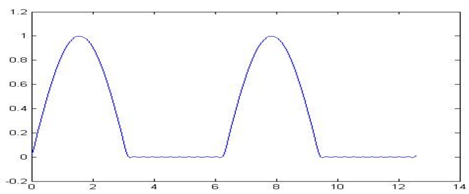
$$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{4}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$$



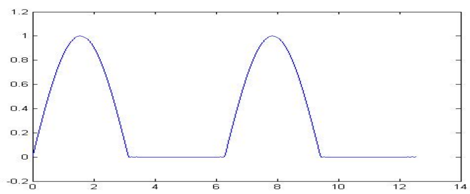
$$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{5}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$$

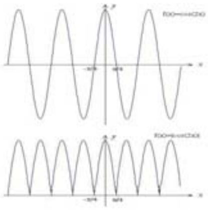


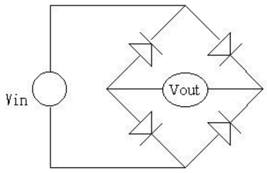
$$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{10}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$$



$$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{20}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$$







Eg. Find the Fourier series of |cos(2*x*)| and calculate $\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$ [交大材研、成大電研]

(Sol.) (a) $f(x)=f\left(x+\frac{\pi}{2}\right)$, $f(x)=\cos(2x)$, $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$

$$2L=\frac{\pi}{2},\ L=\frac{\pi}{4},\ \frac{n\pi x}{L}=4nx\ ,\ \ \frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}\cos(2x)dx=\frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2x dx=\frac{2}{\pi}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}\cos(2x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx=\frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos(2x)\cos(4nx)dx$$

$$=\frac{4}{\pi}\cdot\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}\left[\cos((2+4n)x)+\cos((2-4n)x)\right]dx$$

$$=\frac{2}{\pi}\cdot\left[\left.\frac{\sin(2+4n)x}{2+4n}\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}+\left.\frac{\sin(2-4n)x}{2-4n}\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right]$$

$$=\frac{2}{\pi}\cdot\left[\frac{2\sin\frac{(2n+1)}{2}\pi}{2(2n+1)}+\frac{2\sin\frac{(2n-1)}{2}\pi}{2(2n-1)}\right]=\frac{2}{\pi}\cdot\left[\frac{(-1)^n}{2n+1}+\frac{(-1)^{n+1}}{2n-1}\right]$$

$$=\frac{2}{\pi}\cdot\left[\frac{(-1)^n}{2n+1}-\frac{(-1)^n}{2n-1}\right]=-\frac{4}{\pi}\cdot\frac{(-1)^n}{4n^2-1}=\frac{4}{\pi}\cdot\frac{(-1)^{n+1}}{4n^2-1}$$

$\because$ Even function, $\therefore$ $b_n=0$ $\Rightarrow f(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)=|\cos(2x)|$

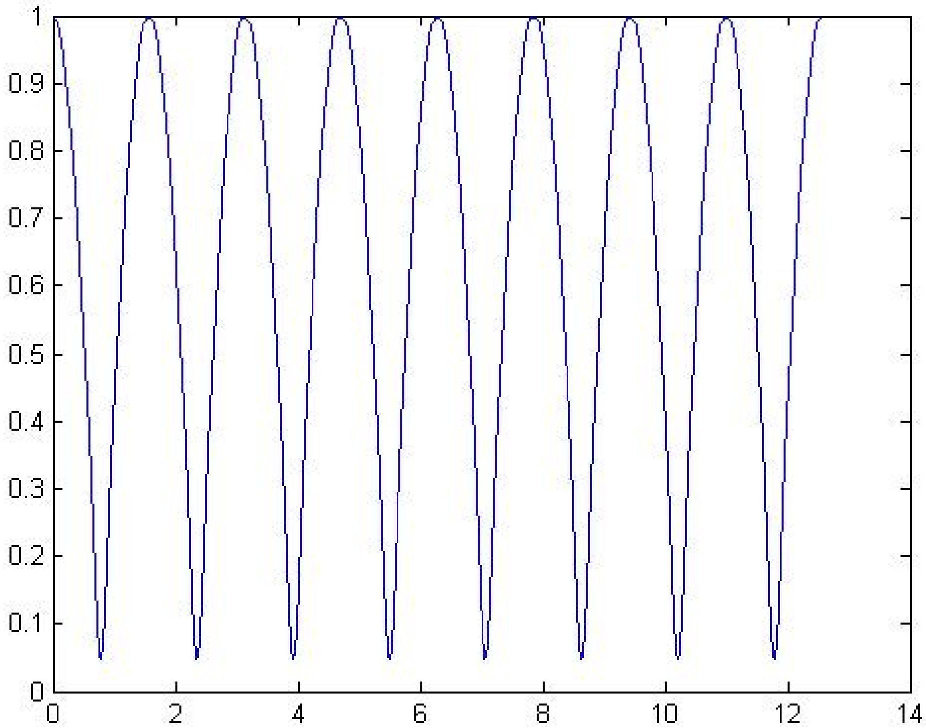
(b) $x=0,\ f(x)=1=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi}{4}\left(1-\frac{2}{\pi}\right)\ \Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}=\frac{\pi}{4}\left(\frac{2}{\pi}-1\right)$$

