Chapter 1 Electromagnetic Field Theory

1-1 Electric Fields and Electric Dipoles

Gauss's law of $\vec{E}$ **:** $\oiint_s \vec{E}\cdot d\vec{S} = \dfrac{Q}{\varepsilon} = \dfrac{1}{\varepsilon}\iiint \rho\, dv'$

$$\xrightarrow[\text{theorm}]{\text{divergence}} \iiint_v \nabla\cdot\vec{E}dv' = \frac{1}{\varepsilon}\iiint_{v'} \rho\, dv' \Rightarrow \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon}$$

and $\varepsilon_0 = \dfrac{1}{36\pi}\times 10^{-9}$ (F/m) in the free space.

For q at $\vec{R}'$, field point at $\vec{R} \Rightarrow \vec{E} = \dfrac{q\hat{a}_{RR'}}{4\pi\varepsilon|\vec{R}-\vec{R}'|^2} = \dfrac{q(\vec{R}-\vec{R}')}{4\pi\varepsilon|\vec{R}-\vec{R}'|^3}$ and $\hat{a}_{RR'} = \dfrac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|}$.

$\vec{E}$ due to a system of discrete charges: $\vec{E} = \dfrac{1}{4\pi\varepsilon}\sum_{k=1}^{n}\dfrac{q_k(\vec{R}-\vec{R}'_k)}{|\vec{R}-\vec{R}'_k|^3}$

Volume source $\rho \Rightarrow \vec{E} = \dfrac{1}{4\pi\varepsilon}\iiint_{v'} \hat{a}_R \dfrac{\rho}{R^2}dv' = \dfrac{1}{4\pi\varepsilon}\iiint_{v'}\dfrac{\rho\vec{R}}{|\vec{R}|^3}dv'$

Surface source $\rho_s \Rightarrow \vec{E} = \dfrac{1}{4\pi\varepsilon}\iint_{s'} \hat{a}_R \dfrac{\rho_s}{R^2}dS'$ Line source $\rho_l \Rightarrow \vec{E} = \dfrac{1}{4\pi\varepsilon}\int_{l'} \hat{a}_R \dfrac{\rho_l}{R^2}dl'$

Eg. Show that Coulomb's law $\vec{F} = \hat{a}_R\dfrac{q_1q_2}{4\pi\varepsilon R^2} = \dfrac{q_1q_2\vec{R}}{4\pi\varepsilon R^3}$**, where** $\hat{a}_R = \dfrac{\vec{R}}{R}$**,** $R = |\vec{R}|$**.**

(Proof) $\because$ $\vec{F} = q_2\vec{E}$ and $\oiint \vec{E}\cdot d\vec{S} = \dfrac{q_1}{\varepsilon} = 4\pi r^2 E$, $\vec{E} = \hat{a}_R\dfrac{q_1}{4\pi\varepsilon R^2} = \dfrac{q_1\vec{R}}{4\pi\varepsilon R^3}$

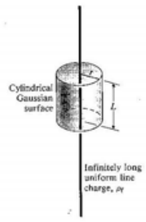
$$\therefore\ \vec{F} = \hat{a}_R\frac{q_1q_2}{4\pi\varepsilon R^2}$$

Eg. Determine the electric field intensity of an infinitely long line charge of a uniform density ρ_l in air.

(Sol.)

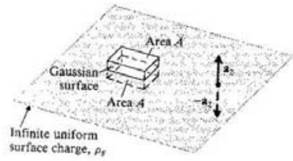
$$\oiint_s \vec{E}\cdot d\vec{S} = \int_0^L\int_0^{2\pi} Erd\phi dz = 2\pi rLE$$

$$2\pi rLE = \frac{\rho_l L}{\varepsilon_0},\quad \vec{E} = \hat{a}_r\frac{\rho_l}{2\pi\varepsilon_0 r}$$



Eg. Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.

(Sol.) $\oint_s \vec{E}\cdot d\vec{S} = 2ES = 2EA$, $2EA = \dfrac{\rho_s A}{\varepsilon_0}$



$$\Rightarrow \vec{E} = \begin{cases} = \hat{z}\dfrac{\rho_s}{2\varepsilon_0}, & z>0 \\ = -\hat{z}\dfrac{\rho_s}{2\varepsilon_0}, & z<0 \end{cases}$$

Eg. A line charge of uniform density ρ_l in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle. [高考]

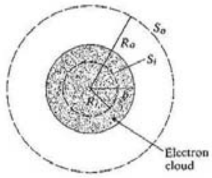
(Sol.) $d\vec{E}_y = -\dfrac{\rho_l(bd\phi)}{4\pi\varepsilon_0 b^2}\cdot\sin\phi$, $\vec{E} = \hat{y}E_y = -\hat{y}\dfrac{\rho_l}{4\pi\varepsilon_0 b}\int_0^\pi \sin\phi\, d\phi = -\hat{y}\dfrac{\rho_l}{2\pi\varepsilon_0 b}$

Eg. Determine the electric field caused by spherical cloud of electrons with a volume charge density ρ=-ρ_0 for $0\le R\le b$ (both ρ_0 and b are positive) and ρ=0 for R>b. [交大電子物理所]

(Sol.)

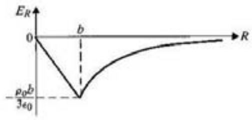
(a) $R\ge b$

$$Q = -\rho_0\frac{4\pi}{3}b^3,\quad \vec{E} = \hat{a}_R\frac{Q}{4\pi\varepsilon_0 R^2} = -\hat{a}_R\frac{\rho_0 b^3}{3\varepsilon_0 R^2}$$



(b) $0\le R\le b$

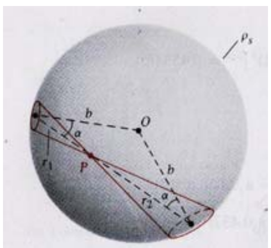
$$\vec{E} = \hat{a}_R E,\quad d\vec{S} = \hat{a}_R dS,\quad \oint_{S_i}\vec{E}\cdot d\vec{S} = E\int_{S_i} dS = E4\pi R^2$$



$$Q = \iiint_v \rho dv = -\rho_0\iiint_v dv = -\rho_0\frac{4\pi}{3}R^3,\quad \vec{E} = -\hat{a}_R\frac{\rho_0 R}{3\varepsilon_0}$$

Eg. A total charge Q is put on a thin spherical shell of radius b. Determine the electrical field intensity at an arbitrary point inside the shell. [台大電研]

(Sol.)



