Chapter 10 Numerical Solutions of Inverse Laplace Transforms

10-1 Liou's Method

Theorem $Y(s)=\dfrac{b_1s^{n-1}+b_2s^{n-2}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{n-1}s+a_n}$ **is equivalent to a linear system as**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \vdots\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} \quad \text{or} \quad \dot{x}=Ax,$$

where

$$\begin{bmatrix}x_1(0)\\ x_2(0)\\ \vdots\\ \vdots\\ x_n(0)\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ a_{n-2} & a_{n-3} & \cdots & 0\\ a_{n-1} & a_{n-2} & \cdots & 1\end{bmatrix}^{-1}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ \vdots\\ b_n\end{bmatrix}, \text{ or } x(0)=T^{-1}b$$

and $x_1(t)=y(t)$, $x_2(t)=y'(t)$, …, $x_n(t)=y^{(n-1)}(t)$. Then $x(t)=e^{At}\cdot x(0)=e^{At}\cdot T^{-1}b$, it is used to obtain $y(t)=x_1(t)$.

(Proof) Solve $y^{(n)}(t)+a_1y^{(n-1)}(t)+a_2y^{(n-2)}(t)+a_3y^{(n-3)}(t)+\ldots+a_{n-1}y'(t)+a_ny(t)=0$

By Laplace transform $Y(s)=L[y(t)]$

$\Rightarrow [s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\ldots-y^{(n-1)}(0)]+a_1[s^{n-1}Y(s)-s^{n-2}y(0)-s^{n-3}y'(0)-\ldots-y^{(n-2)}(0)]$
$+a_2[s^{n-2}Y(s)-s^{n-3}y(0)-s^{n-4}y'(0)-\ldots-y^{(n-3)}(0)]+\ldots+a_{n-1}[sY(s)-y(0)]+a_nY(s)=0$

$\Rightarrow [s^n+a_1s^{n-1}+a_2s^{n-2}+\ldots+a_{n-1}s+a_n]Y(s)=y(0)s^{n-1}+[a_1y(0)+y'(0)]s^{n-2}$
$+[a_2y(0)+a_1y'(0)+y''(0)]s^{n-3}+\ldots+[a_{n-1}y(0)+a_{n-2}y'(0)+a_{n-3}y''(0)+\ldots+y^{(n-1)}(0)]$
$=b_1s^{n-1}+b_2s^{n-2}+b_3s^{n-3}+\ldots+b_{n-1}s+b_n$

$$\Rightarrow Y(s)=\frac{b_1s^{n-1}+b_2s^{n-2}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+a_2s^{n-2}+\cdots+a_{n-1}s+a_n}$$

By $x_1(t)=y(t)$, $x_2(t)=y'(t)$, …, $x_n(t)=y^{(n-1)}(t)\Rightarrow$

$b_1=y(0)=x_1(0)$,

$b_2=a_1y(0)+y'(0)=a_1x_1(0)+x_2(0)$,

$b_3=a_2y(0)+a_1y'(0)+y''(0)=a_2x_1(0)+a_1x_2(0)+x_3(0)$, ...

$b_n=a_{n-1}y(0)+a_{n-2}y'(0)+a_{n-3}y''(0)+\ldots+y^{(n-1)}(0)=a_{n-1}x_1(0)+a_{n-2}x_2(0)+a_{n-3}x_3(0)+\ldots+x_n(0)$,

$$\therefore \begin{bmatrix}b_1\\ b_2\\ \vdots\\ \vdots\\ b_n\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ a_{n-2} & a_{n-3} & \cdots & 0\\ a_{n-1} & a_{n-2} & \cdots & 1\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\\ \vdots\\ \vdots\\ x_n(0)\end{bmatrix}\Rightarrow\begin{bmatrix}x_1(0)\\ x_2(0)\\ \vdots\\ \vdots\\ x_n(0)\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ a_{n-2} & a_{n-3} & \cdots & 0\\ a_{n-1} & a_{n-2} & \cdots & 1\end{bmatrix}^{-1}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ \vdots\\ b_n\end{bmatrix}$$

Eg. $Y(s)=\dfrac{s+1}{s^2+3s-4}$, $y(t)=?$

(Sol.) $y(t)=x_1(t)$, $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 4 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}1\\ -2\end{bmatrix}$

$$\Rightarrow x(t)=e^{At}\cdot x(0)=\begin{bmatrix}\frac{1}{5}e^{-4t}+\frac{4}{5}e^{t} & -\frac{1}{5}e^{-4t}+\frac{1}{5}e^{t}\\ \frac{-4}{5}e^{-4t}+\frac{4}{5}e^{t} & \frac{4}{5}e^{-4t}+\frac{1}{5}e^{t}\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}$$

$\Rightarrow y(t)=\frac{3}{5}e^{-4t}+\frac{2}{5}e^{t}$. **Note:** $e^{At}\approx I_0+\frac{At}{1!}+\frac{A^2t^2}{2!}+\frac{A^3t^3}{3!}+\cdots$.

10-2 Wang's Method

Theorem

$sY(s)=\dfrac{b_1s^n+b_2s^{n-1}+\cdots+b_{n-1}s^2+b_ns}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}=b_1+\dfrac{c_1s^{n-1}+c_2s^{n-2}+\cdots+c_{n-1}s+c_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$, **where** $c_i=b_{i+1}-b_1a_i$, $i=0, 1, \ldots, n$ **and** $b_{n+1}=0$.

It is equivalent to the linear system as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & & 0\\ \vdots & \vdots & 0 & & \vdots\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ \vdots\\ 0\\ 1\end{bmatrix}u(t)\text{, or } \dot{x}=Ax+Bu,$$

where $y(t)=[c_n\ c_{n-1}\ c_{n-2}\cdots c_1]\begin{bmatrix}x_1(t)\\ \vdots\\ x_n(t)\end{bmatrix}+b_1u(t)=Cx+b_1u=C\cdot\int_0^t e^{A\tau}d\tau\cdot B+b_1$

Eg. $Y(s)=\dfrac{s+1}{s^2+3s-4}$, $y(t)=?$

(Sol.) $sY(s)=\dfrac{s^2+s}{s^2+3s-4}=1+\dfrac{-2s+4}{s^2+3s-4}$ is equivalent to

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 4 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t) \text{ and } y(t)=[4\ -2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+u(t)$$

$$\Rightarrow y(t)=[4\ -2]\cdot\int_0^t e^{A\tau}d\tau\cdot\begin{bmatrix}0\\ 1\end{bmatrix}+1=\frac{3}{5}e^{-4t}+\frac{2}{5}e^{t}$$