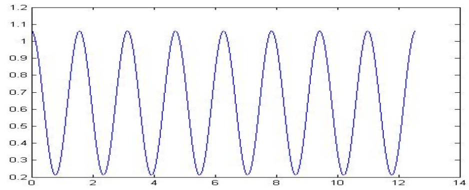
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{6}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx).$

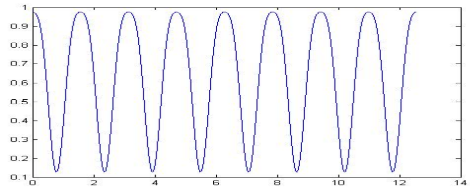
```
>>x = 0:0.001:4*pi; y=2/pi;
>>for n=1:6
y=y+4*(-1)^(n+1)*cos(4*n*x)/(4*n^2-1)/pi
end
>>plot (x,y)
```



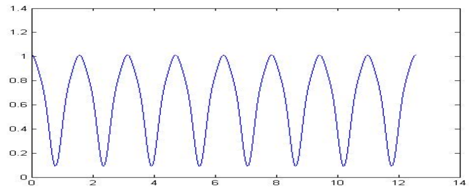
$$\frac{2}{\pi}+\frac{4}{3\pi}\cos(4x)$$



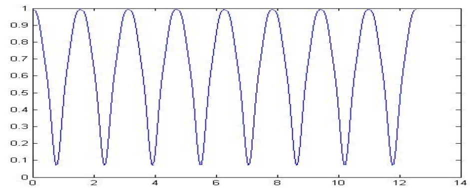
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{2}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



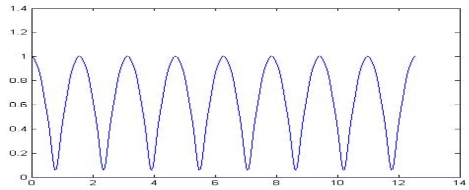
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{3}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



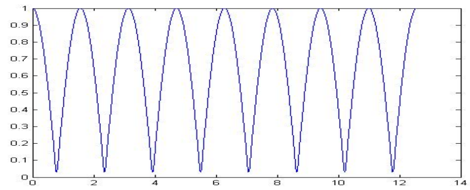
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{4}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



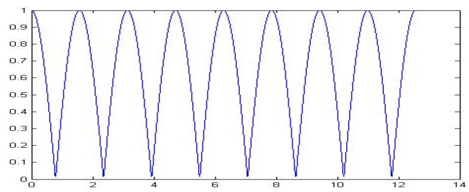
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{5}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



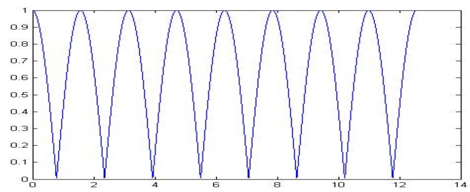
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{6}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



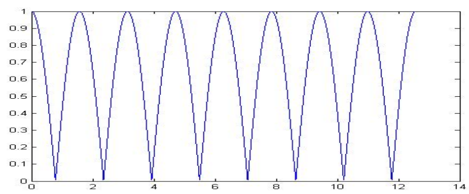
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{10}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



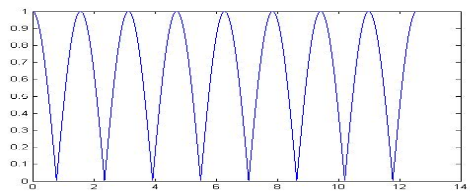
$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{20}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$

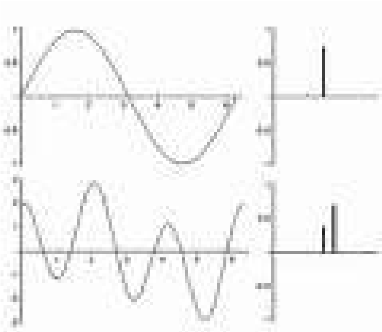


$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{50}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



$$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{100}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$





Discrete spectrum of *f*(*t*):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{j\frac{n\pi t}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}$$

5-2 Fourier Transforms and Inverse Fourier Transforms

Fourier Transform pair defined in Engineering:

$$\begin{cases} F(\omega) = \Im[f(x)] \equiv \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx \\ f(x) = \Im^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega \end{cases}$$, where $\omega=2\pi v$.