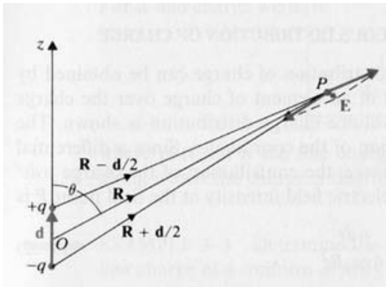
$$\rho_s = \frac{Q}{4\pi b^2},\quad dE = \frac{\rho_s}{4\pi\varepsilon_0}\left(\frac{dS_1}{r_1^{\,2}} - \frac{dS_2}{r_2^{\,2}}\right)$$

$$d\Omega = \frac{dS_1}{r_1^{\,2}}\cos\alpha = \frac{dS_2}{r_2^{\,2}}\cos\alpha,\quad dE = \frac{\rho_s}{4\pi\varepsilon_0}\left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha}\right) = 0$$

Electric dipole: A pair of equal but opposite charges with separation.

$$|\vec{R}-\frac{\vec{d}}{2}|^{-3}=[(\vec{R}-\frac{\vec{d}}{2})\cdot(\vec{R}-\frac{\vec{d}}{2})]^{-3/2}=[R^2-R\cdot d+\frac{d^2}{4}]^{-3/2}$$

$$\cong R^{-3}[1-\frac{\vec{R}\cdot\vec{d}}{R^2}]^{-3/2}\cong R^{-3}[1+\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}]$$



$$|\vec{R}+\frac{\vec{d}}{2}|^{-3}\cong R^{-3}[1-\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}]$$

$$\Rightarrow \vec{E}=\frac{q}{4\pi\varepsilon}\left\{\frac{\vec{R}-\frac{\vec{d}}{2}}{\left|\vec{R}-\frac{\vec{d}}{2}\right|^3}-\frac{\vec{R}+\frac{\vec{d}}{2}}{\left|\vec{R}+\frac{\vec{d}}{2}\right|^3}\right\}\cong\frac{q}{4\pi\varepsilon R^3}\left[3\frac{\vec{R}\cdot\vec{d}}{R^2}\vec{R}-\vec{d}\right]=\frac{1}{4\pi\varepsilon R^3}\left[3\frac{\vec{R}\cdot\vec{P}}{R^2}\vec{R}-\vec{P}\right]$$

$$\left(\quad \vec{p}=\hat{z}\,p=p\left(\hat{a}_R\cos\theta-\hat{a}_\theta\sin\theta\right)\qquad \vec{R}\cdot\vec{p}=p\cos\theta\quad\right)$$

$$\Rightarrow \vec{E}=\frac{p}{4\pi\varepsilon R^3}\left(\hat{a}_R\,2\cos\theta+\hat{a}_\theta\sin\theta\right)$$

Eg. At what value of θ does the electric field intensity of a z-directed dipole have no z-component.

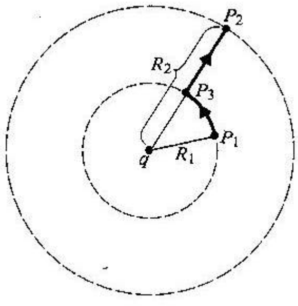
(Sol.) $\vec{E}=\dfrac{\vec{p}}{4\pi\varepsilon_0 R^3}(\hat{a}_r\,2\cos\theta+\hat{a}_\theta\sin\theta)$, $\hat{z}=\hat{a}_r\cos\theta-\hat{a}_\theta\sin\theta$

No z-component $\Rightarrow 2\cos\theta\cdot\cos\theta-\sin\theta\cdot\sin\theta=0\Rightarrow\tan^2\theta=2\Rightarrow\theta=54.7°$ or $125.3°$

1-2 Static Electric Potentials

$$\vec{E} = -\nabla V \quad \Rightarrow V_2 - V_1 = -\int_{p_1}^{p_2} \vec{E} \cdot d\vec{l} \quad \text{and} \quad \nabla^2 V = -\rho/\varepsilon$$

Electric potential due to a point charge:



$$V = -\int_{\infty}^{R} \hat{a}_R \frac{q}{4\pi\varepsilon R^2} \cdot \hat{a}_R \, dR = \frac{q}{4\pi\varepsilon R}$$

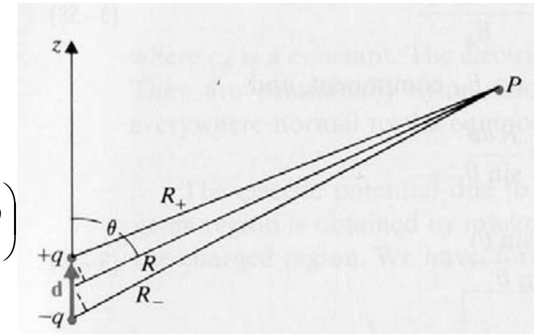
$$V_{21} = V_{p_2} - V_{p_1} = \frac{q}{4\pi\varepsilon}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

Electric potential due to discrete charges:

$$V = \frac{1}{4\pi\varepsilon} \sum_{k=1}^{n} \frac{q_k}{|R - R'_k|}$$

Electric potential due to an electric dipole:

$$V = \frac{q}{4\pi\varepsilon}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$



If $d \ll R$, we have

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

and $$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right)$$

$$V = \frac{qd\cos\theta}{4\pi\varepsilon R^2}, \quad \vec{p} = q\vec{d} \Rightarrow V = \frac{\vec{p} \cdot \hat{a}_R}{4\pi\varepsilon R^2} \quad (V)$$

$$\Rightarrow \vec{E} = -\nabla V = -\hat{a}_R \frac{\partial V}{\partial R} - \hat{a}_\theta \frac{\partial V}{\partial \theta} = \frac{p}{4\pi\varepsilon R^3}(\hat{a}_R \, 2\cos\theta + \hat{a}_\theta \sin\theta)$$

Scalar electric potential due to various charge distributions:

Volume source $\rho \Rightarrow V = \frac{1}{4\pi\varepsilon} \iiint_{v'} \frac{\rho}{R} dv'$.

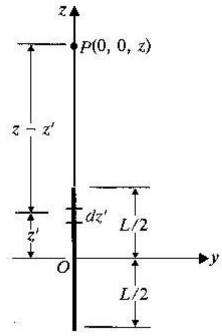
Surface source $\rho_s \Rightarrow V = \frac{1}{4\pi\varepsilon} \iint_{s'} \frac{\rho_s}{R} dS'$

Line source $\rho_l \Rightarrow V = \frac{1}{4\pi\varepsilon} \int \frac{\rho_l}{R} dl'$

Note: 1. V is a scalar, but $\vec{E}$ is a vector.

2. $\vec{E} = -\nabla V$ is valid only in the static EM field.

Eg. Obtain a formula for electrical field intensity along the axis of a uniform line charge of length L. The uniform line-charge density is ρ_l. [高考]



(Sol.) $R = z - z'$, $\quad z > \dfrac{L}{2}$

$$V = \frac{\rho_l}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{dz'}{z-z'} = \frac{\rho_l}{4\pi\varepsilon_0}\ln\left[\frac{z+(L/2)}{z-(L/2)}\right], \quad z > \frac{L}{2}$$

$$\vec{E} = -\hat{z}\frac{dV}{dz} = \hat{z}\frac{\rho_l L}{4\pi\varepsilon_0\left[z^2-(L/2)^2\right]}, \quad z > \frac{L}{2}$$

Eg. A finite line charge of length L carrying uniform line charge density ρ_l is coincident with the x-axis. Determine V and $\vec{E}$ in the plane bisecting the line charge.

