Chapter 4 Numerical Methods in Linear Algebra

4-1 Eigenvalues and Eigenvectors of Matrices

Power method: Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, and $x_1, x_2, \ldots, x_n$ are the corresponding eigenvectors; i.e., $Ax_i=\lambda_i x_i$, $i=1, 2, \ldots, n$. Let $v^{(0)}$ be an arbitrary vector, then

$$v^{(0)} = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \quad v^{(1)} = Av^{(0)} = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n$$

$$v^{(2)} = Av^{(1)} = A^2v^{(0)} = c_1\lambda_1^2x_1 + c_2\lambda_2^2x_2 + \cdots + c_n\lambda_n^2x_n$$

$$\vdots$$

$$v^{(n)} = A^nv^{(0)} = c_1\lambda_1^nx_1 + c_2\lambda_2^nx_2 + \cdots + c_n\lambda_n^nx_n$$

If $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$, $v^{(n)} = A^nv^{(0)} = \lambda_1^n\left[c_1x_1 + c_2x_2\left(\frac{\lambda_2}{\lambda_1}\right)^n + \cdots + c_nx_n\left(\frac{\lambda_n}{\lambda_1}\right)^n\right]$

As $n\to\infty$, $v^{(n)} = A^nv^{(0)} \to \lambda_1^nc_1x_1$ **is a multiple of x_1.**

With an appropriate normalization factor c_1, this method can be used find the dominant eigenvalue λ_1 and its corresponding eigenvector.

Eg. Find the dominant eigenvalue of $A=\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ **and its corresponding eigenvector.**

(Sol.) Select $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $Ax = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 1 \\ 0.2 \end{bmatrix} = 4.6\begin{bmatrix} 1 \\ 0.2174 \\ 0.0435 \end{bmatrix}$

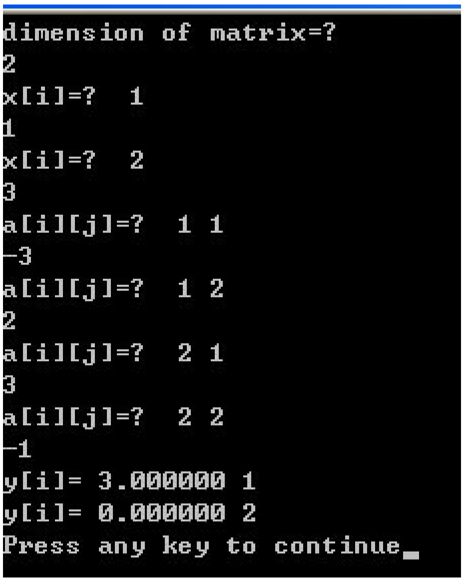
$$A\begin{bmatrix} 1 \\ 0.2174 \\ 0.0435 \end{bmatrix} = \begin{bmatrix} 4.2134 \\ 0.4783 \\ -0.0435 \end{bmatrix} = 4.2134\begin{bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0.1134 \\ -0.0103 \end{bmatrix} = \begin{bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{bmatrix} = 4.1134\begin{bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{bmatrix} = \begin{bmatrix} 4.0526 \\ 0.1077 \\ -0.0025 \end{bmatrix} = 4.0526\begin{bmatrix} 1 \\ 0.0266 \\ -0.0006 \end{bmatrix}, \therefore \lambda\to 4 \text{ and } x\to\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A C++ Program (developed by K. –Y. Lee) of computing the multiplication of a *n*×*n* matrix and a *n*-dimensional vector is listed as follows.

```
#include <stdio.h>
#include <math.h>
main()
{
int i,j,k,nn; float a[100][100],x[100],y[100];
printf("dimension of matrix=?\n");
scanf("%d",&nn);
for(i=0;i<=nn-1;i++)
{printf("x[i]=?    %d \n",i+1); scanf("%f",&x[i]);}
for (i=0;i<=nn-1;i++)
{ for (j=0;j<=nn-1;j++)
    {printf("a[i][j]=?    %d %d \n",i+1,j+1); scanf("%f",&a[i][j]);}
}
for (i=0;i<=nn-1;i++)
{ y[i]=0.;
   for (j=0;j<=nn-1;j++)
   {y[i]=y[i]+a[i][j]*x[j];}
   printf("y[i]= %f %d \n",y[i],i+1);}
  }
```



```
dimension of matrix=?
2
x[i]=?  1
1
x[i]=?  2
3
a[i][j]=?  1 1
-3
a[i][j]=?  1 2
2
a[i][j]=?  2 1
3
a[i][j]=?  2 2
-1
y[i]= 3.000000 1
y[i]= 0.000000 2
Press any key to continue
```