Parseval's Identities for Fourier Transform pairs $\Im[f(x)]=F(\omega)$ and $\Im[g(x)]=G(\omega)$:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) g^*(x) dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) G^*(\omega) d\omega$$

Continuous Spectrum of $f(t)$: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

***Convolution* in Fourier Transform:** $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$

***Correlation* in Fourier Transform:** $f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau) g(\tau - t) d\tau$

Basic theorems of Fourier Transforms $\Im[f(x)]=F(\omega)$ and $\Im[g(x)]= G(\omega)$:

1. $\Im[af(x)+bg(x)]=aF(\omega)+bG(\omega)$
2. $\Im[f(ax)]=[F(\omega/a)]/a,\ a>0$

(Proof) For $a>0$, let $ax=u$

$$\Im[f(ax)]=\int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(ax) dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-j(\frac{\omega}{a})ax} \cdot f(ax) d(ax) = \frac{1}{a}\int_{-\infty}^{\infty} e^{-j(\frac{\omega}{a})u} \cdot f(u) du$$

$$= \frac{1}{a} F[(\frac{\omega}{a})]$$

3. $\Im[f(x)e^{jax}]=F(\omega-a)$

(Proof) $\Im[f(x)e^{jax}]=\int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x) e^{jax} dx = \int_{-\infty}^{\infty} e^{-j(\omega-a)x} \cdot f(x) dx = F(\omega-a)$

4. $\Im[f'(x)]=j\omega F(\omega)$, $\Im[f^{(n)}(x)]=(j\omega)^n F(\omega)$ in case of $f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\ldots=0$

(Proof) $\Im[f'(x)]=\int_{-\infty}^{\infty} e^{-j\omega x} \cdot f'(x) dx = \int_{-\infty}^{\infty} e^{-j\omega x} df(x)$

$= e^{-j\omega x} f(x) \|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-j\omega) e^{-j\omega x} \cdot f(x) dx = e^{-j\omega\infty} f(\infty) - e^{j\omega\infty} f(-\infty) + j\omega \int_{-\infty}^{\infty} e^{-j\omega x} \cdot f(x) dx = j\omega F(\omega)$

By mathematical induction, we have $\Im[f^{(n)}(x)]=(j\omega)^n F(\omega)$ if $f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\ldots=0$.

5. $\Im[f(x)*g(x)]=F(\omega)G(\omega)$
6. $\Im[g^*(x)\bigstar f(x)]=F(\omega)G^*(\omega)$, where $g^*(x)$ and $G^*(\omega)$ are the complex conjugates of $g(x)$ and $G(\omega)$, respectively.

7. $\Im\,[f(x-a)]= e^{-ja\omega}F(\omega)$

(Proof) Let $x-a=u$, $\Im\,[f(x-a)]=\int_{-\infty}^{\infty} e^{-j\omega x}\cdot f(x-a)dx=\int_{-\infty}^{\infty} e^{-j\omega x}\cdot f(x-a)d(x-a)$

$$=e^{-ja\omega}\int_{-\infty}^{\infty} e^{-j\omega(x-a)}\cdot f(x-a)d(x-a)=e^{-ja\omega}\int_{-\infty}^{\infty} e^{-j\omega u}\cdot f(u)du$$

$$= e^{-ja\omega}F(\omega)$$

8. $\Im\,[x^n f(x)]=(j)^n F^{(n)}(\omega)$

(Proof) $\Im\,[xf(x)]=\int_{-\infty}^{\infty} e^{-j\omega x}\cdot xf(x)dx=\int_{-\infty}^{\infty} j\frac{d}{d\omega}e^{-j\omega x}\cdot f(x)dx=j\frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-j\omega x}\cdot f(x)dx$

$$=jF'(\omega)$$

By mathematical induction, we have $\Im\,[x^n f(x)]=(j)^n F^{(n)}(\omega)$.