(Sol.) $$V = \int_{-L/2}^{L/2}\frac{\rho_l dx}{4\pi\varepsilon_0\sqrt{x^2+y^2}} = \frac{\rho_l}{2\pi\varepsilon_0}\left\{\ln\left[\sqrt{\left(\frac{L}{2}\right)^2+y^2}+\frac{L}{2}\right]-\ln y\right\}$$

and $$\vec{E} = -\nabla V = \hat{y}\frac{\rho_l}{2\pi\varepsilon_0 y}\left[\frac{L/2}{\sqrt{(L/2)^2+y^2}}\right]$$

Eg. A charge is distributed uniformly over an $L\times L$ square plate. Determine V and $\vec{E}$ at a point on the axis perpendicular to the plate and through its center.

(Sol.) $\rho_s = \dfrac{Q}{L^2}$, $\quad y^2 \to y^2+z^2$, $\quad V = \dfrac{\rho_s}{2\pi\varepsilon_0}\displaystyle\int_{-L/2}^{L/2}\left\{\ln\left[\sqrt{\left(\frac{L}{2}\right)^2+y^2+z^2}+\frac{L}{2}\right]-\ln\sqrt{y^2+z^2}\right\}dy$

$$= \frac{Q}{\pi\varepsilon_0 L^2}\left\{\frac{L}{2}\ln\left[\frac{\sqrt{2\left(\frac{L}{2}\right)^2+z^2}+\frac{L}{2}}{\sqrt{2\left(\frac{L}{2}\right)^2+z^2}-\frac{L}{2}}\right]-z\cdot\tan^{-1}\left[\frac{\left(\frac{L}{2}\right)^2}{z\sqrt{2\left(\frac{L}{2}\right)^2+z^2}}\right]\right\}$$

$$\vec{E} = -\nabla V = \hat{z}\frac{Q}{\pi\varepsilon_0 L^2}\tan^{-1}\left[\frac{\left(\frac{L}{2}\right)^2}{z\sqrt{2\left(\frac{L}{2}\right)^2+z^2}}\right]$$

Eg. A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine $\vec{E}$ and V as functions of the radial distance R. [高考]

(Sol.) $R>R_o$, $\oint\!\!\oint_s \vec{E}\cdot d\vec{S} = E4\pi R^2 = \dfrac{Q}{\varepsilon_0}$, $E = \dfrac{Q}{4\pi\varepsilon_0 R^2}$, $V = -\int_\infty^R EdR = \dfrac{Q}{4\pi\varepsilon_0 R}$

$R_i<R<R_o$, $E=0$, $V = V\Big|_{R=R_0} = \dfrac{Q}{4\pi\varepsilon_0 R_o}$

$R<R_i$, $E = \dfrac{Q}{4\pi\varepsilon_0 R^2}$, $V = -\int EdR + C = \dfrac{Q}{4\pi\varepsilon_0 R} + C$

$$C = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_o} - \frac{1}{R_i}\right) \Rightarrow V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i}\right)$$

Eg. A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine $\vec{E}$ and V on its axis (a) at a point outside the tube, (b) at a point inside the tube.

(Sol.) $$dV = \int_0^{2\pi} \frac{\rho_s b d\phi' dz'}{4\pi\varepsilon_0\sqrt{b^2+(z-z')^2}} = \frac{\rho_s b dz'}{2\varepsilon_0\sqrt{b^2+(z-z')^2}}$$

(a) $$V_o = \int_0^h \frac{\rho_s b dz'}{2\varepsilon_0\sqrt{b^2+(z-z')^2}} = \frac{b\rho_s}{2\varepsilon_0}\ln\frac{b+\sqrt{b^2+z^2}}{(z-h)+\sqrt{b^2+(z-h)^2}}, \quad \rho_s = \frac{Q}{2\pi bh}$$

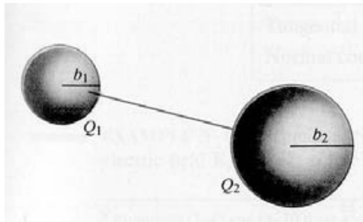
$$\vec{E}_o = -\hat{z}\frac{dV}{dz} = \hat{z}\frac{b\rho_s}{2\varepsilon_0}\left[\frac{1}{\sqrt{b^2+(z-h)^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]$$

(b) $$V_i = \int_0^z \frac{\rho_s b dz'}{2\varepsilon_0\sqrt{b^2+(z-z')^2}} + \int_z^h \frac{\rho_s b dz'}{2\varepsilon_0\sqrt{b^2+(z'-z)^2}}$$

$$= \frac{\rho_s b}{2\varepsilon_0}\ln\frac{\rho_s}{b^2}\left(z+\sqrt{b^2+z^2}\right)\cdot\left[(h-z)+\sqrt{b^2+(h-z)^2}\right]$$

Eg. Consider two spherical conductors with radii b_1 and b_2 (b_2>b_1) that are connected wire. The distance between the conductors is very large in comparison to b_2 so that charges on spherical conductors may be considered uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces?

(Sol.)

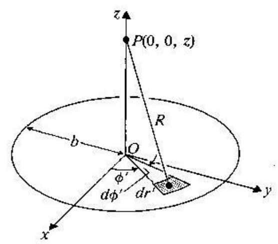


(a) $\dfrac{Q_1}{4\pi\varepsilon_0 b_1} = \dfrac{Q_2}{4\pi\varepsilon_0 b_2}$, $\dfrac{Q_1}{Q_2} = \dfrac{b_1}{b_2}$, $Q_1 + Q_2 = Q$

$$Q_1 = \frac{b_1}{b_1 + b_2} Q \quad and \quad Q_2 = \frac{b_2}{b_1 + b_2} \mathrm{Q}$$

(b) $\mathrm{E}_{1n} = \dfrac{Q_1}{4\pi\varepsilon_0 b_1^2}$ $\quad and \quad$ $E_{2n} = \dfrac{Q_2}{4\pi\varepsilon_0 b_2^2}$, $\dfrac{E_1}{E_2} = \left(\dfrac{b_2}{b_1}\right)^2 \dfrac{Q_1}{Q_2} = \dfrac{b_2}{b_1}$

Eg. Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries uniform surface charge density ρ_s. [高考]



(Sol.) $ds' = r'dr'd\phi'$, $R = \sqrt{z^2 + r'^2}$

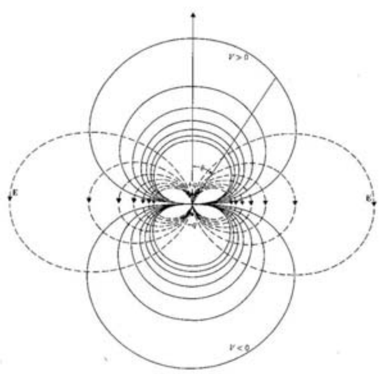
$$V = \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{b} \frac{r'}{\left(z^2 + r'^2\right)^{1/2}} dr'\, d\phi' \quad = \frac{\rho_s}{2\varepsilon_0}\left[\left(z^2 + b^2\right)^{1/2} - |z|\right]$$

$$\vec{E} = -\nabla V = -\hat{z}\frac{\partial V}{\partial z} \quad = \begin{cases} \hat{z}\dfrac{\rho_s}{2\varepsilon_0}\left[1 - z\left(z^2 + b^2\right)^{-1/2}\right], & z > 0 \\ -\hat{z}\dfrac{\rho_s}{2\varepsilon_0}\left[1 + z\left(z^2 + b^2\right)^{-1/2}\right], & z < 0 \end{cases}$$

As z>>1 $\Rightarrow z\left(z^2 + b^2\right)^{-1/2} \cong 1 - \dfrac{b^2}{2z^2}$, $\vec{E} = \hat{z}\dfrac{\left(\pi b^2 \rho_s\right)}{4\pi\varepsilon_0 z^2} = \begin{cases} \hat{z}\dfrac{Q}{4\pi\varepsilon_0 z^2} & z > 0 \\ -\hat{z}\dfrac{Q}{4\pi\varepsilon_0 z^2} & z < 0 \end{cases}$

Eg. Make a two-dimensional sketch of the equipotential lines and the electric field lines for an electric dipole.