A C++ Program (developed by K. –Y. Lee) of computing the multiplication of two *n*×*n* matrices *C*=*AB* is listed as follows.

```
#include <stdio.h>
#include <math.h>
main()
{
int i,j,k,nn; float a[100][100],b[100][100],c[100][100];
```

```
printf("dimension of matrix(<=100)=?\n");
scanf("%d",&nn);
for (i=0;i<=nn-1;i++)
{ for (j=0;j<=nn-1;j++)
   {printf("a[i][j]=? b[i][j]=?%d %d \n",i+1,j+1);
    scanf("%f %f",&a[i][j],&b[i][j]);}
}
for (i=0;i<=nn-1;i++)
{
   for (j=0;j<=nn-1;j++)
   {
      c[i][j]=0.;
      for (k=0;k<=nn-1;k++)
      {c[i][j]= c[i][j]+a[i][k]*b[k][j];}
      printf("c[i][j]= %f %d %d \n",c[i][j],i+1,j+1);
   }
}
}
```

```
dimension of matrix(<=100)=?
2
a[i][j]=? b[i][j]=?1 1
1 4
a[i][j]=? b[i][j]=?1 2
2 -8
a[i][j]=? b[i][j]=?2 1
3 2
a[i][j]=? b[i][j]=?2 2
-5 7
c[i][j]= 8.000000 1 1
c[i][j]= 6.000000 1 2
c[i][j]= 2.000000 2 1
c[i][j]= -59.000000 2 2
Press any key to continue
```

Inverse Power method:

1. To find the eigenvalue of the least magnitude, we can apply the power method to A^{-1} to obtain its eigenvalue of the largest magnitude λ_M. Then $1/\lambda_M$ is the eigenvalue of the least magnitude for A. ($Ax=\lambda x \Leftrightarrow A^{-1}x=\lambda^{-1}x$)

2. To find the eigenvalue near q, i.e., λ-$q\approx 0$, $(A-qI)x=(\lambda-q)x$, λ-q is the eigenvalue of the least magnitude, then apply power method to find the dominant eigenvalue of $(A-qI)^{-1}$, say $\lambda_{(A-qI)^{-1}}$. Then $q+1/\lambda_{(A-qI)^{-1}}$ is the eigenvalue of A near q. One can guess $x^{(0)}$ to obtain q by $q=\dfrac{x^{t(0)}Ax^{(0)}}{x^{t(0)}x^{(0)}}$.

Eg. For $A=\begin{bmatrix}-4 & 14 & 0\\ -5 & 13 & 0\\ -1 & 0 & 2\end{bmatrix}$, find its eigenvalue near 6.

(Sol.) Guess $x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $q=\dfrac{[1\,1\,1]A\begin{bmatrix}1\\1\\1\end{bmatrix}}{[1\,1\,1]\begin{bmatrix}1\\1\\1\end{bmatrix}}=6.3333$

$$\Rightarrow (A-qI)^{-1}x^{(0)} \Rightarrow \cdots \Rightarrow \lambda \approx 6.000005 \leftrightarrow x \approx [1 \quad 0.714286 \quad -0.2499]$$

Eg. Find all eigenvalues of $A=\begin{bmatrix}4 & -1 & 1\\ 1 & 1 & 1\\ -2 & 0 & -6\end{bmatrix}$.