Eg. Find (a) $\Im\,[xe^{-|x|}]$, (b) $\Im\,[e^{-3|x|}]$, (c) $\Im^{-1}\left\{\frac{4}{4+\omega^2}\right\}$, (d) $\int_{-\infty}^{\infty}\frac{\cos\omega}{\omega^2+4}d\omega$, (e) $f(x)$ if $\int_0^{\infty} f(x)\cos(2x)dx = e^{-2}$. [文化電機轉學考]

(Sol.) $\Im\left[e^{-a|x|}\right]=\int_{-\infty}^{\infty} e^{-a|x|}e^{-i\omega x}dx=\int_{-\infty}^{0} e^{ax}e^{-i\omega x}dx+\int_0^{\infty} e^{-ax}e^{-i\omega x}dx$

$$=\left.\frac{e^{(a-i\omega)x}}{a-i\omega}\right|_{-\infty}^{0}+\left.\frac{e^{(-a-i\omega)x}}{-a-i\omega}\right|_{0}^{\infty}=\frac{1}{a-i\omega}+\frac{1}{a+i\omega}=\frac{2a}{a^2+\omega^2}$$

(a) According to $\Im\left[x^n f(x)\right]=(i)^n\frac{d^n}{d\omega^n}F(\omega)$,

$$\Im\left[x\cdot e^{-a|x|}\right]=i\frac{d}{d\omega}\Im\left[e^{-a|x|}\right]=i\frac{d}{d\omega}\left(\frac{2a}{a^2+\omega^2}\right)=\frac{-4ai\omega}{\left(a^2+\omega^2\right)^2}$$

$$a=1,\ \Im\left[xe^{-|x|}\right]=\frac{-i4\omega}{\left(\omega^2+1\right)^2}$$

(b) $a=3$, $\Im\left[e^{-3|x|}\right]=\frac{6}{\omega^2+9}$, (c) $a=2$, $\Im^{-1}\left[\frac{4}{4+\omega^2}\right]=e^{-2|x|}$

(d) $\Im^{-1}\left[\frac{4}{4+\omega^2}\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{4+\omega^2}e^{i\omega x}d\omega=e^{-2|x|}$

$$x=1,\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{4+\omega^2}e^{i\omega}d\omega=e^{-2},\quad \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{4+\omega^2}\left[\cos\omega+i\sin\omega\right]d\omega=e^{-2}$$

$$I_m(\cdots\cdots)=0,\ R_e(\cdots\cdots)\neq 0,\ \therefore\ \int_{-\infty}^{\infty}\frac{\cos\omega}{4+\omega^2}d\omega=\frac{\pi}{2}e^{-2}$$

(e) $\int_{-\infty}^{\infty}\frac{2}{\pi}\cdot\frac{\cos\omega}{\omega^2+4}d\omega=e^{-2}=2\int_0^{\infty}\frac{2}{\pi}\cdot\frac{\cos\omega}{\omega^2+4}d\omega=\int_0^{\infty}\frac{4}{\pi}\frac{\cos\omega}{\omega^2+4}d\omega$

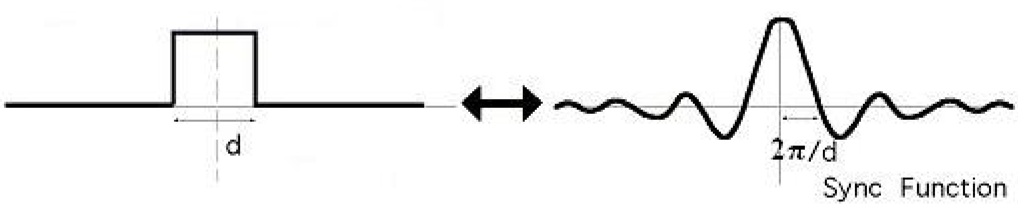
Set $\omega=2x$, $\int_0^{\infty}\frac{4}{\pi}\cdot\frac{\cos 2x}{4x^2+4}\cdot 2dx=e^{-2}=\int_0^{\infty} f(x)\cos(2x)dx$, $\therefore\ f(x)=\frac{2}{\pi}\cdot\frac{1}{(x^2+1)}$

Eg. For two rectangular functions: $f(x)=\begin{cases}1, |x|\le 1\\ 0, |x|\rangle 1\end{cases}$, $g(x)=\begin{cases}1, |x|\le 2\\ 0, |x|\rangle 2\end{cases}$, find (a) $\Im[f(x)]$, (b) $\Im[g(x)]$, and (c) $\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx$. [文化電機轉學考]