(Sol.)



For an electric dipole, $V = \dfrac{qd\cos\theta}{4\pi\varepsilon_0 R^2}$ =constant $\Rightarrow R = c_v\sqrt{\cos\theta}$

$d\vec{l} = k\vec{E}$, where k is a constant.

$$\hat{a}_R\, dR + \hat{a}_\theta\, Rd\theta + \hat{a}_\phi\, R\sin\theta d\phi = k\left(\hat{a}_R\, E_R + \hat{a}_\theta\, E_\theta + \hat{a}_\phi\, E_\phi\right)$$

$$\frac{dR}{E_R} = \frac{Rd\theta}{E_\theta} = \frac{R\sin\theta d\phi}{E_\phi},\quad \frac{dR}{2\cos\theta} = \frac{Rd\theta}{\sin\theta},\quad R = c_E \sin^2\theta$$

1-3 Magnetic Fields

Magnetic field: $\vec{H}=\dfrac{\vec{B}}{\mu}=\dfrac{\vec{B}}{\mu_0}-\vec{M}$, where $\mu_0=4\pi\times10^{-7}$ (A/m) in the free space.

Magnetic flux density: $\vec{B}=\dfrac{\mu_0 I}{4\pi}\oint_C \dfrac{d\ell\times\hat{a}_R}{R^2}$

Ampere's law of $\vec{H}$: $\oint \vec{H}\cdot\mathrm{d}\vec{l}=I \Leftrightarrow \nabla\times\vec{H}=\vec{J}$

$\dfrac{1}{\mu_o}\nabla\times\vec{B}=\vec{J}+\vec{J}_m=\vec{J}+\nabla\times\vec{M}$ or $\nabla\times(\dfrac{\vec{B}}{\mu_0}-\vec{M})=\vec{J}$,

$\vec{H}=\dfrac{\vec{B}}{\mu_0}-\vec{M}\Rightarrow\oint\vec{H}\cdot d\vec{l}=I$, and $\vec{B}=\mu_0(\vec{H}+\vec{M})=\mu_0(1+\chi_m)\vec{H}=\mu_0\mu_r\vec{H}$,

where $\mu_r=1+\chi_m=\dfrac{\mu}{\mu_0}$

Gauss's law of $\vec{B}$: $\nabla\cdot\vec{B}=0\Leftrightarrow \oiint_S \vec{B}\cdot\mathrm{d}\vec{S}=0$

$\because$ $\nabla\cdot\vec{B}=0$, $\exists\vec{A}$ fulfills $\vec{B}=\nabla\times\vec{A}$

$$\nabla\times\vec{B}=\mu\vec{J}=\nabla\times\nabla\times\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

Choose $\nabla\cdot\vec{A}=0\Rightarrow\nabla^2\vec{A}=-\mu\vec{J}$ (**Note:** $\nabla^2 V=-\dfrac{\rho}{\varepsilon}$ is scalar Poisson's equation)

$\because$ $V=\dfrac{1}{4\pi\varepsilon}\iiint_{V'}\dfrac{\rho}{R}dv'$, $\therefore$ $\vec{A}=\dfrac{\mu}{4\pi}\iiint_{V'}\dfrac{\vec{J}}{R}dv'$ (Wb/m)

Magnetic flux: $\boxed{\Phi=\iint_S\vec{B}\cdot d\vec{S}=\iint_S(\nabla\times\vec{A})\cdot d\vec{S}=\oint_C\vec{A}\cdot d\vec{l}' \qquad (Wb)}$

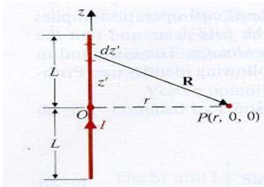
Biot-Savart's law: $\vec{B}=\dfrac{\mu I}{4\pi}\oint\dfrac{d\vec{l}'\times\hat{a}_R}{R^2}=\dfrac{\mu I}{4\pi}\oint_{C'}\dfrac{d\vec{l}'\times\vec{R}}{R^3}$

$\vec{A}=\dfrac{\mu}{4\pi}\iiint_{V'}\dfrac{\vec{J}}{R}\mathrm{d}v'=\dfrac{\mu I}{4\pi}\oint_{C'}\dfrac{d\vec{l}}{R}$, $\because$ $\nabla\times(f\vec{G})=f\nabla\times\vec{G}+(\nabla f)\times\vec{G}$),

$$\therefore\ \vec{B}=\nabla\times\vec{A}=\nabla\times\left[\frac{\mu I}{4\pi}\oint_{C'}\frac{d\vec{l}'}{R}\right]=\frac{\mu I}{4\pi}\oint_{C'}\nabla\times(\frac{d\vec{l}'}{R})=\frac{\mu I}{4\pi}\oint\left[\frac{1}{R}\nabla\times dl'+(\nabla\frac{1}{R})\times dl'\right]=\frac{\mu I}{4\pi}\oint_{C'}\frac{d\vec{l}'\times\hat{a}_R}{R^2}$$

Note: $\begin{cases} \nabla = \hat{x}\dfrac{\partial}{\partial x} + \hat{y}\dfrac{\partial}{\partial y} + \hat{z}\dfrac{\partial}{\partial z} \\ d\vec{l}' = \hat{x}dx' + \hat{y}dy' + \hat{z}dz' \Rightarrow \nabla \times d\vec{l}' = 0 \end{cases}$, and then $d\vec{B} = \dfrac{\mu I}{4\pi}\left(\dfrac{d\vec{l}' \times \hat{a}_R}{R^2}\right) = \dfrac{\mu I}{4\pi}\dfrac{d\vec{l}' \times \vec{R}}{R^3}$

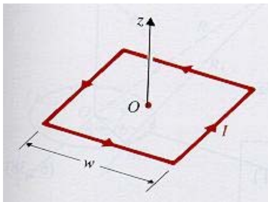
Eg. A direct current *I* flows in a straight wire of length 2*L*. Find the magnetic flux density $\vec{B}$ at a point located at a distance *r* from the wire in the bisecting plane.



(Sol.) $d\vec{l}' \times \vec{R} = \hat{z}dz' \times (\hat{a}_r r - \hat{z}z') = \hat{a}_\phi r dz'$, $R = (z^2 + r^2)^{1/2}$

$$\vec{B} = \hat{a}_\phi \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{r dz'}{(z'^2 + r^2)^{3/2}} = \hat{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2 + r^2}}$$

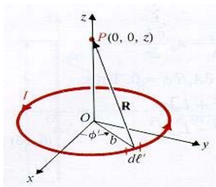
Eg. Find the magnetic flux density at the center of a square loop, with side *w* carrying a direct current *I*.



