Chapter 8 Partial Differential Equations

8-1 Elliptic Partial Differential Equations

Solve $\dfrac{\partial^2 u(x,y)}{\partial x^2}+\dfrac{\partial^2 u(x,y)}{\partial y^2}=f(x,y)$, $\forall\,(x,y)\in R$; $u(x,y)=g(x,y)$, $\forall\,(x,y)\in S$, where $R=\{(x,y): a\leqq x\leqq b,\ c\leqq y\leqq d\}$, and S is the boundary of R.

Set $x_i=a+ih$, $y_j=c+jk$, $h=(b-a)/n$, $k=(d-c)/m$, $i=0, 1, 2, \ldots, n$, and $j=0, 1, 2, \ldots, m$.

Central difference method:

$$\frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2}+\frac{u(x_i,y_{j+1})-2u(x_i,y_j)+u(x_i,y_{j-1})}{k^2}=f(x_i,y_j)$$

and $u(x_0,y_j)=g(x_0,y_j)$, $u(x_n,y_j)=g(x_n,y_j)$, $j=0, 1, 2, \ldots, m-1$; $u(x_i,y_0)=g(x_i,y_0)$, $u(x_i,y_m)=g(x_i,y_m)$, $i=0, 1, 2, \ldots, n-1$

$$\Rightarrow 2\left[\left(\frac{h}{k}\right)^2+1\right]u_{ij}-(u_{i+1,j}+u_{i-1,j})-\left(\frac{h}{k}\right)^2(u_{i,j+1}+u_{i,j-1})=-h^2 f_{ij}$$ **and $u_{0j}=g_{0j}$, $u_{nj}=g_{nj}$, $u_{i0}=g_{i0}$, $u_{im}=g_{im}$.**

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, $0\leqq x\leqq 0.5$, $0\leqq y\leqq 0.5$ with $u(0,y)=u(x,0)=0$, $u(x,1/2)=200x$, $u(1/2,y)=200y$. (Exact solution: $u(x,y)=400xy$)

(Sol.) We set $n=m=4$, and then

$$h=k=\frac{\frac{1}{2}}{4}=\frac{1}{8},\quad u_{ij}=u(x_i,y_j),\quad \begin{matrix}x_i=ih\\ y_j=jk\end{matrix}\ \Rightarrow \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}=0$$

$$\Rightarrow 2\left[\left(\frac{h}{k}\right)^2+1\right]u_{ij}-(u_{i+1,j}+u_{i-1,j})-\left(\frac{h}{k}\right)^2(u_{i,j+1}+u_{i,j-1})=0$$

$$\Rightarrow\begin{cases}4u_{13}-u_{23}-u_{12}=u_{14}=200\times\frac{1}{8}=25\\ 4u_{23}-u_{33}-u_{13}-u_{22}=u_{24}=200\times\frac{2}{8}=50\\ 4u_{33}-u_{23}-u_{32}=u_{43}+u_{34}=200\times\frac{3}{8}+200\times\frac{3}{8}=150\\ 4u_{12}-u_{22}-u_{13}-u_{11}=u_{02}=0\\ 4u_{22}-u_{32}-u_{12}-u_{23}-u_{21}=0\\ 4u_{32}-u_{22}-u_{33}-u_{31}=u_{42}=200\times\frac{2}{8}=50\\ 4u_{11}-u_{21}-u_{12}=u_{01}+u_{10}=0\\ 4u_{21}-u_{31}-u_{11}-u_{22}=u_{20}=0\\ 4u_{31}-u_{21}-u_{32}=u_{30}+u_{41}=0+200\times\frac{1}{8}=25\end{cases}
\Rightarrow\begin{bmatrix}4&-1&0&-1&0&0&0&0&0\\-1&4&-1&0&-1&0&0&0&0\\0&-1&4&0&0&-1&0&0&0\\-1&0&0&4&-1&0&-1&0&0\\0&-1&0&-1&4&-1&0&-1&0\\0&0&-1&0&-1&4&0&0&-1\\0&0&0&-1&0&0&4&-1&0\\0&0&0&0&-1&0&-1&4&-1\\0&0&0&0&0&-1&0&-1&4\end{bmatrix}\begin{bmatrix}u_{13}\\u_{23}\\u_{33}\\u_{12}\\u_{22}\\u_{32}\\u_{11}\\u_{21}\\u_{31}\end{bmatrix}=\begin{bmatrix}25\\50\\150\\0\\0\\50\\0\\0\\25\end{bmatrix}$$

$\Rightarrow u_{13}=18.75$, $u_{23}=37.5$, …, $u_{21}=12.5$, $u_{31}=18.75$

In **MATLAB** language, we can use the following program to solve the partial differential equation:

```
>>A=[4,-1,0,-1,0,0,0,0,0;-1,4,-1,0,-1,0,0,0,0;0,-1,4,0,0,-1,0,0,0;-1,0,0,4,-1,0,-1,0,0;
0,-1,0,-1,4,-1,0,-1,0;0,0,-1,0,-1,4,0,0,-1;0,0,0,-1,0,0,4,-1,0;0,0,0,0,-1,0,-1,4,-1;
0,0,0,0,0,-1,0,-1,4];
>>B=[25;50;150;0;0;50;0;0;25];
>>rref([A,B])
```

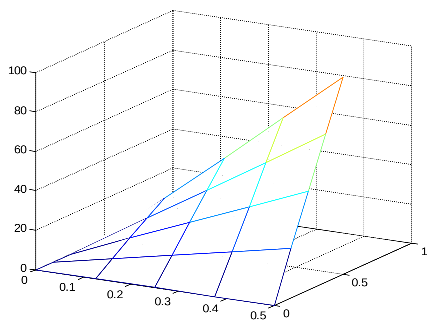
ans =

Columns 1 through 7

1.0000	0	0	0	0	0	0
0	1.0000	0	0	0	0	0
0	0	1.0000	0	0	0	0
0	0	0	1.0000	0	0	0
0	0	0	0	1.0000	0	0
0	0	0	0	0	1.0000	0
0	0	0	0	0	0	1.0000
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 10

0	0	18.7500
0	0	37.5000
0	0	56.2500
0	0	12.5000
0	0	25.0000
0	0	37.5000
0	0	6.2500
1.0000	0	12.5000
0	1.0000	18.7500



8-2 Parabolic Partial Differential Equations

Solve $\frac{\partial u(x,t)}{\partial t} = \alpha^2 \cdot \frac{\partial^2 u(x,t)}{\partial x^2}$, $0 \leqq x \leqq l$, $t>0$, with $u(0,t)=u(l,t)=0$, $u(x,0)=f(x)$.

Set $h=l/m$, $x_i=ih$, $t_j=jk$, $i=0, 1, 2, \ldots, m$, and $j=0, 1, 2, \ldots$

Forward difference method: $\frac{u_{i,j+1}-u_{ij}}{k} - \alpha^2 \cdot \frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2} = 0$

Set $\lambda = \frac{\alpha^2 k}{h^2} \Rightarrow u_{i,j+1} = (1-2\lambda)u_{ij} + \lambda(u_{i+1,j}+u_{i-1,j})$ **and** $u_{0j}=u_{mj}=0$, $u_{i0}=f(x_i)$

$$\Rightarrow \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \lambda & (1-2\lambda) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ \vdots \\ u_{m-1,j} \end{bmatrix}$$

We have

$$\begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \lambda & (1-2\lambda) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix},$$

$$\begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \lambda & (1-2\lambda) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix},$$

$$\begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \lambda & (1-2\lambda) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix},$$

$$\begin{bmatrix} u_{14} \\ u_{24} \\ \vdots \\ \vdots \\ u_{m-1,4} \end{bmatrix} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \lambda & (1-2\lambda) & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix}, \ldots.$$

Backward difference method: $\dfrac{u_{ij}-u_{i,j-1}}{k}-\alpha^2\cdot\dfrac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}=0$

$$\Rightarrow u_{i,j-1}=(1+2\lambda)u_{ij}-\lambda(u_{i+1,j}+u_{i-1,j})$$

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}u_{1j}\\ u_{2j}\\ \vdots\\ \vdots\\ u_{m-1,j}\end{bmatrix}=\begin{bmatrix}u_{1,j-1}\\ u_{2,j-1}\\ \vdots\\ \vdots\\ u_{m-1,j-1}\end{bmatrix}$$, where $u_{i0}=f(x_i)$.

We solve

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}u_{11}\\ u_{21}\\ \vdots\\ \vdots\\ u_{m-1,1}\end{bmatrix}=\begin{bmatrix}f(x_1)\\ f(x_2)\\ \vdots\\ \vdots\\ f(x_{m-1})\end{bmatrix}\Rightarrow\begin{bmatrix}u_{11}\\ u_{21}\\ \vdots\\ \vdots\\ u_{m-1,1}\end{bmatrix}=?$$

and then we solve

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}u_{12}\\ u_{22}\\ \vdots\\ \vdots\\ u_{m-1,2}\end{bmatrix}=\begin{bmatrix}u_{11}\\ u_{21}\\ \vdots\\ \vdots\\ u_{m-1,1}\end{bmatrix}\Rightarrow\begin{bmatrix}u_{12}\\ u_{22}\\ \vdots\\ \vdots\\ u_{m-1,2}\end{bmatrix}=?$$

and then we solve

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}u_{13}\\ u_{23}\\ \vdots\\ \vdots\\ u_{m-1,3}\end{bmatrix}=\begin{bmatrix}u_{12}\\ u_{22}\\ \vdots\\ \vdots\\ u_{m-1,2}\end{bmatrix}\Rightarrow\begin{bmatrix}u_{13}\\ u_{23}\\ \vdots\\ \vdots\\ u_{m-1,3}\end{bmatrix}=?$$

and then we solve

$$\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}\begin{bmatrix}u_{14}\\ u_{24}\\ \vdots\\ \vdots\\ u_{m-1,4}\end{bmatrix}=\begin{bmatrix}u_{13}\\ u_{23}\\ \vdots\\ \vdots\\ u_{m-1,3}\end{bmatrix}\Rightarrow\begin{bmatrix}u_{14}\\ u_{24}\\ \vdots\\ \vdots\\ u_{m-1,4}\end{bmatrix}=?$$

, ….

Eg. Solve $\frac{\partial u(x,t)}{\partial t}-\frac{\partial^2 u(x,t)}{\partial x^2}=0$ **,** $0 \leqq x \leqq 1$**,** $t>0$ **with** $u(0,t)=u(1,t)=0$**, and** $u(x,0)=\sin(\pi x)$**.**

(Sol.) Use the Backward-difference method, and choose $m=10$, $h=0.1$, $k=0.01 \Rightarrow \lambda=1$, $1+2\lambda=3$, $x_i=0.1i$, $t_j=0.01j$, and then we have

$$\begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{91} \end{bmatrix} = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ \vdots \\ u_{90} \end{bmatrix} = \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_2) \\ \vdots \\ \vdots \\ \sin(\pi x_9) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{91} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} \sin(\pi x_1) \\ \sin(\pi x_2) \\ \vdots \\ \vdots \\ \sin(\pi x_9) \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{92} \end{bmatrix} = \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{91} \end{bmatrix} \Rightarrow \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{92} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{91} \end{bmatrix},$$

$$\begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{93} \end{bmatrix} = \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{92} \end{bmatrix} \Rightarrow \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{93} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{92} \end{bmatrix},$$

$$\begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_{14} \\ u_{24} \\ \vdots \\ \vdots \\ u_{94} \end{bmatrix} = \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{93} \end{bmatrix} \Rightarrow \begin{bmatrix} u_{14} \\ u_{24} \\ \vdots \\ \vdots \\ u_{94} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{93} \end{bmatrix},$$

$\cdots$

In **Matlab** language, we can use the following program to solve the partial differential equation:

```
>> A=[3,-1,0,0,0,0,0,0,0;-1,3,-1,0,0,0,0,0,0;0,-1,3,-1,0,0,0,0,0;0,0,-1,3,-1,0,0,0,0;
      0,0,0,-1,3,-1,0,0,0;0,0,0,0,-1,3,-1,0,0;0,0,0,0,0,-1,3,-1,0;0,0,0,0,0,0,-1,3,-1;
      0,0,0,0,0,0,0,-1,3];
>>pi=3.1415926;
>>B=[sin(0.1*pi);sin(0.2*pi);sin(0.3*pi);sin(0.4*pi);sin(0.5*pi);sin(0.6*pi);sin(0.7*pi
);sin(0.8*pi);sin(0.9*pi)];
>>for i=1:5
    C=inv(A)*B
    B=C;
  end
C =
         0.1765
         0.3356
         0.4620
         0.5431
         0.5710
         0.5431
         0.4620
         0.3356
         0.1765

C =
         0.1607
         0.3057
         0.4208
         0.4947
         0.5201
         0.4947
         0.4208
         0.3057
         0.1607

C =
         0.1464
         0.2785
         0.3833
         0.4506
         0.4737
         0.4506
         0.3833
```

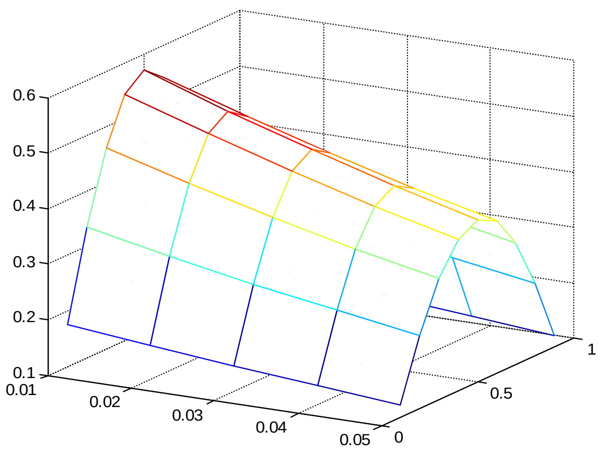
0.2785
0.1464

C =

0.1333
0.2536
0.3491
0.4104
0.4315
0.4104
0.3491
0.2536
0.1333

C =

0.1215
0.2310
0.3180
0.3738
0.3930
0.3738
0.3180
0.2310
0.1215



Richardson's method: $\dfrac{u_{i,j+1}-u_{i,j-1}}{2k}-\alpha^2\dfrac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2}=0$.