(Sol.) (a) $\Im[f(x)]=\int_{-1}^{1}e^{-i\omega x}dx=\frac{e^{-i\omega x}}{-i\omega}\Bigg|_{-1}^{1}=\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}=\frac{2(e^{i\omega}-e^{-i\omega})}{2i\omega}=\frac{2\sin(\omega)}{\omega}$

(b) $\because$ $\Im[f(ax)]=\frac{1}{a}F\left(\frac{\omega}{a}\right)$, $a>0$, $\Im[g(x)]=\Im\left[f\left(\frac{x}{2}\right)\right]=2\cdot\frac{2\sin(2\omega)}{2\omega}=\frac{2\sin(2\omega)}{\omega}$

(c) $f(x)=\Im^{-1}\left\{\frac{2\sin\omega}{\omega}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\omega}{\omega}e^{i\omega x}d\omega$, $f(0)=1\Rightarrow\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi$



Eg. Find $\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$. [成大土木所]

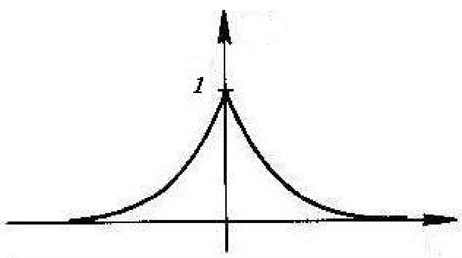
(Sol.) According to $\int_{-\infty}^{\infty}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$,

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin(\omega)}{\omega}\right)^2d\omega=\int_{-1}^{1}1^2dx=2\Rightarrow\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega=\pi$

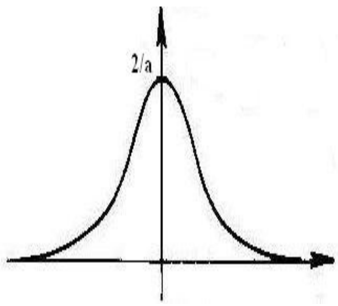
Eg. Find $\Im[e^{-a|x|}]$ and $\Im[e^{-|x|}]$.

(Sol.) $\int_{-\infty}^{\infty}e^{-a|x|}\cdot e^{-j\omega x}dx=\int_{-\infty}^{0}e^{+ax}\cdot e^{-j\omega x}dx+\int_{0}^{\infty}e^{-ax}\cdot e^{-j\omega x}dx$

$=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=\frac{2a}{a^2+\omega^2}=\Im\left[e^{-a|x|}\right]$. For $a=1$, $\Im\left[e^{-|x|}\right]=\frac{2}{1+\omega^2}$



$f(x)=e^{-a|x|}$



$F(\omega)=\frac{2a}{a^2+\omega^2}$

Or, according to $\Im[f(ax)]=\frac{1}{a}F\left(\frac{\omega}{a}\right)$, $\Im\left[e^{-|x|}\right]=\Im\left[e^{-a\left|\frac{x}{a}\right|}\right]=a\cdot\frac{2a}{a^2+(a\omega)^2}=\frac{2}{1+\omega^2}$

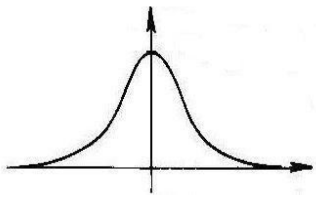
Eg. Find $\Im\left[\dfrac{1}{a^2+x^2}\right]$ **and** $\Im\left[\dfrac{1}{a^2+(x+b)^2}\right]$.

(Sol.) $\Im\left[e^{-a|x|}\right]=\dfrac{2a}{a^2+\omega^2}$, $e^{-a|x|}=\Im^{-1}\left[\dfrac{2a}{a^2+\omega^2}\right]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{2a}{a^2+\omega^2}\cdot e^{i\omega x}d\omega$

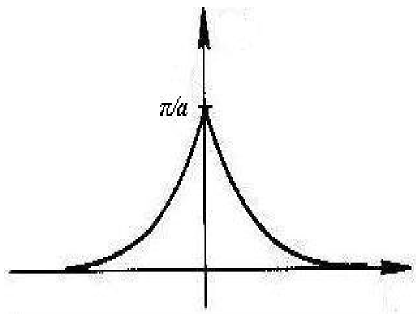
$$=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}e^{-i(-x)\omega}d\omega$$