(Sol.) $L = \dfrac{w}{2}$, $r = \dfrac{w}{2}$ in this case,

$$\vec{B} = 4 \times \hat{z}\frac{\mu_0 I \dfrac{w}{2}}{2\pi\dfrac{w}{2}\sqrt{(\dfrac{w}{2})^2 + (\dfrac{w}{2})^2}} = \hat{z}\frac{2\sqrt{2}\mu_0 I}{\pi w}$$

Eg. Find the magnetic flux density at a point on the axis of a circular loop of radius *b* that a direct current *I*.



(Sol.) $d\vec{l}' = \hat{a}_\phi b d\phi'$, $\vec{R} = \hat{z}z - \hat{a}_r b$, $R = (z^2 + b^2)^{1/2}$

$$d\vec{l}' \times \vec{R} = \hat{a}_r bz d\phi' + \hat{z}b^2 d\phi'$$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{\hat{a}_r bz + \hat{z}b^2}{(z^2 + b^2)^{3/2}}d\phi' = \hat{z}\frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}}$$. In case of z=0, $\vec{B} = \hat{z}\dfrac{\mu_0 I}{2b}$

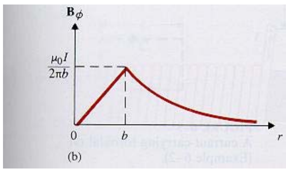
Eg. Determine the magnetic flux density at a point on the axis of a solenoid with radius *b* and length *L*, and with a current in its *N* turns of closely wound coil.

(Sol.) $d\vec{B} = \dfrac{\hat{z}\mu_0 I b^2}{2[(z - z')^2 + b^2]^{\frac{3}{2}}}\left(\dfrac{N}{L}\right)dz'$

$$\vec{B} = \int_0^L d\vec{B} = \frac{\mu_o NI}{2L}\left[\frac{L - z}{\left[(L - z)^2 + b^2\right]^{1/2}} + \frac{z}{\sqrt{z^2 + b^2}}\right] = \frac{\mu_o NI}{2L}\left[\frac{z}{\sqrt{z^2 + b^2}} - \frac{z - L}{\sqrt{(z - L)^2 + b^2}}\right]$$

Ampere's law of $\vec{B}$: $\nabla\times\vec{B}=\mu\vec{J}\Leftrightarrow\oint_c\vec{B}\cdot d\vec{l}=\mu I$, where $\mu=\mu_0$ in the free space.

Eg. An infinitely long, straight conductor with a circular cross section of radius *b* carries a steady current *I*. Determine the magnetic flux density both inside and outside the conductor. [交大光電所]



(Sol.)

(a) Inside the conductor, $r\le b$:

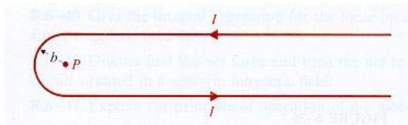
$$\oint_{C1}\vec{B}\cdot d\vec{l}=\int_0^{2\pi}Brd\phi=2\pi rB=\mu_o\left(\frac{\pi r^2}{\pi b^2}\right)I=\mu_o\left(\frac{r}{b}\right)^2 I \Rightarrow \vec{B}=\hat{a}_\phi\frac{\mu_o rI}{2\pi b^2}$$

(b) Outside the conductor: $\oint_{C2}\vec{B}\cdot d\vec{l}=2\pi rB=\mu_o I\Rightarrow\vec{B}=\hat{a}_\phi\frac{\mu_o I}{2\pi r}$

Eg. A long line carrying a current *I* folds back with semicircular bend of radius *b*. Determine magnetic flux density at the center point *P* of the bend. [高考]

(Sol.)

$\vec{B}=\vec{B_1}+\vec{B_2}$, where $\vec{B_1}=2\cdot\hat{z}\frac{\mu_o I}{4\pi b}$, $\vec{B_2}=\hat{z}\frac{\mu_o I}{4b}$



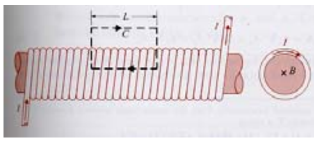
Eg. A current *I* flows in the inner conductor of an infinitely long coaxial line and returns via the outer conductor. The radius of the inner conductor is *a*, and the inner and outer radii of the outer conductor are *b* and *c*, respectively. Find the magnetic flux density $\vec{B}$ for all regions and plot $|\vec{B}|$ versus *r*. [高考電機技師]

(Sol.) $0\le r\le a$, $\vec{B}=\hat{a}_\phi\frac{\mu rI}{2\pi a^2}$, $a\le r\le b$, $\vec{B}=\hat{a}_\phi\frac{\mu I}{2\pi r}$

$b\le r\le c$, $\vec{B}=\hat{a}_\phi\left(\frac{c^2-r^2}{c^2-b^2}\right)\frac{\mu I}{2\pi r}$

Eg. Determine the magnetic flux density inside an infinitely long solenoid with air core having *n* closely wound turns per unit length and carrying a current *I*.

(Sol.) $BL=\mu_o nLI$ => $B=\mu_o nI$

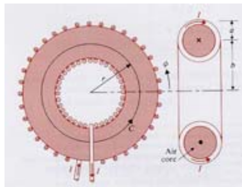


Eg. The figure shows an infinitely long solenoid with air core having *n* closely wound turns per unit length. The windings are slanted at an angle *α* and carry a current *I*. Determine the magnetic flux density both inside and outside the solenoid.

(Sol.) $\vec{B_1}=\begin{cases}0, & 0<r<b\\ \hat{a}_\varphi\dfrac{2\pi\mu_0 bnI\sin\alpha}{2\pi r}=\hat{a}_\phi\dfrac{\mu_0 bnI\sin\alpha}{r}, & r>b\end{cases}$,

$\vec{B_2}=\begin{cases}\hat{z}\mu_o nI\cos\alpha, & 0<r<b\\ 0, & r>b\end{cases}$, $\vec{B}=\vec{B_1}+\vec{B_2}$

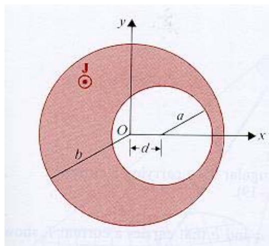
Eg. Determine the magnetic flux density inside a closely wound toroidal coil with an air core having *N* turns and carrying a current *I*. The toroid has a mean radius *b*, and the radius of each turn is *a*.