Note: Forward difference method is conditional stable. Richardson's method has a serious stability problem.

Crank-Nicholson method: $Au^{(j+1)}=Bu^{(j)}$

$$\begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix} \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ \vdots \\ u_{m-1,j} \end{bmatrix}$$

, where

$u_{i0}=f(x_i)$.

We solve

$$\begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix} = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix},$$

and then we solve

$$\begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix} = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix},$$

and then we solve

$$\begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix} = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix},$$

and then we solve

$$\begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & (1+\lambda) & -\frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix} \begin{bmatrix} u_{14} \\ u_{24} \\ \vdots \\ \vdots \\ u_{m-1,4} \end{bmatrix} = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & (1-\lambda) & \frac{\lambda}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix},$$

…

8-3 Hyperbolic Partial Differential Equations

Solve $\dfrac{\partial^2 u(x,t)}{\partial t^2} - \alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = 0$, $0 \leqq x \leqq l$, $t>0$ with $u(0,t)=u(l,t)=0$, $u(x,0)=f(x)$, $\dfrac{\partial u}{\partial t}(x,0)=g(x)$.

Central difference method: **Set $h=l/m$, $x_i=ih$, $t_j=jk$, and $\lambda=\alpha k/h$**

$$\Rightarrow \frac{u_{i,j+1}-2u_{ij}+u_{i,j-1}}{k^2} - \alpha^2 \cdot \frac{u_{i+1,j}-2u_{ij}+u_{i-1,j}}{h^2} = 0 \Rightarrow u_{i,j+1} = 2(1-\lambda^2)u_{ij} + \lambda^2(u_{i+1,j}+u_{i-1,j}) - u_{i,j-1},$$

$u_{i0}=f(x_i)$, $u_{0j}=u_{mj}=0$, and $u_{i1}=u_{i0}+kg(x_i)$ $\approx (1-\lambda^2)u_{i0} + \dfrac{\lambda^2}{2}[f(x_{i+1})+f(x_{i-1})] + kg(x_i)$

$$\Rightarrow \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ \vdots \\ u_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ \vdots \\ u_{m-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ \vdots \\ u_{m-1,j-1} \end{bmatrix}$$

We have

$$\begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix} - \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix},$$

$$\begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix} - \begin{bmatrix} u_{11} \\ u_{21} \\ \vdots \\ \vdots \\ u_{m-1,1} \end{bmatrix},$$

$$\begin{bmatrix} u_{14} \\ u_{24} \\ \vdots \\ \vdots \\ u_{m-1,4} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} u_{13} \\ u_{23} \\ \vdots \\ \vdots \\ u_{m-1,3} \end{bmatrix} - \begin{bmatrix} u_{12} \\ u_{22} \\ \vdots \\ \vdots \\ u_{m-1,2} \end{bmatrix}, \cdots$$

Eg. Solve $\frac{\partial^2 u(x,t)}{\partial t^2}-4\frac{\partial^2 u(x,t)}{\partial x^2}=0$ **,** **$0 \leqq x \leqq 1$, $t>0$ with $u(0,t)=u(1,t)=0$, $u(x,0)=\sin(\pi x)$, and** $\frac{\partial u(x,0)}{\partial t}=0$ **.**

(Sol.) Set $m=10$, $h=0.1$, $k=0.05 \Rightarrow \lambda=1$, $\lambda^2=1$, $2(1-\lambda^2)=0$, $x_i=0.1i$, $t_j=0.05j$

$\Rightarrow u_{i0}=f(x_i)=\sin(\pi x_i)$, $u_{0j}=u_{mj}=0$,

$$u_{i1}\approx\frac{1}{2}[f(x_{i+1})+f(x_{i-1})]+0.05g(x_i)=\frac{1}{2}[\sin(\pi x_{i+1})+\sin(\pi x_{i-1})],$$

and then we have $\begin{bmatrix}u_{10}\\u_{20}\\\vdots\\\vdots\\u_{90}\end{bmatrix}=\begin{bmatrix}\sin(\pi x_1)\\\sin(\pi x_2)\\\vdots\\\vdots\\\sin(\pi x_9)\end{bmatrix}$, $\begin{bmatrix}u_{11}\\u_{21}\\\vdots\\\vdots\\u_{91}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}[\sin(\pi x_2)+\sin(\pi x_0)]\\\frac{1}{2}[\sin(\pi x_3)+\sin(\pi x_1)]\\\vdots\\\vdots\\\frac{1}{2}[\sin(\pi x_{10})+\sin(\pi x_8)]\end{bmatrix}$,

$$\begin{bmatrix}u_{12}\\u_{22}\\\vdots\\\vdots\\u_{92}\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\1&0&1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&0\\\vdots&\ddots&\ddots&\ddots&1\\0&\cdots&0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{2}[\sin(\pi x_2)+\sin(\pi x_0)]\\\frac{1}{2}[\sin(\pi x_3)+\sin(\pi x_1)]\\\vdots\\\vdots\\\frac{1}{2}[\sin(\pi x_{10})+\sin(\pi x_8)]\end{bmatrix}-\begin{bmatrix}\sin(\pi x_1)\\\sin(\pi x_2)\\\vdots\\\vdots\\\sin(\pi x_9)\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}u_{13}\\u_{23}\\\vdots\\\vdots\\u_{93}\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\1&0&1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&0\\\vdots&\ddots&\ddots&\ddots&1\\0&\cdots&0&1&0\end{bmatrix}\begin{bmatrix}u_{12}\\u_{22}\\\vdots\\\vdots\\u_{92}\end{bmatrix}-\begin{bmatrix}\frac{1}{2}[\sin(\pi x_2)+\sin(\pi x_0)]\\\frac{1}{2}[\sin(\pi x_3)+\sin(\pi x_1)]\\\vdots\\\vdots\\\frac{1}{2}[\sin(\pi x_{10})+\sin(\pi x_8)]\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}u_{14}\\u_{24}\\\vdots\\\vdots\\u_{94}\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\1&0&1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&0\\\vdots&\ddots&\ddots&\ddots&1\\0&\cdots&0&1&0\end{bmatrix}\begin{bmatrix}u_{13}\\u_{23}\\\vdots\\\vdots\\u_{93}\end{bmatrix}-\begin{bmatrix}u_{12}\\u_{22}\\\vdots\\\vdots\\u_{92}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}u_{15}\\u_{25}\\\vdots\\\vdots\\u_{95}\end{bmatrix}=\begin{bmatrix}0&1&0&\cdots&0\\1&0&1&\ddots&\vdots\\0&\ddots&\ddots&\ddots&0\\\vdots&\ddots&\ddots&\ddots&1\\0&\cdots&0&1&0\end{bmatrix}\begin{bmatrix}u_{14}\\u_{24}\\\vdots\\\vdots\\u_{94}\end{bmatrix}-\begin{bmatrix}u_{13}\\u_{23}\\\vdots\\\vdots\\u_{93}\end{bmatrix}\Rightarrow\ldots$$

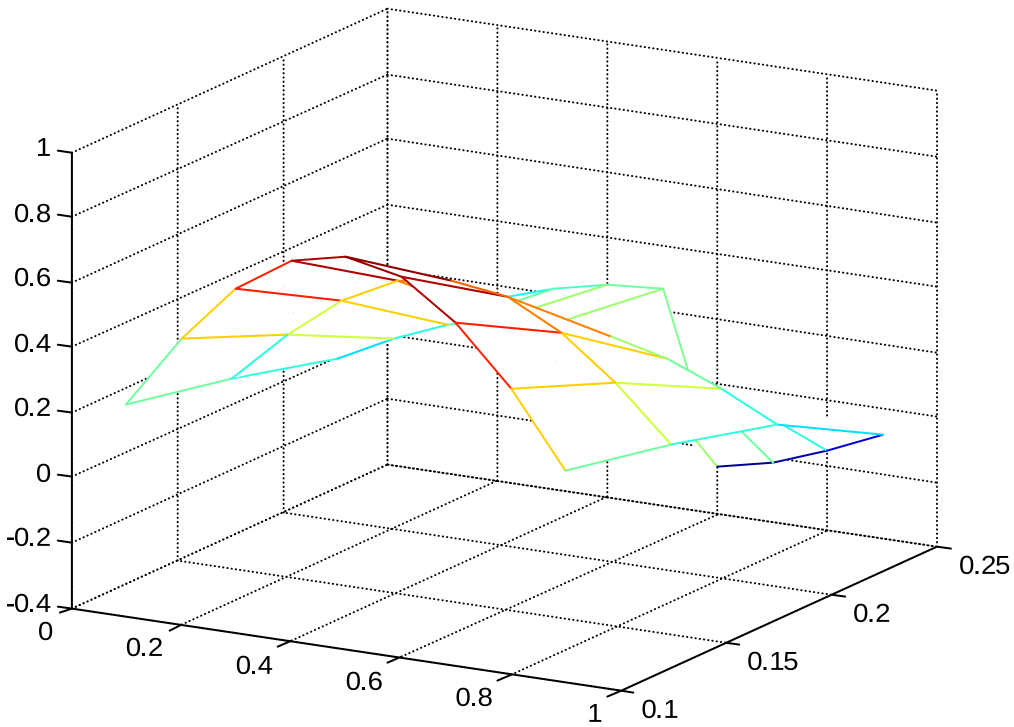
In **Matlab** language, we can use the following program to solve the partial differential equation:

```
>>A=[0,1,0,0,0,0,0,0,0;1,0,1,0,0,0,0,0,0;0,1,0,1,0,0,0,0,0;0,0,1,0,1,0,0,0,0;
    0,0,0,1,0,1,0,0,0;0,0,0,0,1,0,1,0,0;0,0,0,0,0,1,0,1,0;0,0,0,0,0,0,1,0,1;
    0,0,0,0,0,0,0,1,0];
>>pi=3.1415926;
>>B=[sin(0.1*pi);sin(0.2*pi);sin(0.3*pi);sin(0.4*pi);sin(0.5*pi);sin(0.6*pi);sin(0.7*pi
);sin(0.8*pi);sin(0.9*pi)];
>>C=[(0+sin(0.2*pi))/2;(sin(0.1*pi)+sin(0.3*pi))/2;(sin(0.2*pi)+sin(0.4*pi))/2;(sin(0.
3*pi)+sin(0.5*pi))/2;(sin(0.4*pi)+sin(0.6*pi))/2;(sin(0.5*pi)+sin(0.7*pi))/2;(sin(0.6*p
i)+sin(0.8*pi))/2;(sin(0.7*pi)+sin(0.9*pi))/2;(sin(0.8*pi)+sin(pi))/2];
>>for i=2:5
    D=A*C-B
    B=C;
    C=D;
  end
D =

    0.2500
    0.4755
    0.6545
    0.7694
    0.8090
    0.7694
    0.6545
    0.4755
    0.2500

D =

    0.1816
    0.3455
    0.4755
    0.5590
    0.5878
    0.5590
    0.4755
    0.3455
    0.1816

D =

    0.0955
    0.1816
    0.2500
```