(Sol.) $x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow Ax^{(0)}=\begin{bmatrix}4\\3\\-8\end{bmatrix}=-8\begin{bmatrix}-0.5\\-0.375\\1\end{bmatrix}$

$$\Rightarrow A\begin{bmatrix}-0.5\\-0.375\\1\end{bmatrix}=-5\begin{bmatrix}0.125\\-0.025\\1\end{bmatrix} \Rightarrow \cdots \Rightarrow Ax^{(\infty)}=-5.76849\begin{bmatrix}-0.1157\\-0.1306\\1\end{bmatrix}, \quad \therefore \lambda_1=-5.76849$$

$$A^{-1}=-\frac{1}{26}\begin{bmatrix}-6 & -6 & -2\\ 4 & -22 & -3\\ 2 & 2 & 5\end{bmatrix}=\begin{bmatrix}3/13 & 3/13 & 1/13\\ -2/13 & 11/13 & 3/26\\ -1/13 & 1/13 & -5/26\end{bmatrix}$$

$$x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \cdots \Rightarrow A^{-1}x^{(\infty)}=0.7689\begin{bmatrix}0.4121\\1\\-0.1129\end{bmatrix}, \quad \therefore \lambda_2=\frac{1}{0.7689}=1.29923$$

Suppose it has an eigenvalue near q=3, $A-qI=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & 1\\ -2 & 0 & -9\end{bmatrix}$

$$x^{(0)}=\begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \cdots \Rightarrow (A-qI)^{-1}x^{(\infty)}=2.13174\begin{bmatrix}1\\0.31963\\-0.21121\end{bmatrix}, \quad \therefore \lambda_3=3+\frac{1}{2.13174}=3.4691$$

In **Matlab** language, we can use the following instructions to obtain the eigenvalues of a matrix:

```
>>A=[1,4,3;4,9,6;7,1,9]
    A =
        1    4    3
        4    9    6
        7    1    9
>>eig(A)
    ans =
      -1.0205
       5.1344
      14.8861
```

And we can use the following instructions to find out the *ij*-entry, the m^{th} row, and the n^{th} column of a matrix:

```
>>A=[0,1,2;3,4,5]
    A =
        0    1    2
        3    4    5
>>A(2,1)
    ans =
        3
>>A(2,3)
    ans =
        5
>>ROW1=A(1,:)
    ROW1 =
        0    1    2
>>COL2=A(:,2)
    COL2 =
        1
        4
```

4-2 Matrix Inversions

Eg. $A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}$, $A^{-1}=?$

(Sol.) $\left[\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 3 & 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 2 & 0 & 0 & 1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 0 & 3 & -5 & -3 & 1 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\end{array}\right]$

$$\rightarrow\left[\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & -5 & 0 & 1 & -3\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & \frac{2}{5} & -\frac{6}{5}\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5}\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 0 & 2/5 & -1/5\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 0 & -1/5 & 3/5\end{array}\right],\quad A^{-1}=\begin{bmatrix}0 & 2/5 & -1/5\\ -1 & 0 & 1\\ 0 & -1/5 & 3/5\end{bmatrix}$$

In **Matlab** language, we can use the following instructions to obtain the inverse of a matrix:

```
>>A=[2,1;4,3]
     A =
           2        1
           4        3
>>inv(A)
     ans =

          1.5000     -0.5000
         -2.0000      1.0000
```

4-3 Determinants of Matrices

$A \xrightarrow[\text{(Gaussian Elimination)}]{}$ triangular matrix $\Rightarrow$ *det*(*A*)=Product of the diagonal elements

(**Gaussian Elimination:** Add a multiple of a column/row to another column/row.)

Eg. $A=\begin{bmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{bmatrix}$**, *det*(*A*)=?**

(Sol.) $\begin{bmatrix} 1 & 4 & -2 & 3 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & -12 & 5 & -7 \\ 0 & -2 & 4 & -6 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 8/3 & -16/3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -2 & 3 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & -8 \end{bmatrix}$, *det*(*A*)=1 · (-6) · (-3) · (-8)=-144

Eg. $A=\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{bmatrix}$**, *det*(*A*)=?**

(Sol.) $A \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 3 & 3 & 2 \\ 0 & -4 & -1 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 0 & -13 \\ 0 & 0 & 3 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}$

det(*A*)=-[1 · (-1) · 3 · (-13)]=-39

In **Matlab** language, we can use the following instructions to obtain the determinant of a matrix:

```
>>A=[1,3,0;-1,5,2;1,2,1];

det(A)

ans =

10
```

4-4 Factorizations of Matrices

***LU* Factorization:** $A=LU$, where L is a lower triangular matrix and U is an upper triangular matrix. There are many ways.

