Chapter 9 Integral Equations

9-1 Classification of Integral Equations

Fredholm equation: $\alpha(x)y(x)=F(x)+\lambda\int_a^b K(x,t)y(t)dt$

The 1st kind: $\alpha=0 \Rightarrow -F(x)=G(x)=\lambda\int_a^b K(x,t)y(t)dt$

The 2nd kind: $\alpha=1 \Rightarrow y(x)=F(x)+\lambda\int_a^b K(x,t)y(t)dt$

Voterra equation: $\alpha(x)y(x)=F(x)+\lambda\int_a^x K(x,t)y(t)dt$

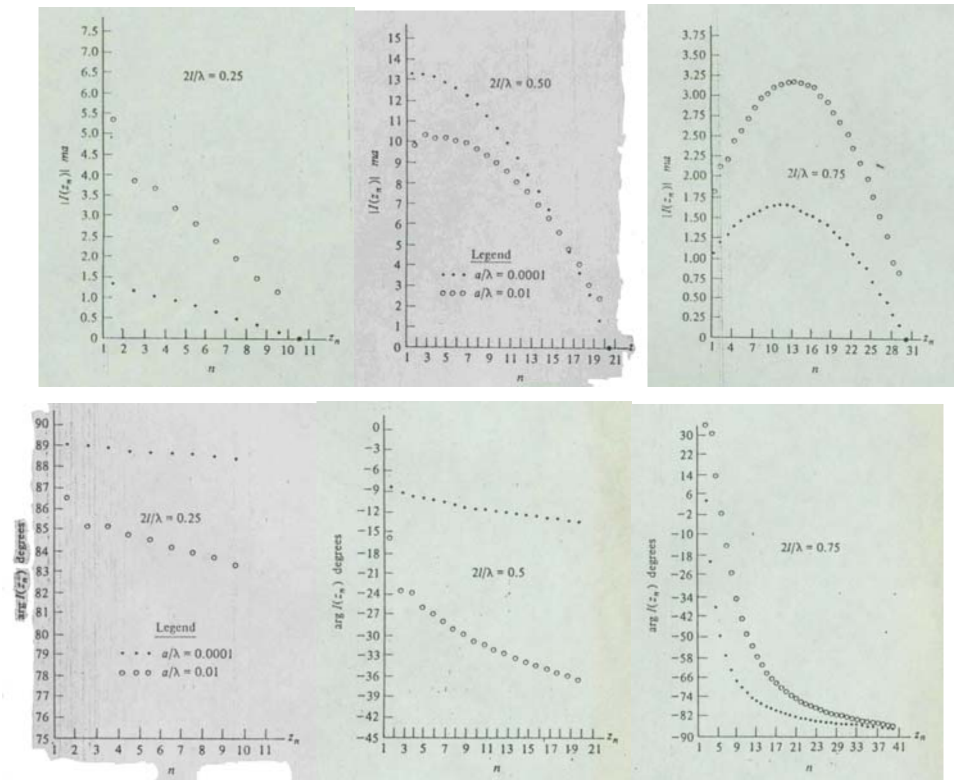
The 1st kind: $\alpha=0 \Rightarrow -F(x)=G(x)=\lambda\int_a^x K(x,t)y(t)dt$

The 2nd kind: $\alpha=1 \Rightarrow y(x)=F(x)+\lambda\int_a^x K(x,t)y(t)dt$

Eg. Solve Hallen's equation: $-i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{V}{2}\sin(k|z|)+C\cos(k|z|)=\int_{-l}^{l} I(z')K(z-z',a)dz'$**,**

where $K(z-z',a)=\dfrac{e^{-ik\sqrt{(z-z')^2+a^2}}}{4\pi\sqrt{(z-z')^2+a^2}}$**,** $k=2\pi/\lambda$ **and** C **must be satisfied by** $I(l)=0$**. This is an example of the first kind of the Fredholm equation.**

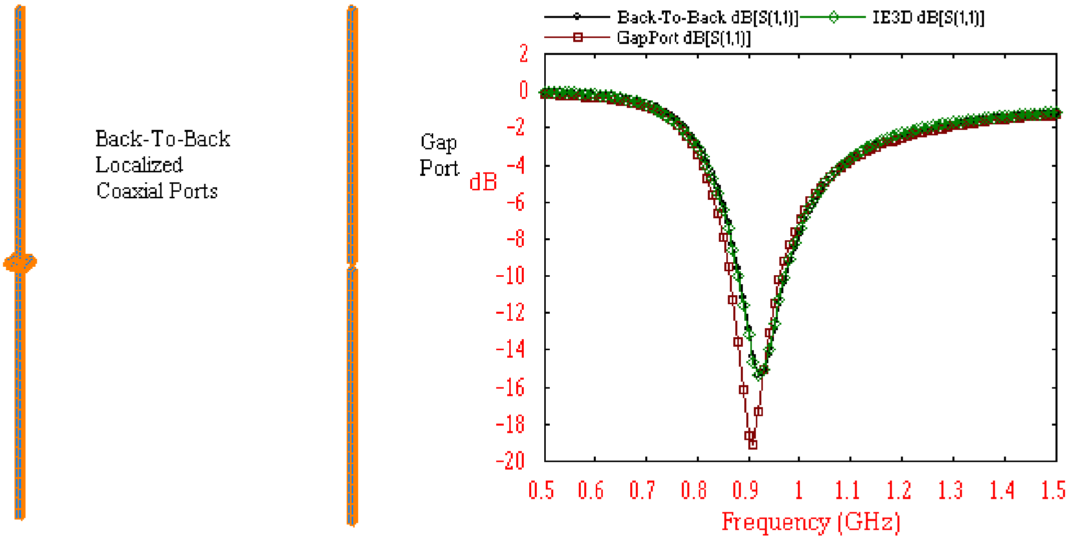
(Sol.)