0.2939
0.3090
0.2939
0.2500
0.1816
0.0955

D =
1.0e-007 *

0.0828
0.1575
0.2168
0.2548
0.2679
-0.2548
-0.2168
-0.1575
-0.0828



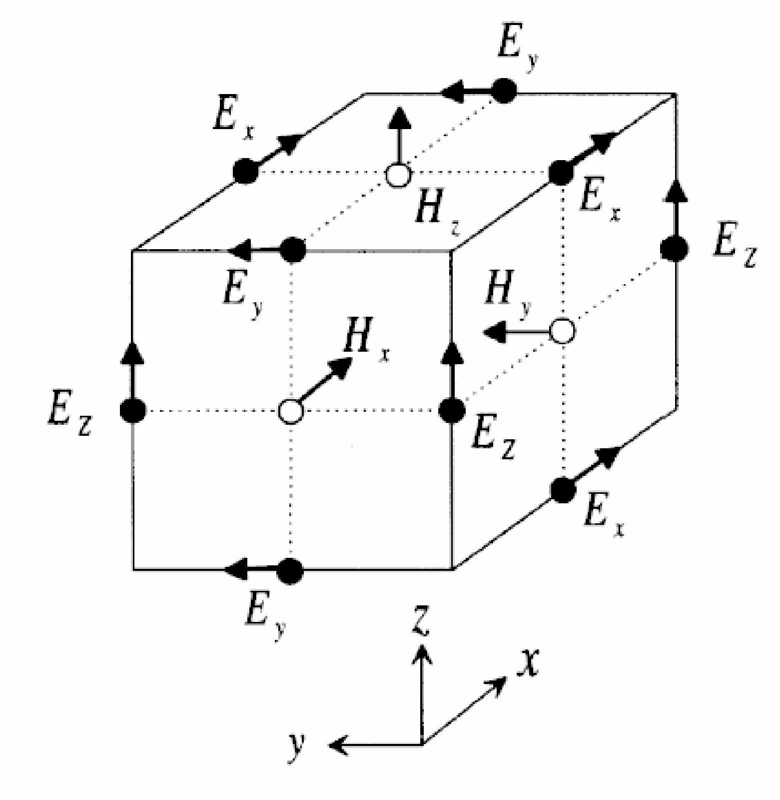
8-4 Finite-Difference Time-Domain (FDTD) Method of Solving Maxwell's Equations

3D Full-vector FDTD method (by Kane S. Yee, 1966):

Maxwell's equations in the source-free region: $\nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}$, $\nabla\times\vec{E}=\mu\frac{\partial\vec{H}}{\partial t}$

$$\Rightarrow \varepsilon\frac{\partial E_x}{\partial t}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z},\quad \varepsilon\frac{\partial E_y}{\partial t}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x},\quad \varepsilon\frac{\partial E_z}{\partial t}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y},$$

$$\mu\frac{\partial H_x}{\partial t}=\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y},\quad \mu\frac{\partial H_y}{\partial t}=\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z},\quad \mu\frac{\partial H_z}{\partial t}=\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}$$



(Yee's cell)

Define the following notations: $E(i\triangle x,j\triangle y,k\triangle z,l\triangle t)\equiv E(i,j,k,l)\equiv E^l(i,j,k)$ and $H(i\triangle x,j\triangle y,k\triangle z,l\triangle t)\equiv H(i,j,k,l)\equiv H^l(i,j,k)$. According to Yee's cell, we have

$E_x\equiv E_x(i+1/2,j,k,l)\equiv E_x^{\,l}(i+1/2,j,k)$,

$E_y\equiv E_y(i,j+1/2,k,l)\equiv E_y^{\,l}(i,j+1/2,k)$,

$E_z\equiv E_z(i,j,k+1/2,l)\equiv E_z^{\,l}(i,j,k+1/2)$,

$H_x\equiv H_x(i,j+1/2,k+1/2,l\text{-}1/2)\equiv H_x^{\,l\text{-}1/2}(i,j+1/2,k+1/2)$,

$H_y\equiv H_y(i+1/2,j,k+1/2,l\text{-}1/2)\equiv H_y^{\,l\text{-}1/2}(i+1/2,j,k+1/2)$,

$H_z\equiv H_z(i+1/2,j+1/2,k,l\text{-}1/2)\equiv H_z^{\,l\text{-}1/2}(i+1/2,j+1/2,k)$,

$$\frac{\partial E_x}{\partial t} \approx \frac{E_x(i+\frac{1}{2},j,k,l+1)-E_x(i+\frac{1}{2},j,k,l)}{\Delta t},$$

$$\frac{\partial E_y}{\partial t} \approx \frac{E_y(i,j+\frac{1}{2},k,l+1)-E_y(i,j+\frac{1}{2},k,l)}{\Delta t},$$

$$\frac{\partial E_z}{\partial t} \approx \frac{E_z(i,j,k+\frac{1}{2},l+1)-E_z(i,j,k+\frac{1}{2},l)}{\Delta t},$$

$$\frac{\partial E_x}{\partial y} \approx \frac{E_x(i+\frac{1}{2},j,k,l)-E_x(i+\frac{1}{2},j-1,k,l)}{\Delta y},$$

$$\frac{\partial E_x}{\partial z} \approx \frac{E_x(i+\frac{1}{2},j,k,l)-E_x(i+\frac{1}{2},j,k-1,l)}{\Delta z},$$

$$\frac{\partial E_y}{\partial x} \approx \frac{E_y(i,j+\frac{1}{2},k,l)-E_y(i-1,j+\frac{1}{2},k,l)}{\Delta x},$$

$$\frac{\partial E_y}{\partial z} \approx \frac{E_y(i,j+\frac{1}{2},k,l)-E_y(i,j+\frac{1}{2},k-1,l)}{\Delta z},$$

$$\frac{\partial E_z}{\partial x} \approx \frac{E_z(i,j,k+\frac{1}{2},l)-E_z(i-1,j,k+\frac{1}{2},l)}{\Delta x},$$

$$\frac{\partial E_z}{\partial y} \approx \frac{E_z(i,j,k+\frac{1}{2},l)-E_z(i,j-1,k+\frac{1}{2},l)}{\Delta y},$$

But

$$\frac{\partial H_x}{\partial t} \approx \frac{H_x(i,j+\frac{1}{2},k+\frac{1}{2},l+\frac{1}{2})-H_x(i,j+\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2})}{\Delta t},$$

$$\frac{\partial H_y}{\partial t} \approx \frac{H_y(i+\frac{1}{2},j,k+\frac{1}{2},l+\frac{1}{2})-H_y(i+\frac{1}{2},j,k+\frac{1}{2},l-\frac{1}{2})}{\Delta t},$$

$$\frac{\partial H_z}{\partial t} \approx \frac{H_z(i+\frac{1}{2},j+\frac{1}{2},k,l+\frac{1}{2})-H_z(i+\frac{1}{2},j+\frac{1}{2},k,l-\frac{1}{2})}{\Delta t},$$

$$\frac{\partial H_x}{\partial y} \approx \frac{H_x(i,j+\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2})-H_x(i,j-\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2})}{\Delta y},$$

$$\frac{\partial H_x}{\partial z} \approx \frac{H_x(i,j+\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2})-H_x(i,j+\frac{1}{2},k-\frac{1}{2},l-\frac{1}{2})}{\Delta z},$$

$$\frac{\partial H_y}{\partial x} \approx \frac{H_y(i+\frac{1}{2}, j, k+\frac{1}{2}, l-\frac{1}{2}) - H_y(i-\frac{1}{2}, j, k+\frac{1}{2}, l-\frac{1}{2})}{\Delta x},$$

$$\frac{\partial H_y}{\partial z} \approx \frac{H_y(i+\frac{1}{2}, j, k+\frac{1}{2}, l-\frac{1}{2}) - H_y(i+\frac{1}{2}, j, k-\frac{1}{2}, l-\frac{1}{2})}{\Delta z},$$

$$\frac{\partial H_z}{\partial x} \approx \frac{H_z(i+\frac{1}{2}, j+\frac{1}{2}, k, l-\frac{1}{2}) - H_z(i-\frac{1}{2}, j+\frac{1}{2}, k, l-\frac{1}{2})}{\Delta x},$$

$$\frac{\partial H_z}{\partial y} \approx \frac{H_z(i+\frac{1}{2}, j+\frac{1}{2}, k, l-\frac{1}{2}) - H_z(i+\frac{1}{2}, j-\frac{1}{2}, k, l-\frac{1}{2})}{\Delta y}$$

$\Rightarrow$

$$E_x^{l+1}(i+\frac{1}{2}, j, k) = E_x^{l}(i+\frac{1}{2}, j, k) + C(i+\frac{1}{2}, j, k)\cdot$$

$$\left[\frac{H_z^{l-\frac{1}{2}}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_z^{l-\frac{1}{2}}(i+\frac{1}{2}, j-\frac{1}{2}, k)}{\Delta y} - \frac{H_y^{l-\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_y^{l-\frac{1}{2}}(i+\frac{1}{2}, j, k-\frac{1}{2})}{\Delta z}\right],$$

$$E_y^{l+1}(i, j+\frac{1}{2}, k) = E_y^{l}(i, j+\frac{1}{2}, k) + C(i, j+\frac{1}{2}, k)\cdot$$

$$\left[\frac{H_x^{l-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) - H_x^{l+\frac{1}{2}}(i, j+\frac{1}{2}, k-\frac{1}{2})}{\Delta z} - \frac{H_z^{l-\frac{1}{2}}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_z^{l-\frac{1}{2}}(i-\frac{1}{2}, j+\frac{1}{2}, k)}{\Delta x}\right],$$

$$E_z^{l+1}(i, j, k+\frac{1}{2}) = E_z^{l}(i, j, k+\frac{1}{2}) + C(i, j, k+\frac{1}{2})\cdot$$

$$\left[\frac{H_y^{l-\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_y^{l-\frac{1}{2}}(i-\frac{1}{2}, j, k+\frac{1}{2})}{\Delta x} - \frac{H_x^{l-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) - H_x^{l-\frac{1}{2}}(i, j-\frac{1}{2}, k+\frac{1}{2})}{\Delta y}\right],$$

$$H_x^{l+\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) = H_x^{l-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) + D(i, j+\frac{1}{2}, k+\frac{1}{2})\cdot$$

$$\left[-\frac{E_z^{l}(i, j+1, k+\frac{1}{2}) - E_z^{l}(i, j, k+\frac{1}{2})}{\Delta y} + \frac{E_y^{l}(i, j+\frac{1}{2}, k+1) - E_y^{l}(i, j+\frac{1}{2}, k)}{\Delta z}\right],$$

$$H_y^{l+\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) = H_y^{l-\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) + D(i+\frac{1}{2}, j, k+\frac{1}{2})\cdot$$

$$\left[\frac{E_z^{l}(i+1, j, k+\frac{1}{2}) - E_z^{l}(i, j, k+\frac{1}{2})}{\Delta x} - \frac{E_x^{l}(i+\frac{1}{2}, j, k+1) - E_x^{l}(i+\frac{1}{2}, j, k)}{\Delta z}\right],$$

$$H_z^{l+\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k)=H_z^{l-\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k)+D(i+\frac{1}{2},j+\frac{1}{2},k)\cdot$$

$$\left[\frac{E_x^l(i+\frac{1}{2},j+1,k)-E_x^l(i+\frac{1}{2},j,k)}{\Delta y}-\frac{E_y^l(i+1,j+\frac{1}{2},k)-E_y^l(i,j+\frac{1}{2},k)}{\Delta x}\right],$$

where $C(i,j,k)=\dfrac{\Delta t}{\varepsilon(i,j,k)}$, $D(i,j,k)=\dfrac{\Delta t}{\mu(i,j,k)}$, and the electromagnetic power distribution $\propto$ $\left|\vec{E}\right|^2=(E_x{}^2+E_y{}^2+E_z{}^2)$.

Stability criterion: $\Delta t\le\dfrac{1}{v_{\max}\sqrt{\dfrac{1}{\Delta x^2}+\dfrac{1}{\Delta y^2}+\dfrac{1}{\Delta z^2}}}$, where $v_{\max}$ is the maximal velocity of EM wave propagating.

