Chapter 7 Boundary-value Problems and Eigenvalue Problems for Ordinary Differential Equations

7-1 Shooting Method

Solve $y''=f(x,y,y')$, $a\leqq x\leqq b$, $y(a)=\alpha$, $y(b)=\beta$.

Set $x_1(a)=y(a)=\alpha$ and guess $x_2(a)=x_1'(a)=y'(a)=s_0$, and apply any numerical method to obtain $y(b)$.

Let $y(b,s)$ denote the solution in $x=b$ with an initial condition $y'(a)=s$. The problem is to determine s so that $y(b,s)-\beta=0$. Then choose s_0 and s_1, by Secant method, $s_k=s_{k-1}-\dfrac{[y(b,s_{k-1})-\beta]\cdot(s_{k-1}-s_{k-2})}{y(b,s_{k-1})-y(b,s_{k-2})}=x_1'(a)=x_2(a)$, $k=2, 3, 4, \ldots$

And apply the numerical method to obtain $y(b,s_k)$. Stop all the procedures in case $y(b,s_k)-\beta=0$.

Eg. Solve $\dfrac{d^2y(t)}{dt^2}=\dfrac{1}{8}(32+2t^3-yy')$, **$1\leqq t\leqq 3$, $y(1)=17$, $y(3)=43/3$.**

(Sol.) Set $x_1(t)=y$, $x_2(t)=x_1'(t)=y'$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} x_2(t) \\ \frac{1}{8}(32+2t^3-x_1(t)x_2(t)) \end{bmatrix}$$

$x_1(1)=17$, $x_1'(1)=s_0 \Rightarrow y(3)=y(3,s_0)$

$x_1(1)=17$, $x_1'(1)=s_1\neq s_0 \Rightarrow y(3)=y(3,s_1)$

$x_1(1)=17$, $s_2=s_1-\dfrac{\left[y(3,s_1)-\frac{43}{3}\right]\cdot(s_1-s_0)}{y(3,s_1)-y(3,s_0)}=x_1'(1)\Rightarrow y(3,s_2)$

$x_1(1)=17$, $s_3=s_2-\dfrac{\left[y(3,s_2)-\frac{43}{3}\right]\cdot(s_2-s_1)}{y(3,s_2)-y(3,s_1)}=x_1'(1)\Rightarrow y(3,s_3)$

$\vdots$

7-2 Finite Difference Method

Solve $y''=f(x,y,y')$, $a\leqq x\leqq b$, $y(a)=\alpha$, $y(b)=\beta$.

Set $y_i=y(x_i)$, $x_i=a+ih$, $h=(b-a)/n$, $i=1, 2, \ldots, n-1 \Rightarrow \dfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}=f\left(x_i, y_i, \dfrac{y_{i+1}-y_{i-1}}{2h}\right)$,

$y_0=\alpha$, $y_n=\beta$

$$\Rightarrow \begin{cases} y_2-2y_1-h^2 f\left(x_1, y_1, \dfrac{y_2-\alpha}{2h}\right)=-\alpha \\ y_3-2y_2+y_1-h^2 f\left(x_2, y_2, \dfrac{y_3-y_1}{2h}\right)=0 \\ y_4-2y_3+y_2-h^2 f\left(x_3, y_3, \dfrac{y_4-y_2}{2h}\right)=0 \\ \quad\vdots \\ y_{n-1}-2y_{n-2}+y_{n-3}-h^2 f\left(x_{n-2}, y_{n-2}, \dfrac{y_{n-1}-y_{n-3}}{2h}\right)=0 \\ -2y_{n-1}+y_{n-2}-h^2 f\left(x_{n-1}, y_{n-1}, \dfrac{\beta-y_{n-2}}{2h}\right)=-\beta \end{cases}$$

Eg. Solve $y''=y^2$, $y(1)=1$, $y(3)=10$, $1\leqq x\leqq 3$.

(Sol.) Boundary conditions: $y_0=1=y(1)$ and $y_n=10=y(3)$

$\dfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}=y_i^2$, $h=(3-1)/n=2/n$, $x_i=1+ih$, $1\leq i\leq n-1$