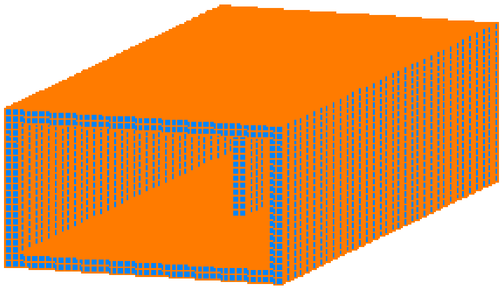


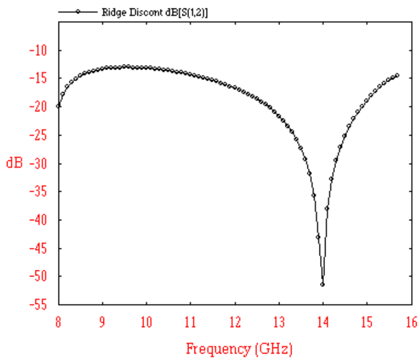
It is shown that the resonant length of the dipole antenna is about 1/4 of wavelength.

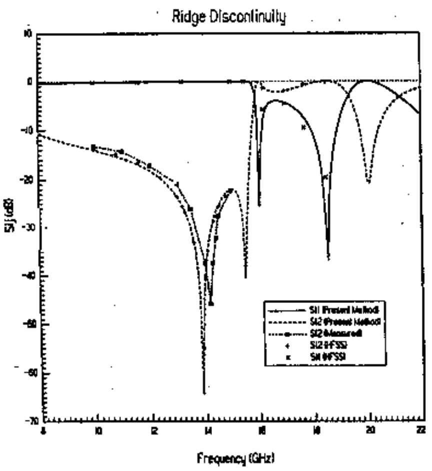
Eg. $f(x)=e^{-x}-\dfrac{1}{2}+\dfrac{e^{-(x+1)}}{2}+\dfrac{1}{2}\int_0^1 (x+1)e^{-xy}f(y)dy$ **is the second kind of the Fredholm equation, which has a solution** $f(x)=e^{-x}$**.**

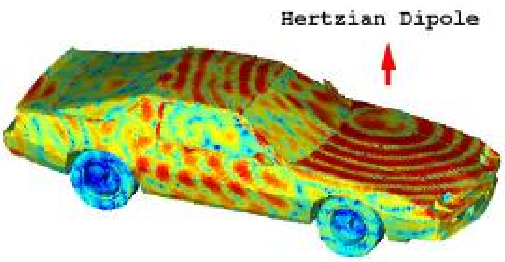
Some examples of solving integral equations:











Irradiation of an 83 Camaro by a one GHz Hertzian dipole source.

Relations between differential and integral equations:

Eg. $\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=f(x)$ **with *y*(*a*)=*C*, *y'*(*a*)=*D*.**

Dummy variables: *x*=*t*, *dx*=*dt*, and then we have $\frac{d^2y}{dt^2}+A(t)\frac{dy}{dt}+B(t)y=f(t)$

$$\int_a^x \frac{d^2y}{dt^2}\cdot dt=\int_a^x f(t)dt-\int_a^x A(t)\frac{dy}{dt}\cdot dt-\int_a^x B(t)y(t)\cdot dt$$

$$=\int_a^x f(t)dt-A(x)y(x)\Big|_a^x+\int_a^x y(t)A'(t)dt-\int_a^x y(t)B(t)dt$$

$$\Rightarrow y'(x)-D=-A(x)y(x)-\int_a^x[B(t)-A'(t)]y(t)dt+\int_a^x f(t)dt+A(a)C$$

Dummy variables: *x*=*s*, *dx*=*ds*

$$\Rightarrow \int_a^x y'(s)ds=[A(a)C+D](x-a)-\int_a^x A(s)y(s)ds$$

$$-\int_a^x\int_a^s[B(t)-A'(t)]y(t)dtds+\int_a^x\int_a^s f(t)dtds$$

$$\Rightarrow y(x)=C+[A(a)C+D](x-a)-\int_a^x\int_t^x[B(t)-A'(t)]y(t)dsdt+\int_a^x\int_t^x f(t)dsdt-\int_a^x A(t)y(t)dt$$

$$=C+[A(a)C+D](x-a)-\int_a^x\{A(t)+(x-t)[B(t)-A'(t)]\}y(t)dt+\int_a^x(x-t)f(t)dt$$

$\Rightarrow y(x)=C+[A(a)C+D](x-a)+\int_a^x(x-t)f(t)dt-\int_a^x\{A(t)+(x-t)[B(t)-A'(t)]\}y(t)dt$ is the 2nd kind of Volterra equation.