3D Scalar FDTD method (by Dr. Wei-Ping Huang, 1991):

Dr. Wei-Ping Huang was born in Beijing. He graduated from MIT and got his PhD degree. Now he is a professor in the ECE Department at the University of Waterloo, Canada. His research group is one of the leading groups internationally in photonics and computer aided design. Dr. Huang is creating new design tools to help develop the highly-efficient photonics systems required for large-scale broadband communication.

Consider a wave equation: $\nabla^2\Psi + \frac{n^2}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{n^2}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$,

where $\Psi(x,y,z,t)$ is the scalar field (approximation) and $n(x,y,z)$ is the refractive index. Using $\Psi(i\triangle x, j\triangle y, k\triangle z, l\triangle t) \equiv \Psi(i,j,k,l) \equiv \Psi^l(i,j,k)$, where $0 \leqq i \leqq I$, $0 \leqq j \leqq J$, and $0 \leqq k \leqq K$.

Let $\frac{\partial^2\Psi}{\partial x^2} = \frac{\Psi^l(i+1,j,k) - 2\Psi^l(i,j,k) + \Psi^l(i-1,j,k)}{\Delta x^2}$,

$\frac{\partial^2\Psi}{\partial y^2} = \frac{\Psi^l(i,j+1,k) - 2\Psi^l(i,j,k) + \Psi^l(i,j-1,k)}{\Delta y^2}$,

$\frac{\partial^2\Psi}{\partial z^2} = \frac{\Psi^l(i,j,k+1) - 2\Psi^l(i,j,k) + \Psi^l(i,j,k-1)}{\Delta z^2}$,

and $\frac{\partial^2\Psi}{\partial t^2} = \frac{\Psi^{l+1}(i,j,k) - 2\Psi^l(i,j,k)^l + \Psi^{l-1}(i,j,k)}{\Delta t^2}$

$$\Rightarrow \Psi^{l+1}(i,j,k) = 2\cdot[1 - \frac{\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2}{n^2(i,j,k)}]\cdot\Psi^l(i,j,k) - \Psi^{l-1}(i,j,k)$$

$$+\frac{\delta_x{}^2}{n^2(i,j,k)}\cdot[\Psi^l(i+1,j,k) + \Psi^l(i-1,j,k)] + \frac{\delta_y{}^2}{n^2(i,j,k)}\cdot[\Psi^l(i,j+1,k) + \Psi^l(i,j-1,k)]$$

$$+\frac{\delta_z{}^2}{n^2(i,j,k)}\cdot[\Psi^l(i,j,k+1) + \Psi^l(i,j,k-1)]$$

where $\delta_x = \frac{c\Delta t}{\Delta x}$, $\delta_y = \frac{c\Delta t}{\Delta y}$, $\delta_z = \frac{c\Delta t}{\Delta z}$.

The electromagnetic power distribution $\propto$ $|\Psi|^2$.

The 3D scalar FDTD method may be incorporated with the Mur's absorbing boundary conditions.

Mur's First-Order Absorbing Boundary Conditions:

At x=0, $\Psi^{l+1}(0,j,k) = \Psi^l(1,j,k) + \frac{c\Delta t - \Delta x}{c\Delta t + \Delta x}\cdot[\Psi^{l+1}(1,j,k) - \Psi^l(0,j,k)]$

$x=I\triangle x$, $\Psi^{l+1}(I,j,k)=\Psi^{l}(I-1,j,k)+\frac{c\Delta t-\Delta x}{c\Delta t+\Delta x}\cdot[\Psi^{l+1}(I-1,j,k)-\Psi^{l}(I,j,k)]$

$y=0$, $\Psi^{l+1}(i,0,k)=\Psi^{l}(i,1,k)+\frac{c\Delta t-\Delta y}{c\Delta t+\Delta y}\cdot[\Psi^{l+1}(i,1,k)-\Psi^{l}(i,0,k)]$

$y=J\triangle y$, $\Psi^{l+1}(i,J,k)=\Psi^{l}(i,J-1,k)+\frac{c\Delta t-\Delta y}{c\Delta t+\Delta y}\cdot[\Psi^{l+1}(i,J-1,k)-\Psi^{l}(i,J,k)]$

$z=0$, $\Psi^{l+1}(i,j,0)=\Psi^{l}(i,j,0)+\frac{c\Delta t-\Delta z}{c\Delta t+\Delta z}\cdot[\Psi^{l+1}(i,j,1)-\Psi^{l}(i,j,0)]$

$z=K\triangle z$, $\Psi^{l+1}(i,j,K)=\Psi^{l}(i,j,K-1)+\frac{c\Delta t-\Delta z}{c\Delta t+\Delta z}\cdot[\Psi^{l+1}(i,j,K-1)-\Psi^{l}(i,j,K)]$

3D Semi-vectorial FDTD method (also by Dr. Wei-Ping Huang):

$$E_x^{l+1}(i,j,k)=2\cdot\{1-\frac{[2-\frac{T(i+1,j,k)+T(i-1,j,k)}{2}]\cdot\delta_x^{\ 2}+\delta_y^{\ 2}+\delta_z^{\ 2}}{n^2(i,j,k)}\}\cdot E_x^{\ l}(i,j,k)-E_x^{\ l-1}(i,j,k)$$

$$+\frac{\delta_x^{\ 2}}{n^2(i,j,k)}\cdot[T(i+1,j,k)\cdot E_x^{\ l}(i+1,j,k)+T(i-1,j,k)\cdot E_x^{\ l}(i-1,j,k)]$$

$$+\frac{\delta_y^{\ 2}}{n^2(i,j,k)}\cdot[E_x^{\ l}(i,j+1,k)+E_x^{\ l}(i,j-1,k)]+\frac{\delta_z^{\ 2}}{n^2(i,j,k)}\cdot[E_x^{\ l}(i,j,k+1)+E_x^{\ l}(i,j,k-1)]$$

$$E_y^{l+1}(i,j,k)=2\cdot\{1-\frac{\delta_x^{\ 2}+[2-\frac{T(i,j+1,k)+T(i,j-1,k)}{2}]\cdot\delta_y^{\ 2}+\delta_z^{\ 2}}{n^2(i,j,k)}\}\cdot E_y^{\ l}(i,j,k)-E_y^{\ l-1}(i,j,k)$$

$$+\frac{\delta_x^{\ 2}}{n^2(i,j,k)}\cdot[E_y^{\ l}(i+1,j,k)+E_y^{\ l}(i-1,j,k)]+\frac{\delta_z^{\ 2}}{n^2(i,j,k)}\cdot[E_y^{\ l}(i,j,k+1)+E_y^{\ l}(i,j,k-1)]$$

$$+\frac{\delta_y^{\ 2}}{n^2(i,j,k)}\cdot[T(i,j+1,k)\cdot E_y^{\ l}(i,j+1,k)+T(i,j-1,k)\cdot E_y^{\ l}(i,j-1,k)]$$

where $T(i\pm1,j,k)=\frac{2n^2(i\pm1,j,k)}{n^2(i\pm1,j,k)+n^2(i,j,k)}$ and $T(i,j\pm1,k)=\frac{2n^2(i,j\pm1,k)}{n^2(i,j\pm1,k)+n^2(i,j,k)}$.

The 3D Semi-vectorial FDTD method may be incorporated with the perfectly matched layer (PML) boundary conditions.

Perfectly matched layer (PML) boundary conditions:

In the x-direction:

$$\Psi^{l+1}(i,j,k) = -\Psi^{l-1}(i,j,k) + 2\left[1 - \frac{\delta_y^{\ 2} + \delta_z^{\ 2}}{n^2(i,j,k)}\right]\Psi^{l}(i,j,k)$$

$$+\frac{\delta_x^{\ 2}}{n^2(i,j,k)}\left[D_{2x}^{\ \ l+\frac{1}{2}}(i,j,k) - D_{2x}^{\ \ l-\frac{1}{2}}(i,j,k)\right] \quad + \frac{\delta_y^{\ 2}}{n^2(i,j,k)}\left[\Psi^{l}(i,j+1,k) + \Psi^{l}(i,j-1,k)\right]$$

$$+\frac{\delta_z^{\ 2}}{n^2(i,j,k)}\left[\Psi^{l}(i,j,k+1) + \Psi^{l}(i,j,k-1)\right],$$ where $\delta_x = \frac{c\Delta t}{\Delta x}$, $\delta_y = \frac{c\Delta t}{\Delta y}$, $\delta_z = \frac{c\Delta t}{\Delta z}$

$$D_{1x}^{\ \ l+1}(i+1,j,k) = \frac{a_x - \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}}{a_x + \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}} \cdot D_{1x}^{\ \ l-1}(i+1,j,k)$$

$$+\frac{1}{a_x + \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}} \cdot \left[\Psi^{l}(i+1,j,k) - \Psi^{l}(i,j,k)\right]$$

$$D_{2x}^{\ \ l+1}(i,j,k) = \frac{a_x - \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}}{a_x + \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}} \cdot D_{2x}^{\ \ l-1}(i,j,k)$$

$$+\frac{1}{a_x + \dfrac{\sigma_x \cdot \Delta t}{\varepsilon_0 n^2(i+1,j,k)}} \cdot \Big[D_{1x}^{\ \ l+1}(i+1,j,k) - D_{1x}^{\ \ l+1}(i-1,j,k)$$

$$- D_{1x}^{\ \ l-1}(i+1,j,k) + D_{1x}^{\ \ l-1}(i-1,j,k)\Big],$$

$$\sigma_x(x) = \left(\frac{x}{d}\right)^m \sigma_{x\max} = -\left(\frac{x}{d}\right)^m \frac{n_x^{\ 2}\varepsilon_0 c(m+1)\ln(R_0)}{2d}$$

$d = n_x \Delta x$: thickness of *PML*, m: order, R_0 : reflection from *PML* at normal incidence

$n_x = 8$, $R_0 = 10^{-2}$, m=4

In the y-direction:

$$\Psi^{l+1}(i,j,k) = -\Psi^{l-1}(i,j,k) + 2\left[1 - \frac{\delta_x{}^2 + \delta_z{}^2}{n^2(i,j,k)}\right]\Psi^{l}(i,j,k)$$

$$+ \frac{\delta_x{}^2}{n^2(i,j,k)}\left[\Psi^{l}(i+1,j,k) + \Psi^{l}(i-1,j,k)\right]$$

$$+ \frac{\delta_y{}^2}{n^2(i,j,k)}\left[D_{2y}{}^{l+\frac{1}{2}}(i,j,k) - D_{2y}{}^{l-\frac{1}{2}}(i,j,k)\right] + \frac{\delta_z{}^2}{n^2(i,j,k)}\left[\Psi^{l}(i,j,k+1) + \Psi^{l}(i,j,k-1)\right],$$

where $\delta_x = \dfrac{c\Delta t}{\Delta x}$, $\delta_y = \dfrac{c\Delta t}{\Delta y}$, $\delta_z = \dfrac{c\Delta t}{\Delta z}$

$$D_{1y}{}^{l+1}(i,j+1,k) = \frac{a_y - \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}}{a_y + \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}} \cdot D_{1y}{}^{l-1}(i,j+1,k)$$

$$+ \frac{1}{a_y + \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}} \cdot \left[\Psi^{l}(i,j+1,k) - \Psi^{l}(i,j,k)\right]$$

$$D_{2y}{}^{l+1}(i,j,k) = \frac{a_y - \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}}{a_y + \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}} \cdot D_{2y}{}^{l-1}(i,j,k)$$

$$+ \frac{1}{a_y + \dfrac{\sigma_y \cdot \Delta t}{\varepsilon_0 n^2(i,j+1,k)}} \cdot \Big[D_{1y}{}^{l+1}(i,j+1,k) - D_{1y}{}^{l+1}(i,j-1,k)$$

$$- D_{1y}{}^{l-1}(i,j+1,k) + D_{1y}{}^{l-1}(i,j-1,k)\Big],$$

$$\sigma_y(y) = \left(\frac{y}{d}\right)^m \sigma_{y\max} = -\left(\frac{y}{d}\right)^m \frac{n_y{}^2 \varepsilon_0 c(m+1)\ln(R_0)}{2d}$$

