Chapter 9 Complex Numbers and Functions

9-1 Complex Numbers and Complex Functions

Complex number z: $z=a+ib=r\angle\theta=r\cos\theta+ir\sin\theta$

Complex conjugate of z: $\bar{z}=a-ib=r\angle-\theta=r\cos\theta-ir\sin\theta$

Theorem $z\in R\Leftrightarrow z=\bar{z}$, z is pure imaginary $\Leftrightarrow z=-\bar{z}$

Theorem $R_e(z)=\dfrac{z+\bar{z}}{2}=a$, $I_m(z)=\dfrac{z-\bar{z}}{2i}=b$

Theorem $\overline{zw}=\bar{z}\,\bar{w}$, $\overline{z+w}=\bar{z}+\bar{w}$, $|z|^2=z\cdot\bar{z}$

Theorem $|zw|=|z||w|$, $\quad$ **arg(zw)=arg(z)+arg(w)**, $\quad$ $|z/w|=|z|/|w|$ $\quad$ **if** $\quad$ $w\neq0$, **arg(z/w)=arg(z)-arg(w)**

Theorem

$$w^2=z=a+ib=r\angle\theta \Leftrightarrow w=\sqrt{r}\angle(\frac{\theta}{2}+n\pi)=\sqrt{r}\cos(\frac{\theta}{2}+n\pi)+i\sqrt{r}\sin(\frac{\theta}{2}+n\pi)$$

Theorem $z\cdot w=R_e(\bar{z}w)=\dfrac{1}{2}(\bar{z}w+z\bar{w})=|z||w|\cos\phi$

$$z\times w=I_m(\bar{z}w)=\frac{1}{2i}(\bar{z}w-z\bar{w})=|z||w|\sin\phi$$

Analytic function $f(z)$: $f(z)$ and $f'(z)$ are continuous and bounded at a certain point.

Theorem For an analytic complex function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$, we have
$f'(z)=\dfrac{df(z)}{dz}=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$.

Cauchy-Riemann Equations to test whether $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic:

$$f(z)=f(x+iy)=u(x,y)+iv(x,y) \text{ is analytic} \Leftrightarrow \begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\[2mm]\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases}$$

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=x^2-y^2$. Find out its imaginary part $v(x,y)$.

(Sol.) $u(x,y)=x^2-y^2 \Rightarrow \begin{cases}\dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y}\\[2mm]\dfrac{\partial u}{\partial y}=-2y=-\dfrac{\partial v}{\partial x}\end{cases} \Rightarrow v(x,y)=2xy+c.$

Eg. Is $f(x,y)=x+y+i(-x+y)$ analytical function?

(Sol.) $u(x,y)=x+y$, $v(x,y)=-x+y \Rightarrow \dfrac{\partial u}{\partial x}=1=\dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y}=1=-\dfrac{\partial v}{\partial x}$, $\therefore$ $f(x,y)$ is analytic.

Eg. Show that $f(z)=z^3$ is analytic.

(Proof) $f(z)=f(x+iy)=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)=u(x,y)+iv(x,y)$

$$\frac{\partial u}{\partial x}=3x^2-3y^2=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-6xy=-\frac{\partial v}{\partial x}$$

Eg. Show that $f(z)=|z|^2=z\bar{z}$ is not analytic at any point $z\neq 0$.

(Proof) $f(z)=f(x+iy)=x^2+y^2=u(x,y)+iv(x,y) \Rightarrow v(x,y)=0$

$\dfrac{\partial u}{\partial x}=2x\neq\dfrac{\partial v}{\partial y}$ if $x\neq 0$, $\dfrac{\partial u}{\partial y}=2y\neq-\dfrac{\partial v}{\partial x}$ if $y\neq 0 \Rightarrow f(z)$ is not analytic at $z\neq 0$

Eg. Is $f(x,y)=x^2-y^2-2y+i(2xy+2x)$ an analytic function? $f(z,\bar{z})=$?

(Sol.) $x=\dfrac{1}{2}(z+\bar{z})$, $y=\dfrac{1}{2i}(z-\bar{z})$

$$f(z,\bar{z})=\frac{1}{4}(z+\bar{z})^2-\frac{1}{(-4)}(z-\bar{z})^2-2\cdot\frac{1}{(2i)}(z-\bar{z})+i\left[\frac{1}{2i}(z+\bar{z})(z-\bar{z})+(z+\bar{z})\right]$$

$$=z^2+2iz\equiv f(z) \text{ is independent of } \bar{z}$$

$$\begin{cases}u(x,y)=x^2-y^2-2y\\v(x,y)=2xy+2x\end{cases} \Rightarrow \begin{cases}\dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y}\\[2mm]\dfrac{\partial u}{\partial y}=-2y-2=-\dfrac{\partial v}{\partial x}\end{cases}, \therefore f(x,y) \text{ is analytic}$$

Eg. Is $f(z)=\bar{z}$ an analytic function?