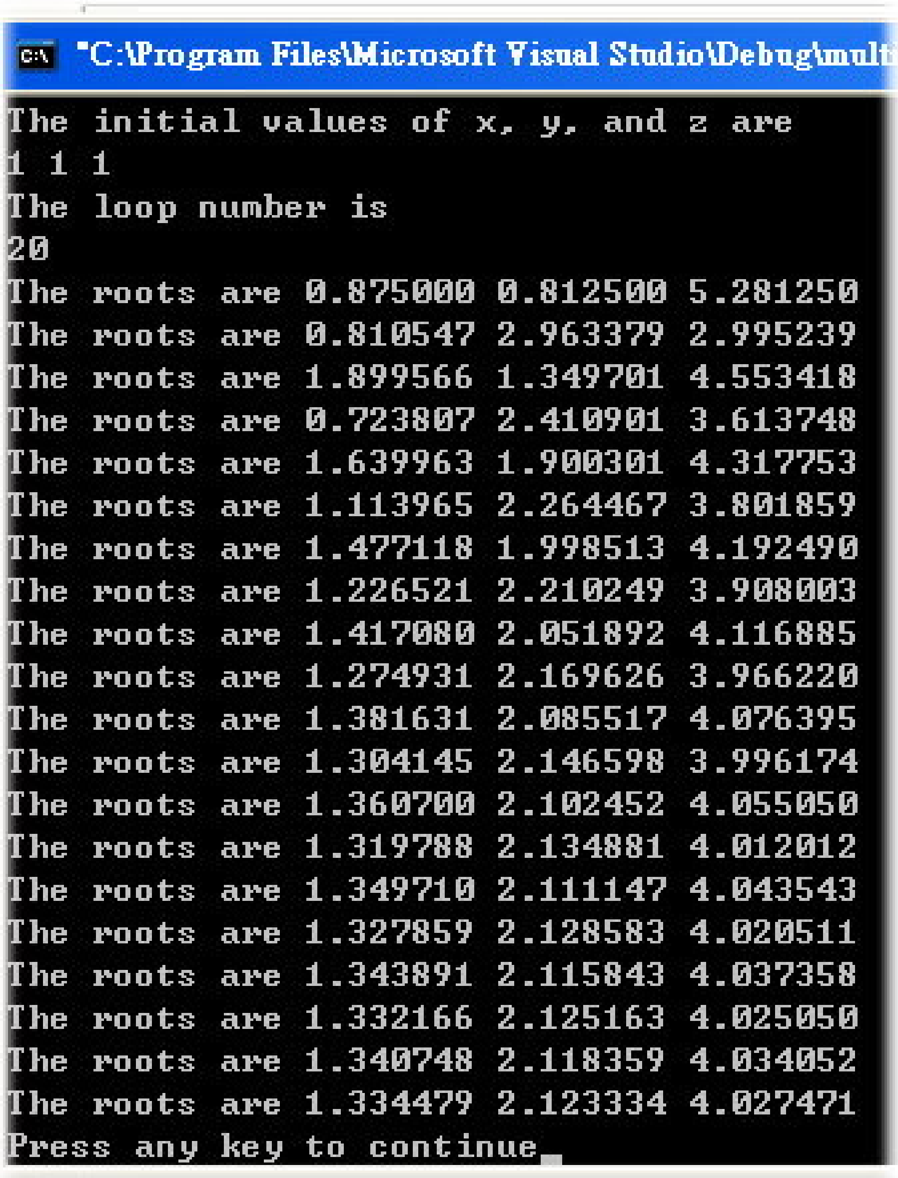
$\Rightarrow y_{i-1}-2y_i-h^2y_i^2+y_{i+1}=0$, $1\leq i\leq n-1$

Choose $n=4$, $h=2/4=0.5 \Rightarrow \begin{cases} 1-2y_1-(0.5)^2y_1^2+y_2=0 \\ y_1-2y_2-(0.5)^2y_2^2+y_3=0 \\ y_2-2y_3-(0.5)^2y_3^2+10=0 \end{cases}$ can be transformed into

$\begin{cases} y_1=[1-0.25y_1^2+y_2]/2 \\ y_2=[y_1-0.25y_2^2+y_3]/2 \\ y_3=[y_2-0.25y_3^2+10]/2 \end{cases}$, and then it can be solved by the following **C++** program.

```
#include <stdio.h>
#include <math.h>
main()
{
int i,lop; float x,y,z;
printf("The initial values of x, y, and z are\n"); scanf("%f %f %f",&x,&y,&z);
printf("The loop number is\n"); scanf("%d",&lop);
for (i=1;i<=lop;i++)
```

```
{x=0.5*(1-0.25*x*x+y); y=0.5*(x-0.25*y*y+z); z=0.5*(y-0.25*z*z+10);
printf("The roots are %f %f %f \n",x,y,z);
    }
}
```



Eg. Solve $y'' - \left(1 - \frac{x}{5}\right)y = x$, **$y(1)=2$, $y(3)=-1$, $1 \leqq x \leqq 3$.**

(Sol.) $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \left(1 - \frac{x_i}{5}\right)y_i = x_i$, $x_i=1+ih$