$d = n_y \Delta y$: thickness of *PML*, m: order

R_0 : reflection from *PML* at normal incidence

$n_y = 8$, $R_0 = 10^{-2}$, $m=4$

In the z-direction:

$$\Psi^{l+1}(i,j,k) = -\Psi^{l-1}(i,j,k) + 2\left[1 - \frac{\delta_x^{\ 2} + \delta_y^{\ 2}}{n^2(i,j,k)}\right]\Psi^l(i,j,k)$$

$$+ \frac{\delta_x^{\ 2}}{n^2(i,j,k)}\left[\Psi^l(i+1,j,k) + \Psi^l(i-1,j,k)\right]$$

$$+ \frac{\delta_y^{\ 2}}{n^2(i,j,k)}\left[\Psi^l(i,j+1,k) + \Psi^l(i,j-1,k)\right] + \frac{\delta_z^{\ 2}}{n^2(i,j,k)}\left[D_{2z}^{\ l+\frac{1}{2}}(i,j,k) - D_{2z}^{\ l-\frac{1}{2}}(i,j,k)\right],$$

where $\delta_x = \dfrac{c\Delta t}{\Delta x}$, $\delta_y = \dfrac{c\Delta t}{\Delta y}$, $\delta_z = \dfrac{c\Delta t}{\Delta z}$

$$D_{1z}^{\ l+1}(i,j,k+1) = \frac{a_z - \dfrac{\sigma_z \cdot \Delta t}{\varepsilon_0 n^2(i,j,k+1)}}{a_z + \dfrac{\sigma_z \cdot \Delta t}{\varepsilon_0 n^2(i,j,k+1)}} \cdot D_{1z}^{\ l-1}(i,j,k+1)$$

$$+ \frac{1}{a_z + \dfrac{\sigma_z \cdot \Delta t}{\varepsilon_0 n^2(i,j,k+1)}} \cdot \left[\Psi^l(i,j,k+1) - \Psi^l(i,j,k)\right]$$

$$\sigma_z(z) = \left(\frac{z}{d}\right)^m \sigma_{z\max} = -\left(\frac{z}{d}\right)^m \frac{n_z^{\ 2}\varepsilon_0 c(m+1)\ln(R_0)}{2d}$$

$$+ \frac{1}{a_z + \dfrac{\sigma_z \cdot \Delta t}{\varepsilon_0 n^2(i,j,k+1)}} \cdot \left[D_{1z}^{\ l+1}(i,j,k+1) - D_{1z}^{\ l+1}(i,j,k-1)\right.$$

$$\left. - D_{1z}^{\ l-1}(i,j,k+1) + D_{1z}^{\ l-1}(i,j,k-1)\right]$$

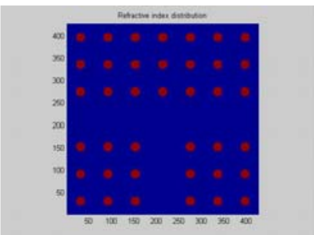
$$\sigma_z(z) = \left(\frac{z}{d}\right)^m \sigma_{z\max} = -\left(\frac{z}{d}\right)^m \frac{n_z^{\ 2}\varepsilon_0 c(m+1)\ln(R_0)}{2d}$$

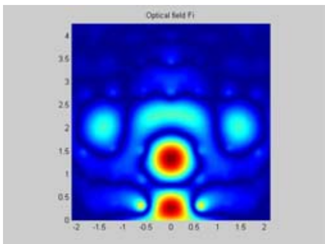
$d = n_z\Delta z$: thickness of *PML*, *m*: order

R_0 : reflection from *PML* at normal incidence

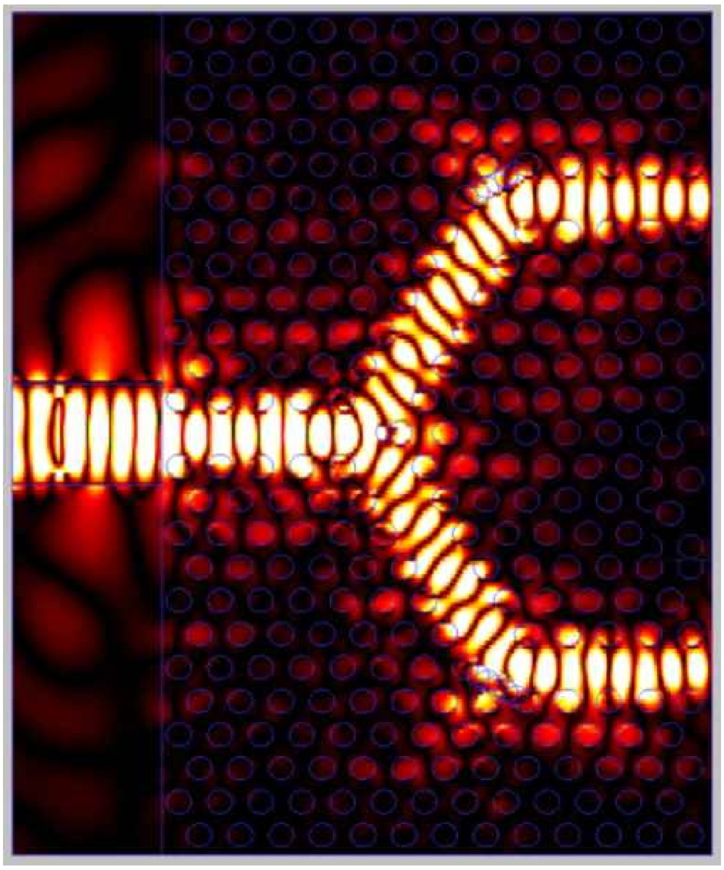
$n_z = 8$, $R_0 = 10^{-2}$, $m=4$

Eg. Use FDTD method to simulate the characteristics of photonic crystal T-junction waveguide. (by Y. -J. Lin, 林宇仁)

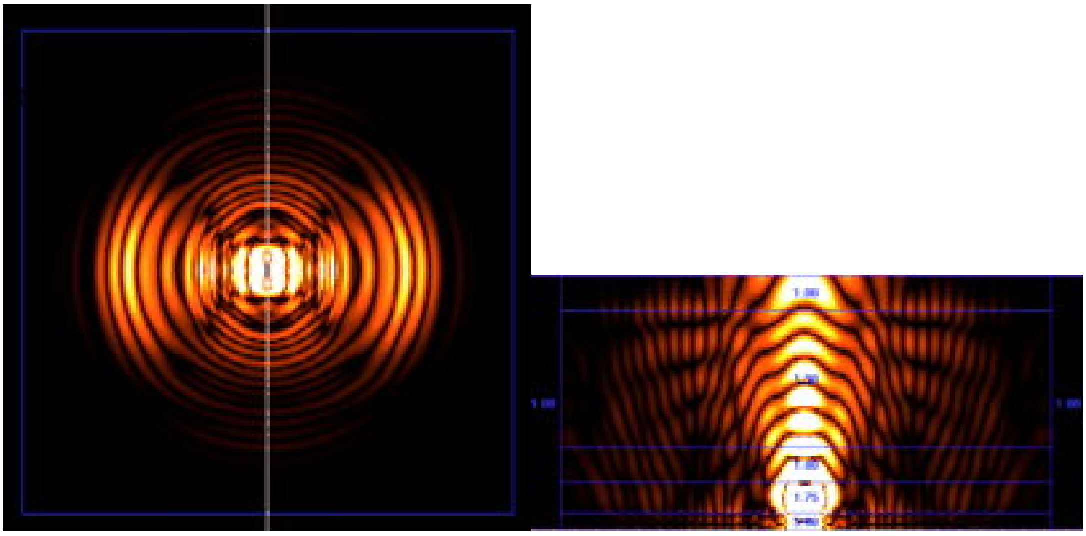




Eg. Use FDTD method to simulate the characteristics of photonic crystal Y-junction waveguide.



Eg. Use FDTD method to simulate the characteristics of LED.



8-5 Finite-difference Beam Propagation Method (FDBPM)

Consider that a light wave propagates along z-directional optical waveguide and set $\vec{E}=\hat{x}E'_x+\hat{y}E'_y+\hat{z}E'_z$ in a source-free region.

***E*-formulation (by Dr. Wei-Ping Huang):**

$$\nabla\times\nabla\times\vec{E}-n^2k^2\vec{E}=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}-n^2k^2\vec{E}=0\Rightarrow\nabla^2\vec{E}+n^2k^2\vec{E}=\nabla(\nabla\cdot\vec{E})$$

$$\Rightarrow\nabla^2(\hat{x}E'_x+\hat{y}E'_y+\hat{z}E'_z)+n^2k^2(\hat{x}E'_x+\hat{y}E'_y+\hat{z}E'_z)=\nabla(\frac{\partial E'_x}{\partial x}+\frac{\partial E'_y}{\partial y}+\frac{\partial E'_z}{\partial z})$$

$\because$ No source, $\therefore\nabla\cdot(\varepsilon\vec{E})=\nabla\cdot(n^2\vec{E})=0\Rightarrow\frac{\partial(n^2E'_x)}{\partial x}+\frac{\partial(n^2E'_y)}{\partial y}+\frac{\partial n^2}{\partial z}E'_z+n^2\frac{\partial E'_z}{\partial z}=0$

If the refractive index profile varies slowly along the z-axis, $\partial n^2/\partial z$ may be neglected. $\Rightarrow\frac{\partial E'_z}{\partial z}\approx-\frac{1}{n^2}[\frac{\partial(n^2E'_x)}{\partial x}+\frac{\partial(n^2E'_y)}{\partial y}]$

$$\nabla^2(\hat{x}E'_x+\hat{y}E'_y+\hat{z}E'_z)+n^2k^2(\hat{x}E'_x+\hat{y}E'_y+\hat{z}E'_z)$$

$$=(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z})(\frac{\partial E'_x}{\partial x}+\frac{\partial E'_y}{\partial y}-\frac{1}{n^2}[\frac{\partial(n^2E'_x)}{\partial x}+\frac{\partial(n^2E'_y)}{\partial y}])$$

$$\Rightarrow\nabla^2E'_x+n^2k^2E'_x=\frac{\partial}{\partial x}\{\frac{\partial E'_x}{\partial x}+\frac{\partial E'_y}{\partial y}-\frac{1}{n^2}[\frac{\partial(n^2E'_x)}{\partial x}+\frac{\partial(n^2E'_y)}{\partial y}]\},$$

and $\nabla^2E'_y+n^2k^2E'_y=\frac{\partial}{\partial y}\{\frac{\partial E'_x}{\partial x}+\frac{\partial E'_y}{\partial y}-\frac{1}{n^2}[\frac{\partial(n^2E'_x)}{\partial x}+\frac{\partial(n^2E'_y)}{\partial y}]\}$.

Assume that $E'_x=E_xe^{-jn_0kz}$ and $E'_y=E_ye^{-jn_0kz}$, where k is the vacuum wave number and $n_0=\beta/k$ is the effective refractive index. With the assumption that the guided lightwave has a slow-varying envelop and is paraxially propagating; i.e., $\left|\frac{\partial^2E_x}{\partial z^2}\right|<<2n_0k\left|\frac{\partial E_x}{\partial z}\right|$, $\left|\frac{\partial^2E_y}{\partial z^2}\right|<<2n_0k\left|\frac{\partial E_y}{\partial z}\right|\Rightarrow\left|\frac{\partial^2E_x}{\partial z^2}\right|$ and $\left|\frac{\partial^2E_y}{\partial z^2}\right|$ are neglected.

And then we have the following coupled equations

$$2jn_0k\cdot\frac{\partial E_x(x,y,z)}{\partial z}=\frac{\partial}{\partial x}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial x}[n^2(x,y,z)E_x(x,y,z)]\}+\frac{\partial^2E_x(x,y,z)}{\partial y^2}$$
$$+[n^2(x,y,z)-n_0^{\ 2}]k^2E_x(x,y,z)$$
$$+\frac{\partial}{\partial x}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial y}[n^2(x,y,z)E_y(x,y,z)]\}-\frac{\partial^2E_y(x,y,z)}{\partial x\partial y}$$

and

$$2jn_0k\cdot\frac{\partial E_y(x,y,z)}{\partial z}=\frac{\partial^2 E_y(x,y,z)}{\partial x^2}+\frac{\partial}{\partial y}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial y}[n^2(x,y,z)E_y(x,y,z)]\}$$
$$+[n^2(x,y,z)-n_0{}^2]k^2E_y(x,y,z)$$
$$+\frac{\partial}{\partial y}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial x}[n^2(x,y,z)E_x(x,y,z)]\}-\frac{\partial^2 E_x(x,y,z)}{\partial y\partial x}$$

These coupled equations are the basic formulas of the full-vector BPM. If we neglect the terms of A_{xy} and A_{yx}, then the equations are reduced to the formulas of the semi-vector BPM. In case the transverse variation of the refractive index is very small, and that is, $\frac{1}{n^2}\frac{\partial}{\partial x}(n^2E_x)\approx\frac{\partial E_x}{\partial x}$ and $\frac{1}{n^2}\frac{\partial}{\partial y}(n^2E_y)\approx\frac{\partial E_y}{\partial y}$.