$\therefore\ \int_{-\infty}^{\infty}\dfrac{1}{a^2+\omega^2}e^{-i(-x)\omega}d\omega=\dfrac{\pi}{a}e^{-a|x|}$. Set $u=-x \Rightarrow \int_{-\infty}^{\infty}\dfrac{1}{a^2+\omega^2}\cdot e^{-iu\omega}d\omega=\dfrac{\pi}{a}e^{-a|u|}$

Set $x=\omega$, $\omega=u$, $\Im\left[\dfrac{1}{a^2+x^2}\right]=\int_{-\infty}^{\infty}\dfrac{1}{a^2+x^2}e^{-i\omega x}dx=\dfrac{\pi}{a}\cdot e^{-a|\omega|}$



f(x)$=\dfrac{1}{a^2+x^2}$



F(ω)$=\dfrac{\pi}{a}e^{-a|\omega|}$

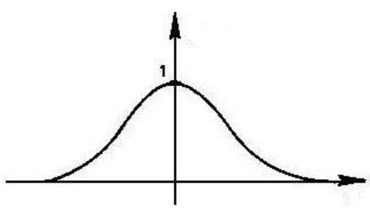
$\because\ \Im[f(x-a)]=e^{-j\omega a}F(\omega)$, $\therefore\ \Im\left[\dfrac{1}{a^2+(x+b)^2}\right]=e^{i\omega b}\cdot\dfrac{\pi}{a}e^{-a|\omega|}$

Eg. Find $\Im(e^{-a^2x^2})$.

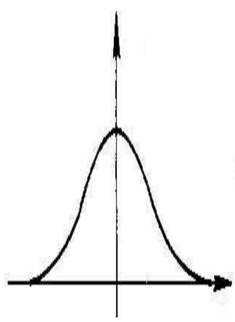
(Sol.) $\int_{-\infty}^{\infty}e^{-a^2x^2}\cdot e^{-j\omega x}dx=\int_{-\infty}^{\infty}e^{-a^2\left(x^2+j\frac{\omega}{a^2}x\right)}dx=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty}e^{-a^2\left[x^2+j\frac{\omega}{a^2}x-\frac{\omega^2}{4a^4}\right]}\cdot dx$

$$=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty}e^{-a^2\left[x+j\frac{\omega}{2a^2}\right]^2}\cdot dx=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty}e^{-a^2u^2}\cdot du\leftarrow\left(u=x+j\frac{\omega}{2a^2}\right)$$

$$=\frac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}\quad\left(Note:\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-a^2(u^2+v^2)}dudv=4\int_{o}^{\infty}\int_{o}^{\frac{\pi}{2}}e^{-a^2r^2}rdrd\theta=\frac{\pi}{a^2}\right)$$



f(x)$=e^{-a^2x^2}$



F(ω)$=\dfrac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}$

Note: $f(x)=\frac{1}{a^2+x^2}$ and $g(x)=e^{-a^2x^2}$ are similar to each other. But their respective Fourier transforms look quite different!

Eg. Determine $\Im[\frac{1}{a+jt}]$. **[台科大電研]**

(Sol.) $\because \Im\left[\frac{1}{a^2+t^2}\right]=\int_{-\infty}^{\infty}\frac{1}{a^2+t^2}e^{-j\omega t}dt=\frac{\pi}{a}\cdot e^{-a|\omega|}$ and $\Im\left[t^n f(t)\right]=(j)^n\frac{d^n}{d\omega^n}F(\omega)$

$\therefore \Im[\frac{1}{a+jt}]=\Im[\frac{a}{a^2+t^2}]-\Im[\frac{jt}{a^2+t^2}]=a\cdot\frac{\pi}{a}e^{-a|\omega|}-j\cdot j\frac{d}{d\omega}[\frac{\pi}{a}e^{-a|\omega|}]=\pi e^{-a|\omega|}\cdot[1-\operatorname{sgn}(\omega)]$

$=\begin{cases}0, & \omega\ge 0\\ 2\pi e^{a\omega}, & \omega<0\end{cases}=2\pi e^{a\omega}\cdot u(-\omega)$

Fourier Transform pair defined in Mathematics:

$$\begin{cases}F(\omega)=\Im[f(x)]\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-j\omega x}dx\\ f(x)=\Im^{-1}[F(\omega)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)e^{j\omega x}d\omega\end{cases}$$

, where $\omega=2\pi\nu$.

Fourier Transform pair defined in Physics/Optics:

$$\begin{cases}G(f)=\Im[g(x)]\equiv\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx\\ g(x)=\Im^{-1}[G(f)]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df\end{cases}$$