(Sol.) $f(z)=f(x+iy)=x-iy=u(x,y)+iv(x,y)$

$$\begin{cases}\dfrac{\partial u}{\partial x}=1\neq\dfrac{\partial v}{\partial y}=-1\\[2mm]\dfrac{\partial u}{\partial y}=0=-\dfrac{\partial v}{\partial x}\end{cases}, \therefore f(z) \text{ is not analytic.}$$

Check $f'(i)=\ell im_{z\to i}\dfrac{f(z)-f(i)}{z-i}=\ell im_{z\to i}\dfrac{\bar{z}-(-i)}{z-i}=\ell im_{z\to i}\dfrac{\bar{z}+i}{z-i}$

(a) if $z=\alpha i$, α is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{-\alpha i+i}{\alpha i-i}=\dfrac{1-\alpha}{\alpha-1}=-1$

(b) if $z=\beta+i$, β is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{\beta-i+i}{\beta+i-i}=1 \Rightarrow f'(i)$ does not exist! $\therefore$ $f(z)=\bar{z}$ is not analytic

Eg. Is $f(x,y)=x^2+iy^2$ an analytic function? $f(z,\bar{z}) = ?$

(Sol.) $f(z,\bar{z}) = \left[\dfrac{1}{2}(z+\bar{z})\right]^2 + i\left[\dfrac{1}{2i}(z-\bar{z})\right]^2 = \dfrac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2) - \dfrac{i}{4}(z^2 - 2z\bar{z} + \bar{z}^2)$

is dependent on $\bar{z}$.

$\begin{aligned}u(x,y) &= x^2 \\ v(x,y) &= y^2\end{aligned} \Rightarrow \begin{cases}\dfrac{\partial u}{\partial x} = 2x \neq \dfrac{\partial v}{\partial y} = 2y \\[2mm] \dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x}\end{cases}$, $\therefore$ $f(x,y)$ is not analytic.

9-2 Complex Operators $\nabla \equiv \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$, $\overline{\nabla} \equiv \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}$, **and** $\nabla^2 = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$

Theorem Let $F(x,y)= G(z,\bar{z})$ be a complex scalar function, and $A(x,y)=P(x,y)+iQ(x,y)= B(z,\bar{z})$ be a complex vector function. Then we have

$\nabla F = 2\dfrac{\partial G}{\partial \bar{z}}$, $\nabla \cdot A = 2\,\mathrm{Re}\left(\dfrac{\partial B}{\partial z}\right)$, $\nabla \times A = 2I_m\left(\dfrac{\partial B}{\partial z}\right)$, **and** $\nabla^2 = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

(Proof) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial x} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$, $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial y} + \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial y} = i\dfrac{\partial}{\partial z} - i\dfrac{\partial}{\partial \bar{z}}$

$\therefore$ $\nabla \equiv \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial \bar{z}}$, $\overline{\nabla} \equiv \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial z} \Rightarrow \nabla F(x,y) = 2\dfrac{\partial}{\partial \bar{z}}\{G(z,\bar{z})\}$

$\nabla \cdot A = R_e(\overline{\nabla}A) = R_e\left(2\dfrac{\partial}{\partial z}B\right) = 2R_e\left(\dfrac{\partial B}{\partial z}\right) = R_e\left\{\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)(P+iQ)\right\} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$

$\nabla \times A = I_m(\overline{\nabla}A) = I_m\left(2\dfrac{\partial}{\partial z}B\right) = 2I_m\left(\dfrac{\partial B}{\partial z}\right) = I_m\left\{\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)(P+iQ)\right\} = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

$\nabla^2 = \nabla \cdot \nabla = R_e(\overline{\nabla}\nabla) = R_e\left(2\dfrac{\partial}{\partial z}2\dfrac{\partial}{\partial \bar{z}}\right) = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$

Eg. Solve $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \dfrac{8}{x^2 + y^2}$.