Choose $n=4 \Rightarrow h=(3-1)/4=0.5 \Rightarrow x_1=1.5$, $x_2=2$, $x_3=2.5$, and $y(1)=2=y_0$, $y(3)= -1= y_4$

$$\Rightarrow y_{i-1} - \left[2 + h^2\left(1 - \frac{x_i}{5}\right)\right]y_i + y_{i+1} = h^2 x_i,\ i=1, 2, 3$$

$$\Rightarrow \begin{cases} y_2 - \left[2 + \left(\frac{1}{2}\right)^2\left(1 - \frac{1.5}{5}\right)\right]y_1 + 2 = \left(\frac{1}{2}\right)^2 \cdot (1.5) \\ y_3 - \left[2 + \left(\frac{1}{2}\right)^2\left(1 - \frac{2}{5}\right)\right]y_2 + y_1 = \left(\frac{1}{2}\right)^2 \cdot 2 \\ -1 - \left[2 + \left(\frac{1}{2}\right)^2\left(1 - \frac{2.5}{5}\right)\right]y_3 + y_2 = \left(\frac{1}{2}\right)^2 \cdot (2.5) \end{cases}$$

$$\Rightarrow \begin{bmatrix} -2.175 & 1 & 0 \\ 1 & -2.15 & 1 \\ 0 & 1 & -2.125 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.675 \\ 0.5 \\ 1.675 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.5748 \\ -0.4248 \\ -0.9881 \end{bmatrix}$$

In **MATLAB** language, we can use the following program to solve the ordinary differential equation:

```
>> A=[-2.175 1 0;1 -2.15 1;0 1 -2.125];
>> B=[-1.675;0.5;1.675];
>> rref([A,B])

ans =

    1.0000         0         0    0.5748
         0    1.0000         0   -0.4248
         0         0    1.0000   -0.9881
```

Eg. Solve $y''=y$, $y(1)=1$, $y(3)=10$, $1\leqq x\leqq 3$.

(Sol.) $\dfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}=y_i$, $h=(3-1)/n$, $x_i=1+ih$, $1\le i\le n-1$

Boundary conditions: $y_0=1=y(1)$ and $y_n=10=y(3)$

$$\Rightarrow\begin{bmatrix} -2-h^2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & -2-h^2 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2-h^2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -2-h^2 & 1 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2-h^2 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \ddots & 0 & 1 & -2-h^2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & -2-h^2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -10 \end{bmatrix}$$

Eg. Solve $y''=y$, $y'(1)=1$, $y'(3)=10$, $1\leqq x\leqq 3$.

(Sol.) $\dfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}=y_i$, $h=(3-1)/n$, $x_i=1+ih$, $1\le i\le n-1$

Boundary conditions: $\dfrac{y_1-y_0}{h}=1$ and $\dfrac{y_n-y_{n-1}}{h}=10$

$$\Rightarrow\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2-h^2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2-h^2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2-h^2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2-h^2 & 1 & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2-h^2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}=\begin{bmatrix} h \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 10h \end{bmatrix}$$

7-3 Eigenvalue Problems

T is a linear operator, $\exists\lambda$ and x such that $T(x)=\lambda x$, then λ is the eigenvalue of T and x is the eigenfunction corresponding to λ.

Eg. For $y''+k^2y=0$ with $y(0)=y(1)=0$, its general solution is $y(x)=a\sin(kx)+b\cos(kx)$.
$y(0)=0 \Rightarrow b=0$, $y(1)=0 \Rightarrow a\sin(k)=0 \Rightarrow k=n\pi$, $n=0, \pm1, \pm2, \ldots$.

$\therefore k^2=n^2\pi^2$ is the eigenvalue of the system $D^2y=\dfrac{d^2y}{dx^2}=-k^2y=\lambda y$.

Eg. Use the finite-difference method to obtain the eigenvalues of $y''+k^2y=0$ with $y(0)=y(1)=0$.

$$\Rightarrow \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+k^2y_i=0,\ h=(1-0)/n,\ i=1, 2, \ldots, n-1 \text{ and } y_0=y_n=0.$$

$$\Rightarrow -y_{i+1}+2y_i-y_{i-1}=h^2k^2y_i$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & 0 & & & \vdots \\ 0 & 0 & -1 & 2 & -1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} = h^2k^2 \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$\Rightarrow AY=\lambda Y \Rightarrow$ Find the eigenvalue λ of A by numerical methods $\Rightarrow k=\left[\lambda/h^2\right]^{1/2}$

In case of $n=5$, we have $h=(1-0)/5=0.2$. And then we can use the following program to obtain the eigenvalues in **MATLAB** language:

```
>> A=[2 -1 0 0;1-1 2 -1 0;0 -1 2 -1;0 0 -1 2];
>> sqrt(eig(A)/0.04)

ans =

    7.0711
    3.8268
    7.0711
    9.2388
```

We can check 3.8268≒π, 7.0711≒2π, and 9.2388≒3π.