

Chapter 7 Boundary-value Problems and Eigenvalue Problems for Ordinary Differential Equations

7-1 Shooting Method

Solve $y''=f(x,y,y')$, $a \leq x \leq b$, $y(a)=\alpha$, $y(b)=\beta$.

Set $x_1(a)=y(a)=\alpha$ and guess $x_2(a)=x_1'(a)=y'(a)=s_0$, and apply any numerical method to obtain $y(b)$.

Let $y(b,s)$ denote the solution in $x=b$ with an initial condition $y'(a)=s$. The problem is to determine s so that $y(b,s)-\beta=0$. Then choose s_0 and s_1 , by Secant method,

$$s_k = s_{k-1} - \frac{[y(b, s_{k-1}) - \beta] \cdot (s_{k-1} - s_{k-2})}{y(b, s_{k-1}) - y(b, s_{k-2})} = x_1'(a) = x_2(a), \quad k=2, 3, 4, \dots$$

And apply the numerical method to obtain $y(b,s_k)$. Stop all the procedures in case $y(b,s_k)-\beta=0$.

Eg. Solve $\frac{d^2 y(t)}{dt^2} = \frac{1}{8}(32 + 2t^3 - yy')$, $1 \leq t \leq 3$, $y(1)=17$, $y(3)=43/3$.

(Sol.) Set $x_1(t)=y$, $x_2(t)=x_1'(t)=y'$

$$\Rightarrow \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{1}{8}(32 + 2t^3 - x_1(t)x_2(t)) \end{bmatrix}$$

$$x_1(1)=17, \quad x_1'(1) = s_0 \Rightarrow y(3) = y(3, s_0)$$

$$x_1(1)=17, \quad x_1'(1) = s_1 \neq s_0 \Rightarrow y(3) = y(3, s_1)$$

$$x_1(1)=17, \quad s_2 = s_1 - \frac{\left[y(3, s_1) - \frac{43}{3} \right] \cdot (s_1 - s_0)}{y(3, s_1) - y(3, s_0)} = x_1'(1) \Rightarrow y(3, s_2)$$

$$x_1(1)=17, \quad s_3 = s_2 - \frac{\left[y(3, s_2) - \frac{43}{3} \right] \cdot (s_2 - s_1)}{y(3, s_2) - y(3, s_1)} = x_1'(1) \Rightarrow y(3, s_3)$$

⋮

7-2 Finite Difference Method

Solve $y''=f(x,y,y')$, $a \leq x \leq b$, $y(a)=\alpha$, $y(b)=\beta$.

Set $y_i=y(x_i)$, $x_i=a+ih$, $h=(b-a)/n$, $i=1, 2, \dots, n-1 \Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$,

$y_0=\alpha$, $y_n=\beta$

$$\Rightarrow \begin{cases} y_2 - 2y_1 - h^2 f\left(x_1, y_1, \frac{y_2 - \alpha}{2h}\right) = -\alpha \\ y_3 - 2y_2 + y_1 - h^2 f\left(x_2, y_2, \frac{y_3 - y_1}{2h}\right) = 0 \\ y_4 - 2y_3 + y_2 - h^2 f\left(x_3, y_3, \frac{y_4 - y_2}{2h}\right) = 0 \\ \vdots \\ y_{n-1} - 2y_{n-2} + y_{n-3} - h^2 f\left(x_{n-2}, y_{n-2}, \frac{y_{n-1} - y_{n-3}}{2h}\right) = 0 \\ -2y_{n-1} + y_{n-2} - h^2 f\left(x_{n-1}, y_{n-1}, \frac{\beta - y_{n-2}}{2h}\right) = -\beta \end{cases}$$

Eg. Solve $y''=y^2$, $y(1)=1$, $y(3)=10$, $1 \leq x \leq 3$.