(Sol.) $\oint \vec{B}\cdot d\vec{l}=2\pi rB=\mu_0 NI$

$\vec{B}=\hat{a}_\phi B=\hat{a}_\phi\dfrac{\mu_0 NI}{2\pi r}$, $(b-a)<r<(b+a)$, $\vec{B}=0$ for $r<(b-a)$ and $r>(b+a)$

Eg. In certain experiments it is desirable to have a region of constant magnetic flux density. This can be created in an off-center cylindrical cavity. The uniform axial current density is $\vec{J}=\hat{z}J$. Find the magnitude and direction of $\vec{B}$ in the cylindrical cavity whose axis is displaced from that of the conducting part by a distance *d*. [台大電研、清大電研、中原電機]



(Sol.) $\vec{J}=\hat{z}J$, $\oint \vec{B}\cdot d\vec{l}=\mu_o I$

If no hole exists,

$$2\pi r_1 B_{\phi 1}=\mu_o\pi r_1^2 J\Rightarrow B_{\phi 1}=\frac{\mu_o r_1}{2}J\Rightarrow\begin{cases}B_{x1}=\dfrac{-\mu_o J}{2}y_1\\ B_{y1}=\dfrac{\mu_o J}{2}x_1\end{cases}$$

For $-\vec{J}$ in the hole potion, $B_{\phi 2}=-\dfrac{\mu_o r_2}{2}J\Rightarrow\begin{cases}B_{x2}=\dfrac{\mu_o J}{2}y_2\\ B_{y2}=-\dfrac{\mu_o J}{2}x_2\end{cases}$

At $y_1=y_2$ and $x_1=x_2+d\Rightarrow B_x=B_{x1}+B_{x2}=0$, and $B_y=B_{y1}+B_{y2}=\dfrac{\mu_o J}{2}d$

1-4 Electromagnetic Forces

Lorentz force equation: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Electric force: $\vec{F}_e = q\vec{E}$. **Magnetic force:** $\vec{F}_m = q\vec{v} \times \vec{\mathrm{B}}$

Eg. An electron is injected with an initial velocity $\vec{v_o} = \hat{y}v_o$ into a region where both an electric field and a magnetic field $\vec{B}$ exist. Describe the motion of the electron if $\vec{E} = \hat{z}E_o$ and $\vec{B} = \hat{x}B_o$. Discuss the effect of the relative magnitude of E_0 and B_0 on the electron paths in parts.

(Sol.) $m\dfrac{\partial \vec{v}}{\partial t} = -e(\vec{E} + \vec{v} \times \vec{B})$,

$$\begin{cases} \vec{E} = \hat{z}E_o \\ \vec{B} = \hat{x}B_o \end{cases} \Rightarrow \begin{cases} \dfrac{\partial v_x}{\partial t} = 0 \\ \dfrac{\partial v_y}{\partial t} = -\dfrac{e}{m}B_o v_z \\ \dfrac{\partial v_z}{\partial t} = -\dfrac{e}{m}(E_o - B_o v_y) \end{cases} \Rightarrow \begin{cases} v_x = 0 \\ v_y = (v_o - \dfrac{E_o}{B_o})\cos\omega_0 t + \dfrac{E_o}{B_o} \\ v_z = (\dfrac{E_o}{R_o} - v_o)\sin\omega_0 t ; \omega_0 = \dfrac{e}{m}B_o \end{cases}$$

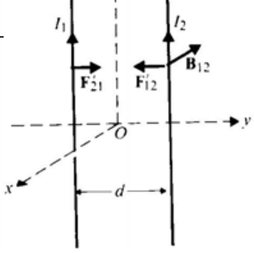
$$(\text{If } v_o \neq \frac{E_o}{B_o}) \Rightarrow \begin{cases} x = 0 \\ y = \dfrac{c_2}{\omega_o}\sin\omega_o t + \dfrac{E_o}{B_o}t \\ z = -\dfrac{c_2}{\omega_o}(1 - \cos\omega_o t), \ c_2 = v_o - \dfrac{E_o}{B_o} \end{cases} \Rightarrow (y - \frac{E_o}{B_o}t)^2 + (z + \frac{c_2}{\omega_o})^2 = (\frac{c_2}{\omega_o})^2$$

Magnetic force due to $\vec{B}$ and I:

$$F_m = q\vec{v} \times \vec{B} \Rightarrow dF_m = dq\frac{d\vec{\ell}}{dt} \times \vec{B} = \frac{dq}{dt}d\vec{\ell} \times \vec{B} = Id\vec{\ell} \times \vec{B}, \ \therefore \ F_m = I\oint_C d\vec{\ell} \times \vec{B}$$

Eg. Determine the force per unit length between two infinitely long parallel conducting wires carrying currents I_1 and I_2 in the same direction. The wires are separated by a distance d. [清大電研]

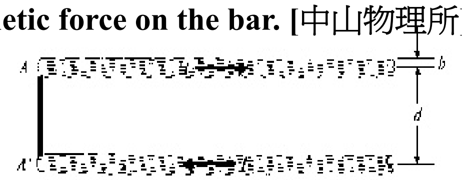
(Sol.) $\vec{F}_{12}' = I_2(\hat{z} \times \vec{B}_{12})$, $\quad \vec{B}_{12} = -\hat{x}\dfrac{\mu_0 I_1}{2\pi d} \Rightarrow \vec{F}_{12}' = -\hat{y}\dfrac{\mu_0 I_1 I_2}{2\pi d}$



Eg. Calculate the force per unit length on each of three equidistant, infinitely long, parallel wires d apart, each carrying a current of I in the same direction. Specify the direction of the force.

(Sol.) $I_1 = I_2 = I_3 = I$, $\vec{B}_2 = \hat{x}2B_{12}\cos 30^\circ = \hat{x}\dfrac{\sqrt{3}\mu_0 I}{2\pi d}$, $\vec{f}_2 = -\hat{z}I \times \vec{B}_2 = -\hat{y}IB_2 = -\hat{y}\dfrac{\sqrt{3}\mu_0 I^2}{2\pi d}$

Eg. The bar AA', serves as a conducting path for the current I in two very long parallel lines. The lines have a radius b and are spaced at a distance d apart. Find the direction and the magnitude of the magnetic force on the bar. [中山物理所]