The formulas of the semi-vector BPM can be simplified into the following Fresnel equation, which is the formula of the scalar BPM.

3D Full-vector BPM involving the Runge-Kutta algorithm (by Y. -J. Lin):

The transverse components of the optical field fulfill the following coupled equations

$$2jn_0k\cdot\frac{\partial E_x(x,y,z)}{\partial z}=\frac{\partial}{\partial x}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial x}[n^2(x,y,z)E_x(x,y,z)]\}+\frac{\partial^2 E_x(x,y,z)}{\partial y^2}$$
$$+[n^2(x,y,z)-n_0{}^2]k^2E_x(x,y,z)$$
$$+\frac{\partial}{\partial x}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial y}[n^2(x,y,z)E_y(x,y,z)]\}-\frac{\partial^2 E_y(x,y,z)}{\partial x\partial y}$$

and

$$2jn_0k\cdot\frac{\partial E_y(x,y,z)}{\partial z}=\frac{\partial^2 E_y(x,y,z)}{\partial x^2}+\frac{\partial}{\partial y}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial y}[n^2(x,y,z)E_y(x,y,z)]\}$$
$$+[n^2(x,y,z)-n_0{}^2]k^2E_y(x,y,z)$$
$$+\frac{\partial}{\partial y}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial x}[n^2(x,y,z)E_x(x,y,z)]\}-\frac{\partial^2 E_x(x,y,z)}{\partial y\partial x}$$

According to the Runge-Kutta method, the transverse components of the propagating field are expressed by

$$\begin{cases} E_x(m,n,l+1) = \left\{ \begin{aligned} & E_x(m,n,l) + \frac{K_1(m,n,l)+2K_2(m,n,l)+2K_3(m,n,l)+K_4(m,n,l)}{6} \\ & + \left[E_y(m,n,l) + \frac{O_1(m,n,l)+2O_2(m,n,l)+2O_3(m,n,l)+O_4(m,n,l)}{6} \right] \end{aligned} \right\} \\ E_y(m,n,l+1) = \left\{ \begin{aligned} & E_y(m,n,l) + \frac{J_1(m,n,l)+2J_2(m,n,l)+2J_3(m,n,l)+J_4(m,n,l)}{6} \\ & + \left[E_x(m,n,l) + \frac{P_1(m,n,l)+2P_2(m,n,l)+2P_3(m,n,l)+P_4(m,n,l)}{6} \right] \end{aligned} \right\} \end{cases},$$

where

$$\begin{aligned} K_1(m,n,l) = \frac{\Delta z}{2jn_0k} \cdot \{ & \frac{a}{\Delta x^2}\left[E_x(m+1,n,l)\right] + \frac{-b}{\Delta x^2}\left[E_x(m,n,l)\right] + \frac{c}{\Delta x^2}\left[E_x(m-1,n,l)\right] \\ & + \frac{1}{\Delta y^2}\left[E_x(m,n+1,l)\right] + \frac{-2}{\Delta y^2}\left[E_x(m,n,l)\right] + \frac{1}{\Delta y^2}\left[E_x(m,n-1,l)\right] \\ & + \left[n^2(m,n,l) - n_0^{\ 2}\right] \cdot k^2 \cdot \left[E_x(m,n,l)\right] \} \end{aligned}$$

$$\begin{aligned} K_2(m,n,l) = \frac{\Delta z}{2jn_0k} \cdot \{ & \frac{a}{\Delta x^2}\left[E_x(m+1,n,l) + \frac{K_1(m+1,n,l)}{2}\right] \\ & + \frac{-b}{\Delta x^2}\left[E_x(m,n,l) + \frac{K_1(m,n,l)}{2}\right] + \frac{c}{\Delta x^2}\left[E_x(m-1,n,l) + \frac{K_1(m-1,n,l)}{2}\right] \\ & + \frac{1}{\Delta y^2}\left[E_x(m,n+1,l) + \frac{K_1(m,n+1,l)}{2}\right] + \frac{-2}{\Delta y^2}\left[E_x(m,n,l) + \frac{K_1(m,n,l)}{2}\right] \\ & + \frac{1}{\Delta y^2}\left[E_x(m,n-1,l) + \frac{K_1(m,n-1,l)}{2}\right] \\ & + \left[n^2(m,n,l) - n_0^{\ 2}\right] \cdot k^2 \cdot \left[E_x(m,n,l) + \frac{K_1(m,n,l)}{2}\right] \} \end{aligned}$$

$$\begin{aligned} K_3(m,n,l) = \frac{\Delta z}{2jn_0k} \cdot \{ & \frac{a}{\Delta x^2}\left[E_x(m+1,n,l) + \frac{K_2(m+1,n,l)}{2}\right] \\ & + \frac{-b}{\Delta x^2}\left[\hat{E}_x(m,n,l) + \frac{K_2(m,n,l)}{2}\right] + \frac{c}{\Delta x^2}\left[\hat{E}_x(m-1,n,l) + \frac{K_2(m-1,n,l)}{2}\right] \\ & + \frac{1}{\Delta y^2}\left[E_x(m,n+1,l) + \frac{K_2(m,n+1,l)}{2}\right] + \frac{-2}{\Delta y^2}\left[E_x(m,n,l) + \frac{K_2(m,n,l)}{2}\right] \\ & + \frac{1}{\Delta y^2}\left[E_x(m,n-1,l) + \frac{K_2(m,n-1,l)}{2}\right] \\ & + \left[n^2(m,n,l) - n_0^{\ 2}\right] \cdot k^2 \cdot \left[E_x(m,n,l) + \frac{K_2(m,n,l)}{2}\right] \} \end{aligned}$$

$$K_4(m,n,l) = \frac{\Delta z}{2jn_0k} \cdot \{ \frac{a}{\Delta x^2}\left[E_x(m+1,n,l) + K_3(m+1,n,l)\right]$$

$$+\frac{-b}{\Delta x^2}\left[E_x(m,n,l)+K_3(m,n,l)\right]+\frac{c}{\Delta x^2}\left[E_x(m-1,n,l)+K_3(m-1,n,l)\right]$$

$$+\frac{1}{\Delta y^2}\left[E_x(m,n+1,l)+K_3(m,n+1,l)\right]+\frac{-2}{\Delta y^2}\left[E_x(m,n,l)+K_3(m,n,l)\right]$$

$$+\frac{1}{\Delta y^2}\left[E_x(m,n-1,l)+K_3(m,n-1,l)\right]$$

$$+\left[n^2(m,n,l)-n_0{}^2\right]\cdot k^2\cdot\left[E_x(m,n,l)+K_3(m,n,l)\right]\}$$

$$O_1(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{\ E_y(m+1,n+1,l)\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+E_y(m,n+1,l)\cdot\left[-2\frac{n^2(m,n+1,l)}{n^2(m,n,l)}+2\right]+E_y(m-1,n+1,l)\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m-1,n,l)}-1\right]$$

$$+E_y(m+1,n-1,l)\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m+1,n,l)}-1\right]+E_y(m,n-1,l)\cdot\left[-2\frac{n^2(m,n-1,l)}{n^2(m,n,l)}+2\right]$$

$$+\ E_y(m-1,n-1,l)\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m-1,n,l)}-1\right]\}\ -\ 2E_y(m+1,n,l)\ +\ 4E_y(m,n,l)\ -\ 2E_y(m-1,n,l)$$

$$O_2(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{$$

$$\left[E_y(m+1,n+1,l)+\frac{O_1(m+1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n+1,l)+\frac{O_1(m,n+1,l)}{2}\right]\cdot\left[-2\frac{n^2(m,n+1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n+1,l)+\frac{O_1(m-1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m-1,n,l)}-1\right]$$

$$+\left[E_y(m+1,n-1,l)+\frac{O_1(m+1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n-1,l)+\frac{O_1(m,n-1,l)}{2}\right]\cdot\left[-2\frac{n^2(m,n-1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n-1,l)+\frac{O_1(m-1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m-1,n,l)}-1\right]\}$$

$$-2\cdot[E_y(m+1,n,l)+O_1(m+1,n,l)/2]+4\cdot[E_y(m,n,l)+O_1(m,n,l)/2]$$

$$-2\cdot[E_y(m-1,n,l)+O_1(m-1,n,l)/2]$$

$$O_3(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{$$

$$\left[E_y(m+1,n+1,l)+\frac{O_2(m+1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n+1,l)+\frac{O_2(m,n+1,l)}{2}\right]\cdot\left[-2\frac{n^2(m,n+1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n+1,l)+\frac{O_2(m-1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m-1,n,l)}-1\right]$$

$$+\left[E_y(m+1,n-1,l)+\frac{O_2(m+1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n-1,l)+\frac{O_2(m,n-1,l)}{2}\right]\cdot\left[-2\frac{n^2(m,n-1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n-1,l)+\frac{O_2(m-1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m-1,n,l)}-1\right]\}$$

$$-2\cdot[E_y(m+1,n,l)+O_2(m+1,n,l)/2]+4\cdot[E_y(m,n,l)+O_2(m,n,l)/2]$$

$$-2\cdot[E_y(m-1,n,l)+O_2(m-1,n,l)/2]$$

$$O_4(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{$$

$$\left[E_y(m+1,n+1,l)+O_3(m+1,n+1,l)\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n+1,l)+O_3(m,n+1,l)\right]\cdot\left[-2\frac{n^2(m,n+1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n+1,l)+O_3(m-1,n+1,l)\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m-1,n,l)}-1\right]$$

$$+\left[E_y(m+1,n-1,l)+O_3(m+1,n-1,l)\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m+1,n,l)}-1\right]$$

$$+\left[E_y(m,n-1,l)+O_3(m,n-1,l)\right]\cdot\left[-2\frac{n^2(m,n-1,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_y(m-1,n-1,l)+O_3(m-1,n-1,l)\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m-1,n,l)}-1\right]\}$$

$$-2\cdot[E_y(m+1,n,l)+O_3(m+1,n,l)]+4\cdot[E_y(m,n,l)+O_3(m,n,l)]$$

$$-2\cdot[E_y(m-1,n,l)+O_3(m-1,n,l)]$$

$$J_1(m,n,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{1}{\Delta x^2}\left[E_y(m+1,n,l)\right]+\frac{-2}{\Delta x^2}\left[E_y(m,n,l)\right]$$

$$+\frac{1}{\Delta x^2}\left[E_y(m-1,n,l)\right]+\frac{d}{\Delta y^2}\left[E_y(m,n+1,l)\right]+\frac{-e}{\Delta y^2}\left[E_y(m,n,l)\right]+$$

$$\frac{f}{\Delta y^2}\left[E_y(m,n-1,l)\right]$$

$$+\left[n^2(m,n,l)-n_0{}^2\right]\cdot k^2\cdot\left[E_y(m,n,l)\right]\}$$

$$J_2(m,n,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{1}{\Delta x^2}\left[E_y(m+1,n,l)+\frac{J_1(m+1,n,l)}{2}\right]$$

$$+\frac{-2}{\Delta x^2}\left[E_y(m,n,l)+\frac{J_1(m,n,l)}{2}\right]+\frac{1}{\Delta x^2}\left[E_y(m-1,n,l)+\frac{J_1(m-1,n,l)}{2}\right]$$

$$+\frac{d}{\Delta y^2}\left[E_y(m,n+1,l)+\frac{J_1(m,n+1,l)}{2}\right]+\frac{-e}{\Delta y^2}\left[E_y(m,n,l)+\frac{J_1(m,n,l)}{2}\right]$$

$$+\frac{f}{\Delta y^2}\left[E_y(m,n-1,l)+\frac{J_1(m,n-1,l)}{2}\right]$$

$$+\left[n^2(m,n,l)-n_0{}^2\right]\cdot k^2\cdot\left[E_y(m,n,l)+\frac{J_1(m,n,l)}{2}\right]\}$$

$$J_3(m,n,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{1}{\Delta x^2}\left[E_y(m+1,n,l)+\frac{J_2(m+1,n,l)}{2}\right]$$