Eg. Transform $\frac{d^2y}{dx^2}+\lambda y=f(x)$ **with *y*(0)=1, *y'*(0)=0 into an integral equation.**

(Sol.) *A*(*x*)=0, *B*(*x*)=λ, *C*=1, *D*=0

$$\Rightarrow y(x)=1+\int_0^x(x-t)f(t)dt-\int_0^x(x-t)\lambda y(t)dt$$

$$\Rightarrow y(x)=1-\int_0^x(t-x)f(t)dt+\lambda\int_0^x(t-x)y(t)dt$$

9-2 Moment Method

Solve a Fredholm equation $F(x)=\alpha(x)y(x)-\lambda\int_a^b K(x,t)y(t)dt$.

Let $y(x)=\sum_{n=1}^{\infty} c_n f_n(x) \approx \sum_{n=1}^{N} c_n f_n(x)$

$$\Rightarrow F(x)\approx\sum_{n=1}^{N} c_n\alpha(x)f_n(x)-\lambda\int_a^b K(x,t)\cdot\sum_{n=1}^{N} c_n f_n(t)dt=\sum_{n=1}^{N} c_n[\alpha(x)f_n(x)-\lambda\int_a^b K(x,t)f_n(t)dt]$$

Choose distinct $x_1, x_2, \ldots, x_N$

$$F(x_1)=\sum_{n=1}^{N} c_n A_n(x_1)=c_1A_1(x_1)+c_2A_2(x_1)+\cdots+c_NA_N(x_1)$$

$$F(x_2)=\sum_{n=1}^{N} c_n A_n(x_2)=c_1A_1(x_2)+c_2A_2(x_2)+\cdots+c_NA_N(x_2)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$F(x_N)=\sum_{n=1}^{N} c_n A_n(x_N)=c_1A_1(x_N)+c_2A_2(x_N)+\cdots+c_NA_N(x_N)$$

Or, $$\begin{bmatrix} F(x_1) \\ F(x_2) \\ \vdots \\ F(x_N) \end{bmatrix}=\begin{bmatrix} A_1(x_1) & A_2(x_1) & \cdots & A_N(x_1) \\ A_1(x_2) & A_2(x_2) & \cdots & A_N(x_2) \\ \vdots & \vdots & & \vdots \\ A_1(x_N) & A_2(x_N) & \cdots & A_N(x_N) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}\Rightarrow\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}=?$$

where $A_n(x_m)=\alpha(x_m)f_n(x_m)-\lambda\int_a^b K(x_m,t)f_n(t)dt$, $1\le n,\ m\le N$.

Solve a Volterra equation $F(x)=\alpha(x)y(x)-\lambda\int_a^x K(x,t)y(t)dt$.

Let $y(x)=\sum_{n=1}^{\infty} c_n f_n(x) \approx \sum_{n=1}^{N} c_n f_n(x)$

$$\Rightarrow F(x_i)\approx\sum_{n=1}^{N} c_n\alpha(x_i)f_n(x_i)-\lambda\int_a^{x_i} K(x_i,t)\cdot\sum_{n=1}^{N} c_n f_n(t)dt$$

$$=\sum_{n=1}^{N} c_n\cdot[\alpha(x_i)f_n(x_i)-\lambda\int_a^{x_i} K(x_i,t)f_n(t)dt]$$

Or, $$\begin{bmatrix} F(x_1) \\ F(x_2) \\ \vdots \\ F(x_N) \end{bmatrix}=\begin{bmatrix} B_1(x_1) & B_2(x_1) & \cdots & B_N(x_1) \\ B_1(x_2) & B_2(x_2) & \cdots & B_N(x_2) \\ \vdots & \vdots & & \vdots \\ B_1(x_N) & B_2(x_N) & \cdots & B_N(x_N) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}\Rightarrow\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}=?$$

where $B_n(x_m)=\alpha(x_m)f_n(x_m)-\lambda\int_a^{x_m} K(x_m,t)f_n(t)dt$, $1\le n,\ m\le N$.

Eg. Let $y(x)=\sum_{n=1}^{N} c_n f_n(x)=\sum_{n=1}^{N} c_n x^{n-1}$ and solve $y(x)=\int_0^1 K(x,t)y(t)dt-x$, where

$$K(x,t)=\begin{cases} x(1-t), & x<t \\ t(1-x), & x\ge t \end{cases}.$$