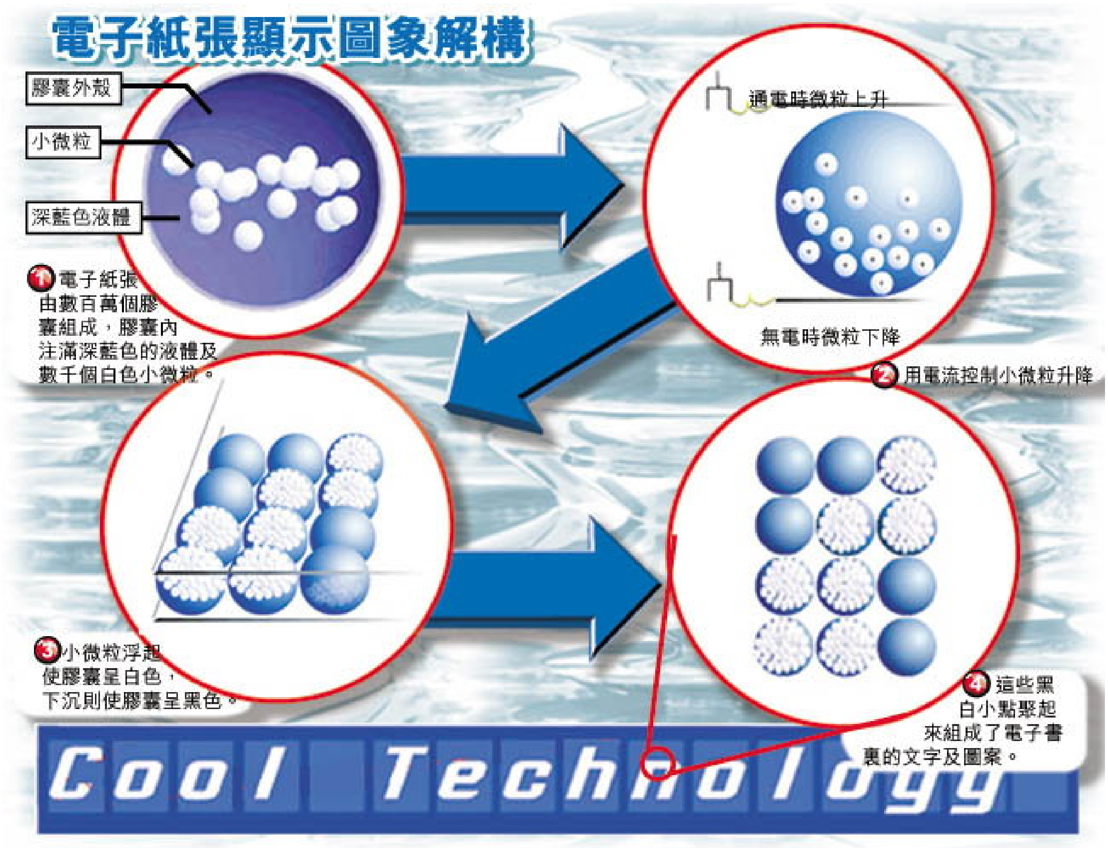


(Sol.)

$$\vec{B} = -\hat{z}\frac{\mu_0 I}{4\pi}\left(\frac{1}{y} + \frac{1}{d-y}\right), \qquad d\vec{\ell} = \hat{y}dy$$

$$\Rightarrow d\vec{F} = Id\vec{\ell} \times \vec{B} = -\hat{x}\frac{\mu_0 I^2}{4\pi}\left(\frac{1}{y} + \frac{1}{d-y}\right)dy \Rightarrow \vec{F} = \int_b^{d-b} d\vec{F} = -\hat{x}\frac{\mu_0 I^2}{2\pi}\ln\left(\frac{d}{b} - 1\right)$$

Application of the electric forces: The *e*-paper



Scalar electric potential function: $V = \frac{1}{4\pi\varepsilon}\iiint_{V'} \frac{\rho}{R}\, dv'$

Vector magnetic potential function: $\vec{A} = \frac{\mu}{4\pi}\iiint_{V'} \frac{\vec{J}}{R}\, dv'$

Retarded potentials:

$$V(R,t) = \frac{1}{4\pi\varepsilon}\iiint_{V'} \frac{\rho(t - R/v)}{R}\, dv', \quad \vec{A}(R,t) = \frac{\mu}{4\pi}\iiint_{V'} \frac{\vec{J}(t - R/v)}{R}\, dv'$$

1-5 Faraday's Law and Magnetic Dipoles

Faraday's law: $\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ or $\displaystyle\oint_C \vec{E}\cdot d\vec{\ell}=-\iint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$

$$\because\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\nabla\times\vec{A})\Rightarrow\nabla\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$$

$$\therefore\ \exists\, V \text{ fulfills } \vec{E}+\frac{\partial\vec{A}}{\partial t}=-\nabla V\Rightarrow\vec{E}=-\nabla V-\frac{\partial\vec{A}}{\partial t}$$

Note: In static field: $\vec{E}=-\nabla V$, but in time-varying field: $\vec{E}=-\nabla V-\dfrac{\partial\vec{A}}{\partial t}$

Emf: $V=\oint\vec{E}\cdot d\vec{l}$. **Magnetic flux:** $\displaystyle\Phi=\oiint_S\vec{B}\cdot d\vec{S}$, $\displaystyle\oint_C\vec{E}\cdot d\vec{\ell}=-\iint_S\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}\Rightarrow V=-\frac{d\Phi}{dt}$

Motional *emf*: $\displaystyle V'=\oint_C(\vec{v}\times\vec{B})\cdot d\vec{l}$ $(Volt)$

$$\vec{F}_m=q\vec{v}\times\vec{B}\Rightarrow\frac{\vec{F}_m}{q}=\vec{v}\times\vec{B}\equiv-\vec{E}_m\Rightarrow V'=-\oint_C\vec{E}_m\cdot d\vec{l}=\oint(\vec{v}\times\vec{B})\cdot d\vec{l}$$

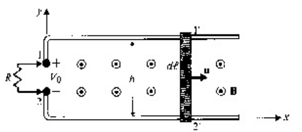
Eg. A circular loop of *N* turns of conducting wire line in the *xy*-plane with its center at the origin of a magnetic field specified by $\vec{B}=\hat{z}B_0\cos(\pi r/2b)\sin\omega t$, where *b* is the radius of the loop and *ω* is the angular frequency. Find the *emf* induced in the loop.

(Sol.) $$\Phi=\int_0^b[\hat{z}B_0\cos(\frac{\pi r}{2b})\sin(\omega t)]\cdot[\hat{z}2\pi r dr]=\frac{8b^2}{\pi}(\frac{\pi}{2}-1)B_0\sin(\omega t)$$

$$V=-\frac{Nd\Phi}{dt}=-\frac{8N}{\pi}b^2(\frac{\pi}{2}-1)B_0\omega\cos(\omega t)$$

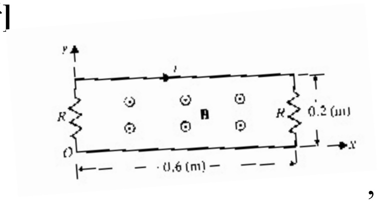
Eg. A metal bar slides over a pair of conducting rails in a uniform magnetic field $\vec{B}=\hat{z}B_0$ with a constant velocity *v*. (a) Determine the open-circuit voltage V_0 that appears across terminals 1 and 2. (b) Assuming that a resistance *R* is connected between the two terminals, find the electric power dissipated in *R*. Neglect the electric resistance of the metal bar and of the conducting rails. [交大電子物理所]

(Sol). (a) $V_0=V_1-V_2=\int_{2'}^{1'}(\hat{x}v\times\hat{z}B_0)\cdot(\hat{y}dl)=-vB_0h$ (V)



(b) $P_e = I^2 R = (\frac{vB_0h}{R})^2 R = \frac{(vB_0h)^2}{R} \quad (W)$

Eg. The circuit in Fig. is situated in a magnetic field $\vec{B} = \hat{z}3\cos(5\pi 10^7 t - \frac{2}{3}\pi x)$ μ*T*. Assuming *R*=15Ω, find the current *i*. [中山物理所]



(Sol.) $\Phi = \int_0^{0.6} 3\cos(5\pi \times 10^7 t - \frac{2}{3}\pi x) 10^{-6} \cdot (0.2dx)$

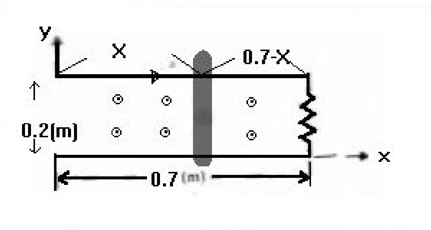
$V = -\frac{d\Phi}{dt} = 45[\cos(5\pi \times 10^7 t - \frac{2}{3}\pi 0.6) - \cos(5\pi 10^9 t)]$,

$i = \frac{V}{2R} = 1.76\sin(5\pi 10^7 t - 0.2\pi)$

Eg. A conducting sliding bar oscillates over two parallel conducting rails in a sinusoidally varying magnetic field $\vec{B} = \hat{z}5\cos(\omega t)$ *T*. The position of the sliding bar is given by *x*=0.35(1-cos*ωt*), and the rails are terminated in a resistance *R*=0.2 Ω. Find *i*.