$$+\frac{-2}{\Delta x^2}\left[E_y(m,n,l)+\frac{J_2(m,n,l)}{2}\right]+\frac{1}{\Delta x^2}\left[E_y(m-1,n,l)+\frac{J_2(m-1,n,l)}{2}\right]$$

$$+\frac{d}{\Delta y^2}\left[E_y(m,n+1,l)+\frac{J_2(m,n+1,l)}{2}\right]+\frac{-e}{\Delta y^2}\left[E_y(m,n,l)+\frac{J_2(m,n,l)}{2}\right]$$

$$+\frac{f}{\Delta y^2}\left[E_y(m,n-1,l)+\frac{J_2(m,n-1,l)}{2}\right]$$

$$+\left[n^2(m,n,l)-n_0{}^2\right]\cdot k^2\cdot\left[E_y(m,n,l)+\frac{J_2(m,n,l)}{2}\right]\}$$

$$J_4(m,n,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{1}{\Delta x^2}\left[E_y(m+1,n,l)+J_3(m+1,n,l)\right]$$

$$+\frac{-2}{\Delta x^2}\left[E_y(m,n,l)+J_3(m,n,l)\right]+\frac{1}{\Delta x^2}\left[E_y(m-1,n,l)+J_3(m-1,n,l)\right]$$

$$+\frac{d}{\Delta y^2}\left[E_y(m,n+1,l)+J_3(m,n+1,l)\right]+\frac{-e}{\Delta y^2}\left[E_y(m,n,l)+J_3(m,n,l)\right]$$

$$+\frac{f}{\Delta y^2}\left[E_y(m,n-1,l)+J_3(m,n-1,l)\right]$$

$$+\left[n^2(m,n,l)-n_0{}^2\right]\cdot k^2\cdot\left[E_y(m,n,l)+J_3(m,n,l)\right]\}$$

$$P_1(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{\ E_x(m+1,n+1,l)\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+E_x(m+1,n,l)\cdot\left[-2\frac{n^2(m+1,n,l)}{n^2(m,n,l)}+2\right]+E_x(m+1,n-1,l)\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m,n-1,l)}-1\right]$$

$$+E_x(m-1,n+1,l)\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m,n+1,l)}-1\right]+E_x(m-1,n,l)\cdot\left[-2\frac{n^2(m-1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+E_x(m-1,n-1,l)\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m,n-1,l)}-1\right]\}-2E_x(m,n+1,l)+4E_x(m,n,l)$$

$$-2E_x(m,n-1,l)$$

$$P_2(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{$$

$$\left[E_x(m+1,n+1,l)+\frac{P_1(m+1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m+1,n,l)+\frac{P_1(m+1,n,l)}{2}\right]\cdot\left[-2\frac{n^2(m+1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m+1,n-1,l)+\frac{P_1(m+1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m1,n-1,l)}-1\right]$$

$$+\left[E_x(m-1,n+1,l)+\frac{P_1(m-1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m-1,n,l)+\frac{P_1(m-1,n,l)}{2}\right]\cdot\left[-2\frac{n^2(m-1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m-1,n-1,l)+\frac{P_1(m-1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m,n-1,l)}-1\right]\}$$

$$-2\cdot[E_x(m,n+1,l)+P_1(m,n+1,l)/2]+4\cdot[E_x(m,n,l)+P_1(m,n,l)/2]$$

$$-2\cdot[E_x(m,n-1,l)+P_1(m,n-1,l)/2]$$

$$P_3(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{$$

$$\left[E_x(m+1,n+1,l)+\frac{P_2(m+1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m+1,n,l)+\frac{P_2(m+1,n,l)}{2}\right]\cdot\left[-2\frac{n^2(m+1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m+1,n-1,l)+\frac{P_2(m+1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m1,n-1,l)}-1\right]$$

$$+\left[E_x(m-1,n+1,l)+\frac{P_2(m-1,n+1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m-1,n,l)+\frac{P_2(m-1,n,l)}{2}\right]\cdot\left[-2\frac{n^2(m-1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m-1,n-1,l)+\frac{P_2(m-1,n-1,l)}{2}\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m,n-1,l)}-1\right]\}$$

$$-2\cdot[E_x(m,n+1,l)+P_2(m,n+1,l)/2]+4\cdot[E_x(m,n,l)+P_2(m,n,l)/2]$$

$$-2\cdot[E_x(m,n-1,l)+P_2(m,n-1,l)/2]$$

$$P_4(m,n,l)=\frac{\Delta z}{8jn_0k\Delta x\Delta y}\{\left[E_x(m+1,n+1,l)+P_3(m+1,n+1,l)\right]\cdot\left[\frac{n^2(m+1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m+1,n,l)+P_3(m+1,n,l)\right]\cdot\left[-2\frac{n^2(m+1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m+1,n-1,l)+P_3(m+1,n-1,l)\right]\cdot\left[\frac{n^2(m+1,n-1,l)}{n^2(m,n-1,l)}-1\right]$$

$$+\left[E_x(m-1,n+1,l)+P_3(m-1,n+1,l)\right]\cdot\left[\frac{n^2(m-1,n+1,l)}{n^2(m,n+1,l)}-1\right]$$

$$+\left[E_x(m-1,n,l)+P_3(m-1,n,l)\right]\cdot\left[-2\frac{n^2(m-1,n,l)}{n^2(m,n,l)}+2\right]$$

$$+\left[E_x(m-1,n-1,l)+P_3(m-1,n-1,l)\right]\cdot\left[\frac{n^2(m-1,n-1,l)}{n^2(m,n-1,l)}-1\right]\}$$

$$-2\cdot[E_x(m,n+1,l)+P_3(m,n+1,l)]+4\cdot[E_x(m,n,l)+P_3(m,n,l)]$$

$$-2\cdot[E_x(m,n-1,l)+P_3(m,n-1,l)]$$

$$a=\frac{1}{2}+\frac{n^2(m+1,n,l)}{2n^2(m,n,l)},\quad b=1+\frac{n^2(m,n,l)}{2}\cdot\left[\frac{1}{n^2(m+1,n,l)}+\frac{1}{n^2(m-1,n,l)}\right],$$

$$c=\frac{1}{2}+\frac{n^2(m-1,n,l)}{2n^2(m,n,l)},\quad d=\frac{1}{2}+\frac{n^2(m,n+1,l)}{2n^2(m,n,l)},$$

$$e=1+\frac{n^2(m,n,l)}{2}\cdot\left[\frac{1}{n^2(m,n+1,l)}+\frac{1}{n^2(m,n-1,l)}\right],\quad f=\frac{1}{2}+\frac{n^2(m,n-1,l)}{2n^2(m,n,l)}$$

Power distribution $\propto \left|\vec{E}\right|^2=(E_x{}^2+E_y{}^2)$

3D Semi-vector BPM involving the Runge-Kutta algorithm (by K. -Y. Lee):

The transverse components of the optical field $E_x(x,y,z)$ and $E_y(x,y,z)$ fulfill

$$2jn_0k\cdot\frac{\partial E_x(x,y,z)}{\partial z}=\frac{\partial}{\partial x}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial x}[n^2(x,y,z)E_x(x,y,z)]\}+\frac{\partial^2 E_x(x,y,z)}{\partial y^2}+[n^2(x,y,z)-n_0^{\ 2}]k^2E_x(x,y,z)$$

and

$$2jn_0k\cdot\frac{\partial E_y(x,y,z)}{\partial z}=\frac{\partial^2 E_y(x,y,z)}{\partial x^2}+\frac{\partial}{\partial y}\{\frac{1}{n^2(x,y,z)}\cdot\frac{\partial}{\partial y}[n^2(x,y,z)E_y(x,y,z)]\}+[n^2(x,y,z)-n_0^{\ 2}]k^2E_y(x,y,z)$$

The Runge-Kutta algorithm can be directly to obtain the propagating field components as

$$E_x(m,i,l+1)=E_x(m,i,l)+\frac{K_1(m,i,l)+2K_2(m,i,l)+2K_3(m,i,l)+K_4(m,i,l)}{6}$$

and

$$E_y(m,i,l+1)=E_y(m,i,l)+\frac{J_1(m,i,l)+2J_2(m,i,l)+2J_3(m,i,l)+J_4(m,i,l)}{6},$$

where

$$K_1(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{aE_x(m+1,i,l)-bE_x(m,i,l)+cE_x(m-1,i,l)}{\Delta x^2}+\frac{E_x(m,i+1,l)-2E_x(m,i,l)+E_x(m,i-1,l)}{\Delta y^2}+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot E_x(m,i,l)\}$$

$$K_2(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{a[E_x(m+1,i,l)+\frac{K_1(m+1,i,l)}{2}]-b[E_x(m,i,l)+\frac{K_1(m,i,l)}{2}]+c[E_x(m-1,i,l)+\frac{K_1(m-1,i,l)}{2}]}{\Delta x^2}$$

$$+\frac{E_x(m,i+1,l)+\frac{K_1(m,i+1,l)}{2}-2[E_x(m,i,l)+\frac{K_1(m,i,l)}{2}]+E_x(m,i-1,l)+\frac{K_1(m,i-1,l)}{2}}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_x(m,i,l)+\frac{K_1(m,i,l)}{2}]\}$$

$$K_3(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{a[E_x(m+1,i,l)+\frac{K_2(m+1,i,l)}{2}]-b[E_x(m,i,l)+\frac{K_2(m,i,l)}{2}]+c[E_x(m-1,i,l)+\frac{K_2(m-1,i,l)}{2}]}{\Delta x^2}$$

$$+\frac{E_x(m,i+1,l)+\frac{K_2(m,i+1,l)}{2}-2[E_x(m,i,l)+\frac{K_2(m,i,l)}{2}]+E_x(m,i-1,l)+\frac{K_2(m,i-1,l)}{2}}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_x(m,i,l)+\frac{K_2(m,i,l)}{2}]\}$$

$$K_4(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{a[E_x(m+1,i,l)+K_3(m+1,i,l)]-b[E_x(m,i,l)+K_3(m,i,l)]+c[E_x(m-1,i,l)+K_3(m,i,l)]}{\Delta x^2}$$

$$+\frac{E_x(m,i+1,l)+K_3(m,i+1,l)-2[E_x(m,i,l)+K_3(m,i,l)]+E_x(m,i-1,l)+K_3(m,i-1,l)}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_x(m,i,l)+K_3(m,i,l)]\}$$

$$J_1(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{E_y(m+1,i,l)-2E_y(m,i,l)+E_y(m-1,i,l)}{\Delta x^2}$$

$$+\frac{dE_y(m,i+1,l)-eE_y(m,i,l)+fE_y(m,i-1,l)}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot E_y(m,i,l)\},$$

$$J_2(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{E_y(m+1,i,l)+\frac{J_1(m+1,i,l)}{2}]-2[E_y(m,i,l)+\frac{J_1(m,i,l)}{2}]+E_y(m-1,i,l)+\frac{J_1(m-1,i,l)}{2}}{\Delta x^2}$$

$$+\frac{d[E_y(m,i+1,l)+\frac{J_1(m,i+1,l)}{2}]-e[E_y(m,i,l)+\frac{J_1(m,i,l)}{2}]+f[E_y(m,i-1,l)+\frac{J_1(m,i-1,l)}{2}]}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_y(m,i,l)+\frac{J_1(m,i,l)}{2}]\}$$

$$J_3(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{E_y(m+1,i,l)+\frac{J_2(m+1,i,l)}{2}-2[E_y(m,i,l)+\frac{J_2(m,i,l)}{2}]+E_y(m-1,i,l)+\frac{J_2(m-1,i,l)}{2}]}{\Delta x^2}$$

$$+\frac{d[E_y(m,i+1,l)+\frac{J_2(m,i+1,l)}{2}]-e[E_y(m,i,l)+\frac{J_2(m,i,l)}{2}]+f[E_y(m,i-1,l)+\frac{J_2(m,i-1,l)}{2}]}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_y(m,i,l)+\frac{J_2(m,i,l)}{2}]\}$$

$$J_4(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{E_y(m+1,i,l)+J_3(m+1,i,l)-2[E_y(m,i,l)+J_3(m,i,l)]+E_y(m-1,i,l)+J_3(m,i,l)}{\Delta x^2}$$

$$+\frac{d[E_y(m,i+1,l)+J_3(m,i+1,l)]-e[E_y(m,i,l)+J_3(m,i,l)]+f[E_y(m,i-1,l)+J_3(m,i-1,l)]}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0^{\ 2}]\cdot k^2\cdot[E_y(m,i,l)+J_3(m,i,l)]\}$$

$$a=\frac{1}{2}+\frac{n^2(m+1,i,l)}{2n^2(m,i,l)},\quad b=1+\frac{n^2(m,i,l)}{2}\cdot[\frac{1}{n^2(m+1,i,l)}+\frac{1}{n^2(m-1,i,l)}],\quad c=\frac{1}{2}+\frac{n^2(m-1,i,l)}{2n^2(m,i,l)},$$

$$d=\frac{1}{2}+\frac{n^2(m,i+1,l)}{2n^2(m,i,l)},\quad e=1+\frac{n^2(m,i,l)}{2}\cdot[\frac{1}{n^2(m,i+1,l)}+\frac{1}{n^2(m,i-1,l)}],\quad f=\frac{1}{2}+\frac{n^2(m,i-1,l)}{2n^2(m,i,l)}$$