(Sol.) Boundary conditions: $y_0=1=y(1)$ and $y_n=10=y(3)$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i^2, \quad h=(3-1)/n=2/n, \quad x_i=1+ih, \quad 1 \leq i \leq n-1$$

$$\Rightarrow y_{i-1} - 2y_i - h^2 y_i^2 + y_{i+1} = 0, \quad 1 \leq i \leq n-1$$

Choose $n=4$, $h=2/4=0.5 \Rightarrow \begin{cases} 1 - 2y_1 - (0.5)^2 y_1^2 + y_2 = 0 \\ y_1 - 2y_2 - (0.5)^2 y_2^2 + y_3 = 0 \\ y_2 - 2y_3 - (0.5)^2 y_3^2 + 10 = 0 \end{cases}$ can be transformed into

$$\begin{cases} y_1 = [1 - 0.25y_1^2 + y_2]/2 \\ y_2 = [y_1 - 0.25y_2^2 + y_3]/2, \text{ and then it can be solved by the following C++ program.} \\ y_3 = [y_2 - 0.25y_3^2 + 10]/2 \end{cases}$$

```
#include <stdio.h>
```

```
#include <math.h>
```

```
main()
```

```
{
```

```
int i,lop; float x,y,z;
```

```
printf("The initial values of x, y, and z are\n"); scanf("%f %f %f",&x,&y,&z);
```

```
printf("The loop number is\n"); scanf("%d",&lop);
```

```
for (i=1;i<=lop;i++)
```

```

    {x=0.5*(1-0.25*x*x+y); y=0.5*(x-0.25*y*y+z); z=0.5*(y-0.25*z*z+10);
printf("The roots are %f %f %f\n",x,y,z);
    }
}

```

```

C:\Program Files\Microsoft Visual Studio\Debug\multi
The initial values of x, y, and z are
1 1 1
The loop number is
20
The roots are 0.875000 0.812500 5.281250
The roots are 0.810547 2.963379 2.995239
The roots are 1.899566 1.349701 4.553418
The roots are 0.723807 2.410901 3.613748
The roots are 1.639963 1.900301 4.317753
The roots are 1.113965 2.264467 3.801859
The roots are 1.477118 1.998513 4.192490
The roots are 1.226521 2.210249 3.908003
The roots are 1.417080 2.051892 4.116885
The roots are 1.274931 2.169626 3.966220
The roots are 1.381631 2.085517 4.076395
The roots are 1.304145 2.146598 3.996174
The roots are 1.360700 2.102452 4.055050
The roots are 1.319788 2.134881 4.012012
The roots are 1.349710 2.111147 4.043543
The roots are 1.327859 2.128583 4.020511
The roots are 1.343891 2.115843 4.037358
The roots are 1.332166 2.125163 4.025050
The roots are 1.340748 2.118359 4.034052
The roots are 1.334479 2.123334 4.027471
Press any key to continue

```

Eg. Solve $y'' - \left(1 - \frac{x}{5}\right)y = x, y(1)=2, y(3)=-1, 1 \leq x \leq 3.$

(Sol.) $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \left(1 - \frac{x_i}{5}\right)y_i = x_i, x_i=1+ih$

Choose $n=4 \Rightarrow h=(3-1)/4=0.5 \Rightarrow x_1=1.5, x_2=2, x_3=2.5$, and $y(1)=2=y_0, y(3)=-1=y_4$

$$\Rightarrow y_{i-1} - \left[2 + h^2 \left(1 - \frac{x_i}{5}\right)\right] y_i + y_{i+1} = h^2 x_i, i=1, 2, 3$$

$$\Rightarrow \begin{cases} y_2 - \left[2 + \left(\frac{1}{2}\right)^2 \left(1 - \frac{1.5}{5}\right)\right] y_1 + 2 = \left(\frac{1}{2}\right)^2 \cdot (1.5) \\ y_3 - \left[2 + \left(\frac{1}{2}\right)^2 \left(1 - \frac{2}{5}\right)\right] y_2 + y_1 = \left(\frac{1}{2}\right)^2 \cdot 2 \\ -1 - \left[2 + \left(\frac{1}{2}\right)^2 \left(1 - \frac{2.5}{5}\right)\right] y_3 + y_2 = \left(\frac{1}{2}\right)^2 \cdot (2.5) \end{cases}$$

$$\Rightarrow \begin{bmatrix} -2.175 & 1 & 0 \\ 1 & -2.15 & 1 \\ 0 & 1 & -2.125 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1.675 \\ 0.5 \\ 1.675 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.5748 \\ -0.4248 \\ -0.9881 \end{bmatrix}$$