Algorithm 1

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}=\begin{bmatrix} \ell_{11} & 0 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \ell_{i1}=a_{i1},\ 1\le i\le n \\ u_{1j}=a_{1j}/\ell_{11}=a_{1j}/a_{11},\ 1<j\le n \\ \ell_{ij}=a_{ij}-\sum_{k=1}^{j-1}\ell_{ik}u_{kj},\ \ j\le i \\ u_{ij}=\left[a_{ij}-\sum_{k=1}^{i-1}\ell_{ik}u_{kj}\right]\Big/\ell_{ii},\ \ i<j \end{cases}$$

Eg. $A=\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix}=LU$, $L=?$ $U=?$

(Sol.) $\ell_{11}=3,\ \ell_{21}=1,\ \ell_{31}=2,\ u_{12}=-1/3,\ u_{13}=2/3$

$$\ell_{22}=2-1\cdot\left(-\frac{1}{3}\right)=\frac{7}{3},\ \ \ell_{32}=-2-2\cdot\left(-\frac{1}{3}\right)=-\frac{4}{3}$$

$$u_{23}=\left[3-1\cdot\frac{2}{3}\right]\Big/\left(\frac{7}{3}\right)=1,\ \ \ell_{33}=-1-2\cdot\frac{2}{3}-\left(-\frac{4}{3}\right)\cdot 1=-1$$

$$L=\begin{bmatrix} 3 & 0 & 0 \\ 1 & 7/3 & 0 \\ 2 & -4/3 & -1 \end{bmatrix},\ U=\begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Algorithm 2

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix},\ U=\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$\Rightarrow u_{1j}=a_{1j},\ 1\le j\le n,\ \ \ell_{i1}=a_{i1}/u_{11}=a_{i1}/a_{11},\ 1<\mathrm{i}\le \mathrm{n}$$

$$u_{ij}=a_{ij}-\sum_{k=1}^{i-1}\ell_{ik}u_{kj},\ \ j\ge i,\ \ \ell_{ij}=\left[a_{ij}-\sum_{k=1}^{j-1}\ell_{ik}u_{kj}\right]\Big/u_{jj},\ \ i>j$$

In **Matlab** language, we can use the following instructions to obtain the *LU* factorization of a matrix. However, the answer may be incorrect.

```
>>A=[5,2,3;4,2,5;8,7,2]
    A =
         5    2    3
         4    2    5
         8    7    2
>>[L,U]=lu(A)
    L =
        0.6250    1.0000         0
        0.5000    0.6316    1.0000
        1.0000         0         0

    U =
        8.0000    7.0000    2.0000
             0   -2.3750    1.7500
             0         0    2.8947
```

Note: *L* is **not** an upper triangular matrix in this case.

QR Factorization: $A=QR$, where Q is an orthogonal matrix and R is an upper triangular matrix.

Let the columns of A be $A_1, A_2, A_3, \ldots, A_n$. And then

$$A_1 = |v_1|\, w_1,\quad A_2 = <A_2, w_1> w_1 + |v_2|\, w_2,\quad A_3 = <A_3, w_1> w_1 + <A_3, w_2> w_2 + |v_3|\, w_3$$

$$\Rightarrow A^t = \begin{bmatrix} A_1{}^t \\ A_2{}^t \\ \vdots \\ A_n{}^t \end{bmatrix} = \underset{(R^t)}{\begin{bmatrix} |v_1| & 0 & 0 & 0 & \cdots & 0 \\ <A_2, w_1> & |v_2| & 0 & \ddots & \ddots & \vdots \\ <A_3, w_1> & <A_3, w_2> & |v_3| & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & |v_n| \end{bmatrix}} \underset{(Q^t)}{\begin{bmatrix} w_1{}^t \\ w_2{}^t \\ \vdots \\ w_n{}^t \end{bmatrix}} \Leftrightarrow A = QR$$

Eg. Find the QR factorization of the matrix $A=\begin{bmatrix} -1 & 2 & -1 \\ 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$.