Note that coefficients b and e approach 2, but a, c, d, and f approach 1 as the transverse variations of the refractive index profiles are very small. In this case, the above semivectorial formulae for two types of polarizations are reduced into the same scalar expressions.

$$\text{Power distribution} \propto \left|\vec{E}\right|^2 = E_x{}^2 \text{ or } E_y{}^2$$

3D Scalar BPM involving the Runge-Kutta algorithm (by K. -Y. Lee):

The scalar optical field $\varphi(x,y,z)$ can be described by the following Fresnel equation

$$2jn_0k\cdot\frac{\partial\phi(x,y,z)}{\partial z}=\frac{\partial^2\phi(x,y,z)}{\partial x^2}+\frac{\partial^2\phi(x,y,z)}{\partial y^2}+[n^2(x,y,z)-n_0{}^2]k^2\phi(x,y,z),$$

Utilizing the finite difference scheme and the Runge-Kutta method, the propagating optical field at $(x=m\Delta x, y=i\Delta y, z=(l+1)\Delta z)$ is expressed by

$$\phi(m,i,l+1)=\phi(m,i,l)+\frac{K_1(m,i,l)+2K_2(m,i,l)+2K_3(m,i,l)+K_4(m,i,l)}{6},$$

where

$$K_1(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{\phi(m+1,i,l)-2\phi(m,i,l)+\phi(m-1,i,l)}{\Delta x^2}$$

$$+\frac{\phi(m,i+1,l)-2\phi(m,i,l)+\phi(m,i-1,l)}{\Delta y^2}+[n^2(m,i,l)-n_0{}^2]\cdot k^2\cdot\phi(m,i,l)\}$$

$$K_2(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{\phi(m+1,i,l)+\frac{K_1(m+1,i,l)}{2}-2\phi(m,i,l)-K_1(m,i,l)+\phi(m-1,i,l)+\frac{K_1(m-1,i,l)}{2}}{\Delta x^2}$$

$$+\frac{\phi(m,i+1,l)+\frac{K_1(m,i+1,l)}{2}-2\phi(m,i,l)-K_1(m,i,l)+\phi(m,i-1,l)+\frac{K_1(m,i-1,l)}{2}}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0{}^2]\cdot k^2\cdot[\phi(m,i,l)+\frac{K_1(m,i,l)}{2}]\}$$

$$K_3(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{\phi(m+1,i,l)+\frac{K_2(m+1,i,l)}{2}-2\phi(m,i,l)-K_2(m,i,l)+\phi(m-1,i,l)+\frac{K_2(m-1,i,l)}{2}}{\Delta x^2}$$

$$+\frac{\phi(m,i+1,l)+\frac{K_2(m,i+1,l)}{2}-2\phi(m,i,l)-K_2(m,i,l)+\phi(m,i-1,l)+\frac{K_2(m,i-1,l)}{2}}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0{}^2]\cdot k^2\cdot[\phi(m,i,l)+\frac{K_2(m,i,l)}{2}]\}$$

and

$$K_4(m,i,l)=\frac{\Delta z}{2jn_0k}\cdot\{\frac{\phi(m+1,i,l)+K_3(m+1,i,l)-2\phi(m,i,l)-2K_3(m,i,l)+\phi(m-1,i,l)+K_3(m,i,l)}{\Delta x^2}$$

$$+\frac{\phi(m,i+1,l)+K_3(m,i+1,l)-2\phi(m,i,l)-2K_3(m,i,l)+\phi(m,i-1,l)+K_3(m,i-1,l)}{\Delta y^2}$$

$$+[n^2(m,i,l)-n_0{}^2]\cdot k^2\cdot[\phi(m,i,l)+K_3(m,i,l)]\}$$

On the other hand, the **transparent boundary condition** (**TBC**) is implemented in the computer programming for reducing extra numerical errors due to undesired wave reflections at the edges of computational windows.

Transparent Boundary Conditions (TBC):

Define α_{+x}, α_{-x}, β_{+x}, β_{-x}, α_{+y}, α_{-y}, β_{+y}, and β_{-y} as

$$\alpha_{\pm x}+j\beta_{\pm x}\equiv\ln[\frac{\phi(\pm M\mp 1,i,l)}{\phi(\pm M\mp 2,i,l)}] \text{ and } \alpha_{\pm y}+j\beta_{\pm y}\equiv\ln[\frac{\phi(m,\pm I\mp 1,l)}{\phi(m,\pm I\mp 2,l)}]$$

TBC shows that

$$\phi(\pm M,i,l)=\begin{cases}\dfrac{\phi(\pm M\mp 1,i,l)^2}{\phi(\pm M\mp 2,i,l)} & ,\alpha_{\pm x}<0\\ \phi(\pm M\mp 1,i,l)\cdot\exp(j\beta_{\pm x}) & ,\alpha_{\pm x}\ge 0\end{cases}$$

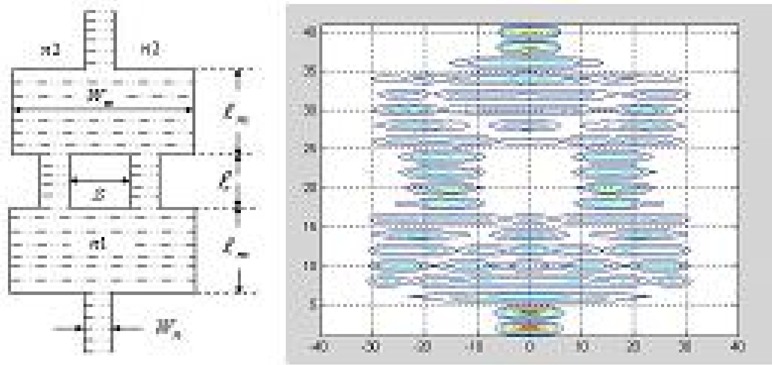
and

$$\phi(m,\pm I,l)=\begin{cases}\dfrac{\phi(m,\pm I\mp 1,l)^2}{\phi(m,\pm I\mp 2,l)} & ,\alpha_{\pm y}<0\\ \phi(m,\pm I\mp 1,l)\cdot\exp(j\beta_{\pm y}) & ,\alpha_{\pm y}\ge 0\end{cases}$$

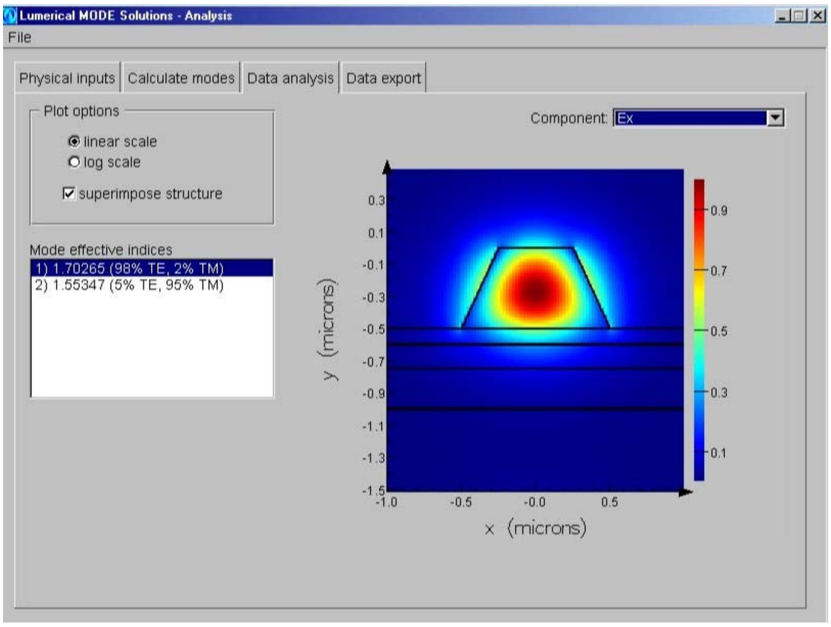
Similarly, the values $K_1(\pm M,i,l)$, $K_2(\pm M,i,l)$, $K_3(\pm M,i,l)$, $K_4(\pm M,i,l)$, $K_1(m,\pm I,l)$, $K_2(m,\pm I,l)$, $K_3(m,\pm I,l)$, and $K_4(m,\pm I,l)$ at the edges of the computation windows are obtained by the same rules.

Power distribution $\propto|\phi|^2$

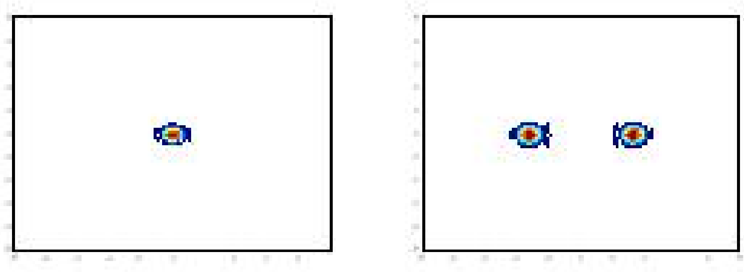
Eg. A laser beam passes an MMI waveguide device. (by K. Huang, 黃建智)



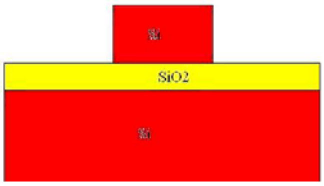
Eg. Simulation of the fundamental mode of a single-mode ridge waveguide.



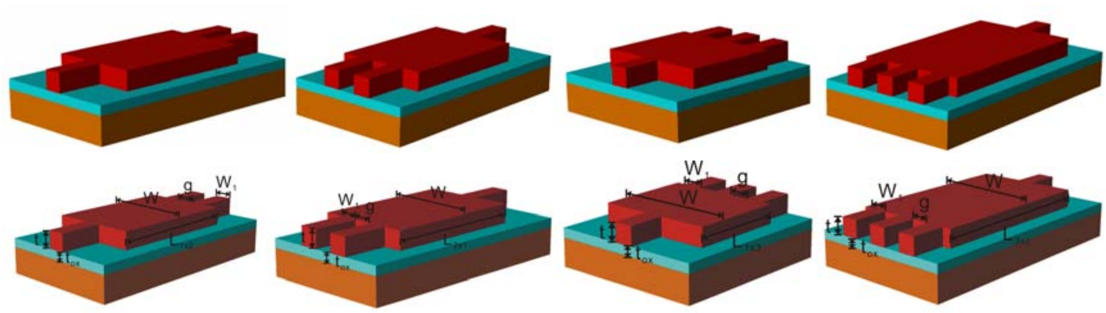
Eg. Simulation of a 10°-*Y*-junction 3D waveguide. (by K. –Y. Lee)



Input mode Output light spots



Eg. For an n×1 MMI optical combiner and a 1×n MMI optical splitter, the BPM simulation shows that $L_{n\times1}/L_{1\times n} \approx n$-1, where $L_{n\times1}$ is the wide section length of the n×1 MMI optical combiner and $L_{1\times n}$ is the wide section length of the 1×n MMI optical splitter, respectively. Consider a SOI waveguide of n_{core}=3.45, n_{SiO2}=1.5, λ=1.55μm. The thickness of oxide is 2μm. The input single-mode waveguide has a 5μm×5μm cross section.



1×2 MMI splitter 2×1 MMI combiner 1×3 MMI splitter 3×1 MMI combiner 3D

BPM simulation (by J. –R. Sze *et al.*):

n	2	3	4	5
W	40μm	40μm	40μm	40μm
g	15μm	8.333μm	5μm	3μm
$L_{n\times1}$	1772μm	2380μm	2684μm	2846μm
$L_{1\times n}$	1805μm	1197μm	893μm	731μm
$L_{n\times1}/L_{1\times n}$	**0.98≒1**	**1.98≒2**	**3.005≒3**	**3.89≒4**