(Sol.) $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\dfrac{\partial^2 U}{\partial z \partial \bar{z}} = \dfrac{8}{x^2 + y^2} = \dfrac{8}{z\bar{z}}$

$\Rightarrow U(z,\bar{z}) = 2\ln|z|\cdot\ln|\bar{z}| + F_1(z) + F_2(\bar{z})$

$\Rightarrow U(x,y) = 2\left[\ell n\sqrt{x^2+y^2} + i\tan^{-1}\left(\dfrac{y}{x}\right)\right]\cdot\left[\ell n\sqrt{x^2+y^2} - i\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

$+ F_1(x+iy) + F_2(x-iy)$

$= 2\left[\ell n\sqrt{x^2+y^2}\right]^2 + 2\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]^2 + F_1(x+iy) + F_2(x-iy)$

Green's theorem $\oint_c P(x,y)dx + Q(x,y)dy = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dxdy$

General Green's theorem $\oint_c P(z,\bar{z})dz + Q(z,\bar{z})d\bar{z} = 2i \iint_R \left(\dfrac{\partial P}{\partial \bar{z}} - \dfrac{\partial Q}{\partial z} \right) dA$

Eg. Show that $A = \dfrac{1}{4i} \oint_c \bar{z}dz - zd\bar{z}$.

(Proof) $\dfrac{1}{4i} \oint_c \bar{z}dz - zd\bar{z} = \dfrac{1}{4i} \cdot 2i \iint_R [1-(-1)]dxdy = \iint_R dxdy = A$

9-3 Elementary Functions

For $z=x+iy$, where $x,y \in R$

Exponential function: $e^z = e^x[\cos(y)+i\sin(y)]$

Theorem $|e^z| = e^x$.

Theorem $e^z = e^w \Leftrightarrow z = w + 2n\pi i$, where n is an integer.

Eg. Solve $e^{4z} = 1 - i\sqrt{3}$. 【台大機研】

(Sol.) $e^{4z} = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}} = e^{\ln 2 - i\frac{\pi}{3}+i2n\pi} \Rightarrow 4z = \ln 2 + i\left(2n\pi - \dfrac{\pi}{3} \right)$

$\Rightarrow z = \dfrac{1}{4}\left[\ln 2 + i\left(2n\pi - \dfrac{\pi}{3} \right) \right]$

Eg. Find e^z **for (a)** $z = -\dfrac{i\pi}{4}$ **, (b)** $z = 3 + \dfrac{\pi}{2}i$. 【清大材料所】

(Ans.) $\dfrac{1-i}{\sqrt{2}}$, ie^3

Trigonometric functions: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{1}{2i}[e^{i(x+iy)} - e^{-i(x+iy)}]$

$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

$\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{1}{2}[e^{i(x+iy)} + e^{-i(x+iy)}]$

$= \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

Theorem $\sin(iy)=i\sinh(y)$ **and** $\cos(iy)=\cosh(y)$.

Hyperbolic functions: $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$, $\cosh(z) = \dfrac{e^z + e^{-z}}{2}$

Logarithm function: $\log(z) = \log|z| + i\arg(z) + 2n\pi i = \log|z| + i\arg(z)$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\dfrac{y}{x}\right)$

Eg. $(-i)^i$=? and $(-1)^{2i}$=?

(Sol.) $(-i)^i = e^{i\ell n(-i)} = e^{i(-i\pi/2)} = e^{\frac{\pi}{2}}$, $(-1)^{2i} = e^{2i\ell n(-1)} = e^{2i(i\pi)} = e^{-2\pi}$

Inverse trigonometric functions:

$$\sin^{-1} z = k\pi + (-1)^k \cdot \left[\frac{\pi}{2} - iLog(z + \sqrt{1-z^2})\right], \quad k = 0,\pm1,\pm2,\cdots$$

$$\cos^{-1} z = 2k\pi + i(-1)^k \cdot Log(z + \sqrt{1-z^2}), \quad k = 0,\pm1,\pm2\cdots, \quad n = 1,2$$

$$\tan^{-1} z = k\pi + \frac{1}{2i} Log\left(\frac{1+iz}{1-iz}\right), \quad k = 0,\pm1,\pm2\cdots$$

$$\sinh^{-1} z = k\pi i - (-1)^k \cdot \left\{Log\left[iz + \sqrt{1+z^2}\right] + \frac{\pi}{2}i\right\}$$

$$\cosh^{-1} z = 2k\pi i + (-1)^n \cdot Log\left[z + \sqrt{1-z^2}\right]$$

Eg. Show that $\tan^{-1}z = \dfrac{i}{2}\ln[\dfrac{i+z}{i-z}]$. 【清大電研、清大動機研】

(Sol.) Let $w = \tan^{-1}z$,

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{2}{e^{iw} + e^{-iw}} = \frac{1}{i} \cdot \frac{e^{2iw} - 1}{e^{2iw} + 1} \Rightarrow e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$\Rightarrow 2iw = \ell n \frac{i-z}{i+z} \Rightarrow w = \tan^{-1} z = \frac{1}{2i}\ln[\frac{i-z}{i+z}] = -\frac{i}{2}\ln[\frac{i-z}{i+z}] = \frac{i}{2}\ln[\frac{i+z}{i-z}]$$