In **MATLAB** language, we can use the following program to solve the ordinary differential equation:

```
>> A=[-2.175 1 0;1 -2.15 1;0 1 -2.125];
>> B=[-1.675;0.5;1.675];
>> rref([A,B])
```

ans =

```
1.0000    0    0    0.5748
0    1.0000    0   -0.4248
0    0    1.0000   -0.9881
```

Eg. Solve $y''=y, y(1)=1, y(3)=10, 1 \leq x \leq 3$.

(Sol.) $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i, h=(3-1)/n, x_i=1+ih, 1 \leq i \leq n-1$

Boundary conditions: $y_0=1=y(1)$ and $y_n=10=y(3)$

$$\Rightarrow \begin{bmatrix} -2-h^2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & -2-h^2 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -2-h^2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -2-h^2 & 1 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2-h^2 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \ddots & 0 & 1 & -2-h^2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & -2-h^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -10 \end{bmatrix}$$

Eg. Solve $y''=y, y'(1)=1, y'(3)=10, 1 \leq x \leq 3$.

(Sol.) $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = y_i, h=(3-1)/n, x_i=1+ih, 1 \leq i \leq n-1$

Boundary conditions: $\frac{y_1 - y_0}{h} = 1$ and $\frac{y_n - y_{n-1}}{h} = 10$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2-h^2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2-h^2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2-h^2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2-h^2 & 1 & \ddots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2-h^2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} h \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 10h \end{bmatrix}$$

7-3 Eigenvalue Problems

T is a linear operator, $\exists \lambda$ and x such that $T(x)=\lambda x$, then λ is the eigenvalue of T and x is the eigenfunction corresponding to λ .

Eg. For $y''+k^2y=0$ with $y(0)=y(1)=0$, its general solution is $y(x)=a\sin(kx)+b\cos(kx)$.
 $y(0)=0 \Rightarrow b=0, y(1)=0 \Rightarrow a\sin(k)=0 \Rightarrow k=n\pi, n=0, \pm 1, \pm 2, \dots$

$\therefore k^2=n^2\pi^2$ is the eigenvalue of the system $D^2y=\frac{d^2y}{dx^2}=-k^2y=\lambda y$.

Eg. Use the finite-difference method to obtain the eigenvalues of $y''+k^2y=0$ with $y(0)=y(1)=0$.

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + k^2 y_i = 0, h=(1-0)/n, i=1, 2, \dots, n-1 \text{ and } y_0=y_n=0.$$

$$\Rightarrow -y_{i+1} + 2y_i - y_{i-1} = h^2 k^2 y_i$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & 0 & & & \vdots \\ 0 & 0 & -1 & 2 & -1 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & \dots & \dots & \dots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix} = h^2 k^2 \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$\Rightarrow AY=\lambda Y \Rightarrow \text{Find the eigenvalue } \lambda \text{ of } A \text{ by numerical methods} \Rightarrow k = \left[\lambda/h^2 \right]^{1/2}$$

In case of $n=5$, we have $h=(1-0)/5=0.2$. And then we can use the following program to obtain the eigenvalues in **MATLAB** language:

```
>> A=[2 -1 0 0;1-1 2 -1 0;0 -1 2 -1;0 0 -1 2];
>> sqrt(eig(A)/0.04)
```

ans =

```
7.0711
3.8268
7.0711
9.2388
```

We can check $3.8268 \doteq \pi$, $7.0711 \doteq 2\pi$, and $9.2388 \doteq 3\pi$.