(Sol.) $A_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$$A_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \sqrt{3}\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \sqrt{3} w_1 \Rightarrow v_1 = \sqrt{3},\quad w_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$|v_2|\, w_2 = A_2 - <A_2, w_1> w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right)\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left(-\frac{1}{\sqrt{3}}\right) \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{\sqrt{42}}{3}\begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} = |v_2|\, w_2 \Rightarrow v_2 = \frac{\sqrt{42}}{3},\quad w_2 = \begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

$$|v_3|\, w_3 = A_3 - <A_3, w_1> w_1 - <A_3, w_2> w_2$$

$$= \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right)\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}\right)\begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{5}{\sqrt{3}}\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \frac{5}{\sqrt{42}}\begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -5/3 \\ 5/3 \\ 5/3 \end{bmatrix} - \begin{bmatrix} 25/42 \\ 20/42 \\ 5/42 \end{bmatrix} = \begin{bmatrix} 1/14 \\ -2/14 \\ 3/14 \end{bmatrix} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} = |v_3|\, w_3$$

$$\Rightarrow v_3 = \frac{1}{\sqrt{14}},\quad w_3 = \begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix},\quad A = QR = \begin{bmatrix} -1/\sqrt{3} & 5/\sqrt{42} & 1/\sqrt{14} \\ 1/\sqrt{3} & 4/\sqrt{42} & -2/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{42} & 3/\sqrt{14} \end{bmatrix}\begin{bmatrix} \sqrt{3} & -1/\sqrt{3} & 5/\sqrt{3} \\ 0 & \sqrt{42}/3 & 5/\sqrt{42} \\ 0 & 0 & 1/\sqrt{14} \end{bmatrix}$$

4-5 Gaussian Elimination

Eg. Solve $\begin{cases} x_1 - x_2 + 2x_3 - x_4 = -8 \\ 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20 \\ x_1 + x_2 + x_3 = -2 \\ x_1 - x_2 + 4x_3 + 3x_4 = 4 \end{cases}$.

(Sol.) $$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$x_4 = 4/2 = 2$, $x_3 = [-4-(-1)\cdot 2]/(-1) = 2$

$x_2 = [6-2-(-1)\cdot 2]/2 = 3$

$x_1 = [-8-(-1)\cdot 2 - 2\cdot 2 - (-1)\cdot 3/1 = -7$

Eg. Solve $\begin{cases} 0.003x_1 + 59.14x_2 = 59.17 \\ 5.291x_1 - 6.13x_2 = 46.78 \end{cases}$.

The exact solution is $\begin{cases} x_1 = 10 \\ x_2 = 1 \end{cases}$, $\dfrac{5.291}{0.003} = 1763.66\cdots \approx 1764$

$\Rightarrow (-6.13 - 59.14 \times 1764)x_2 = 46.78 - 1764 \times 59.17 \Rightarrow -104300x_2 \approx -104400$

$\Rightarrow x_2 \approx 1.001 \approx 1 \Rightarrow x_1 \approx [59.17 - 59.14 \times 1.001]/0.003 \approx -10 \leftarrow$ inaccurate

Pivot procedure: $\max\{|a_{11}^{(0)}|, |a_{21}^{(0)}|\} = \max\{|0.003|, |5.291|\} = 5.291$, $E_2 \leftrightarrow E_1$

$\Rightarrow \begin{cases} 5.291x_1 - 6.130x_2 = 46.78 \\ 0.003x_1 + 59.74x_2 = 59.17 \end{cases}$,

$\dfrac{0.003}{5.291} = 0.0005670 \Rightarrow (59.74 - (-6.13) \times 0.000567)x_2$

$= 59.17 - 0.000567 \times 46.78$

$\Rightarrow 59.14x_2 \approx 59.14 \Rightarrow x_2 = 1 \Rightarrow x_1 = [46.78 + 1 \times 6.13]/5.291 = 10$

Eg. An example of FORTRAN Program of solving a system of linear equations

$$\begin{cases} x+3y+5z=2 \\ 2x+4y+6z=1 \\ x+2y+z=3 \end{cases}$$ **by Gaussian elimination method**

```
C+++++++++++++++++++    NUM=3 IN THIS EXAMPLE ++++++++++++++++++++++++
C+++ COEFFICIENTS: C(1:NUM,1:NUM), CONSTANT TERMS:C(1:NUM,NUM+1) +++
C+++++++++++++ DIMENSION OF INPUT: C(NUM+1,NUM+1) +++++++++++++++++++
C+++++++++++++++ DIMENSION OF OUTPUT: S(NUM) ++++++++++++++++++++++
          DIMENSION C(4,4),S(3)
          C(1,1)=1.
          C(1,2)=3.
          C(1,3)=5.
          C(1,4)=2.
          C(2,1)=2.
          C(2,2)=4.
          C(2,3)=6.
          C(2,4)=1.
          C(3,1)=1.
          C(3,2)=2.
          C(3,3)=1.
          C(3,4)=3.
          NUM=3
          CALL SOLMATR (NUM,C,S)
          WRITE (*,*) S(1),S(2),S(3)
          STOP
          END

          SUBROUTINE SOLMATR (NUM,C,S)
           DIMENSION C(4,4),S(3)
           Z=0.
           DO 200 I=1, NUM
              IF (C(I,I).NE.Z) GOTO 220
                 DO 230 J=I+1,NUM
                    IF (C(J,I).NE.Z) GOTO 250
230           CONTINUE
250           CALL PIVOT(I,J,NUM,C)
220           DIV=C(I,I)
              DO 270 J= 1,NUM+1
                C(I,J)=C(I,J)/DIV
270           CONTINUE
              DO 290 II=1,NUM
                 IF (II.EQ.I) GOTO 290
                   RR= C(II,I)
                   DO 310 K=1, NUM+1
                     C(II,K)=C(II,K)-RR*C(I,K)
310                CONTINUE
290            CONTINUE
```

```
200     CONTINUE
        DO 330 I=1, NUM
           S(I)=C(I,NUM+1)
330     CONTINUE
        RETURN
       END

       SUBROUTINE PIVOT(I,J,NUM,C)
        DIMENSION C(4,4)
        DO 400 K=1, NUM+1
          TRAN=C(I,K)
          C(I,K)=C(J,K)
          C(J,K)=TRAN
400     CONTINUE
        RETURN
       END
```

```
 -3.750000         4.000000       -1.250000
Press any key to continue
```

In **Matlab** language, we can use the following instructions to solve a linear system of equations:

$$\begin{cases} x+3y+5z=2 \\ 2x+4y+6z=1 \\ x+2y+z=3 \end{cases}$$

```
>>A=[1,3,5;2,4,6;1,2,1];
>>B=[2,1,3]'; % [2,1,3]' is the transpose of [2,1,3]
>>rref([A,B])

ans =
    1.0000         0         0   -3.7500
         0    1.0000         0    4.0000
         0         0    1.0000   -1.2500
```

4-6 Iterative Methods for Solving Systems of Linear Equations

Jacobi iterative method:

Eg. Solve $\begin{cases} 10x_1 - x_2 + 2x_3 = 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 = 25 \\ 2x_1 - x_2 + 10x_3 - x_4 = -11 \\ 3x_2 - x_3 + 8x_4 = 15 \end{cases}$.

(Sol.) $$\begin{cases} x_1^{(k)} = \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\ x_2^{(k)} = \frac{1}{11}x_1^{(k-1)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\ x_3^{(k)} = -\frac{1}{5}x_1^{(k-1)} + \frac{1}{10}x_2^{(k-1)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\ x_4^{(k)} = -\frac{3}{8}x_2^{(k-1)} + \frac{1}{8}x_3^{(k-1)} + \frac{15}{8} \end{cases}$$

$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = (0,0,0,0) \Rightarrow \cdots$

$\Rightarrow (x_1^{(10)}, x_2^{(10)}, x_3^{(10)}, x_4^{(10)}) = (1.0001, 1.9998, -0.9998, 0.9998)$

Gauss-Seidel iterative method:

Eg. For the same systems as above, solve it by the Gauss-Seidel method.

(Sol) $$\begin{cases} x_1^{(k)} = \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} + \frac{3}{5} \\ x_2^{(k)} = \frac{1}{11}x_1^{(k)} + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\ x_3^{(k)} = -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\ x_4^{(k)} = -\frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} + \frac{15}{8} \end{cases}$$

$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = (0,0,0,0) \Rightarrow \cdots$

$\Rightarrow (x_1^{(5)}, x_2^{(5)}, x_3^{(5)}, x_4^{(5)}) = (1.000, 2.000, -1.000, 1.000)$