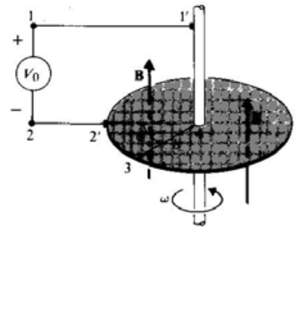
(Sol.) $\Phi = 5\cos\omega t \cdot 0.2(0.7 - x)$, *x*=0.35(1-cos*ωt*), $i = -\frac{1}{R}\frac{d\Phi}{dt}$

$\Rightarrow i = 1.75\omega \sin\omega t \cdot (1 + 2\cos\omega t)$



Eg. The Faraday disk generator consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic field of flux density $B = \hat{z}B_0$ that is parallel to the axis of rotation. Brush contacts are the open-circuit voltage of the generator if the radius of the disk is *b*.

(Sol.) $V_0 = \oint(\vec{v} \times \vec{B}) \cdot d\vec{l} = \int_3^4 [(\hat{a}_\phi r\omega) \times \hat{z}B_0] \cdot (\hat{a}_r dr) = \omega B_0 \int_b^0 r dr = \frac{\omega B_0 b^2}{2} \quad (V)$

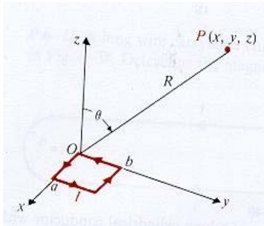


Magnetic dipole moment: $\vec{m} = \hat{z}IS = \hat{z}m$, where S is the area of the loop that carries I and $m=IS$.

Vector potential of a magnetic dipole: $\vec{A} = \dfrac{\mu_0 \vec{m}\times\hat{a}_R}{4\pi R^2}$, where

$$\vec{B} = \nabla\times\vec{A} = \frac{\mu_o m}{4\pi R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right)$$

Eg. For the small rectangular loop with sides *a* and *b* that carries a current *I*. Find the vector magnetic potential $\vec{A}$ at a distant point *P*(*x*,*y*,*z*). And determine the magnetic flux density $\vec{B}$ and $\vec{A}$. [交大光電所]



(Sol.)

$\vec{A} = \dfrac{\mu_0 \vec{m}\times\hat{a}_R}{4\pi R^2}$, where $m=Iab$, $\vec{B} = \nabla\times\vec{A} = \dfrac{\mu_o m}{4\pi R^3}\left(\hat{a}_R 2\cos\theta + \hat{a}_\theta \sin\theta\right)$

Magnetization vector: $\vec{M} = \lim\limits_{\Delta V\to 0}\dfrac{\sum_{k=1}^{\Delta V}\overrightarrow{m_k}}{\Delta V}$ (A/m), where $\vec{m}_k$ is the magnetic dipole moment of an atom.

$$d\vec{A} = \frac{\mu_o \overrightarrow{M}\times\hat{a}_R}{4\pi R^2}dv' = \frac{\mu_o}{4\pi}\overrightarrow{M}\times\nabla'(\frac{1}{R})dv' = \frac{\mu_o}{4\pi}\left[\frac{1}{R}\nabla'\times\overrightarrow{M} - \nabla'\times\left(\frac{\overrightarrow{M}}{R}\right)\right]$$

$$\Rightarrow \vec{A} = \frac{\mu_o}{4\pi}\iiint_{V'}\frac{\nabla'\times\overrightarrow{M}}{R}dv' - \frac{\mu_o}{4\pi}\iiint_{V'}\nabla\times\left(\frac{\overrightarrow{M}}{R}\right)dv' \qquad \left(\iiint_{V'}\nabla'\times\vec{F}\,\mathrm{d}v' = -\oint\!\!\oint_{s}\vec{F}\times d\vec{S}'\right)$$

$$= \frac{\mu_o}{4\pi}\iiint_{V'}\frac{\nabla'\times\overrightarrow{M}}{R}\mathrm{d}v' + \frac{\mu_o}{4\pi}\oint\!\!\oint\frac{\overrightarrow{M}\times\hat{a}_n'}{R}\mathrm{d}S'$$

∴ Magnetization volume current density: $\boxed{\vec{J}_m = \nabla\times\overrightarrow{M}}$ (A/m^2)

Magnetization surface current density: $\boxed{\vec{J}_{ms} = \overrightarrow{M}\times\hat{a}_n}$ (A/m)

Equivalent Magnetization Charge Densities:

$$V_m = \frac{1}{4\pi}\oint\!\!\oint_{S'}\frac{\overrightarrow{M}\cdot\hat{a}_n'}{R}\mathrm{d}S' + \frac{1}{4\pi}\iiint_{V'} - \frac{(\nabla'\cdot\overrightarrow{M})}{R}\mathrm{d}v'$$ (**Note:** $V = \dfrac{1}{4\pi\varepsilon_o}\oint\!\!\oint\dfrac{\vec{P}\cdot\hat{a}_n'}{R}dS' + \dfrac{1}{4\pi\varepsilon_o}\iiint_{V'}\dfrac{-\left(\nabla'\cdot\vec{P}\right)}{R}dv'$)

Define the magnetization surface charge density as $\boxed{\rho_{ms} = \overrightarrow{M}\cdot\hat{a}_n}$ and the magnetization volume charge density as $\boxed{\rho_m = -\nabla\cdot\overrightarrow{M}}$

Eg. A circular rod of magnetic material with permeability μ is inserted coaxially in the long solenoid. The radius a of the rod is less than the inner radius b of the solenoid. The solenoid's winding has n turns per unit length and carries a current I. (a) Find the values of $\vec{B}$, $\vec{H}$, and $\vec{M}$ inside the solenoid for $r<a$ and for $a<r<b$. (b) What are the equivalent magnetization current densities J_m and J_{ms} for the magnetized rod? [清大電研]

(Sol.) (a) $r<a$: $\vec{H}=\hat{z}nI$, $\vec{B}=\hat{z}\mu nI$, $\vec{M}=\dfrac{\vec{B}}{\mu_0}-\vec{H}=\hat{z}(\dfrac{\mu}{\mu_0}-1)nI$,

$a<r<b$: $\vec{H}=\hat{z}nI$, $\vec{B}=\hat{z}\mu_o nI$, $\vec{M}=0$

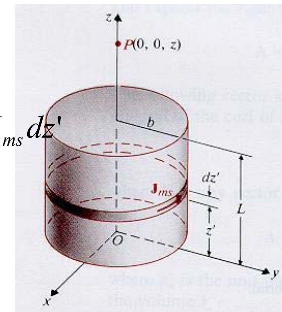
(b) $\vec{J}_m=\nabla'\times\vec{M}=0$, $\vec{J}_{ms}=\vec{M}\times\hat{a}_n=(\hat{z}\times\hat{a}_r)(\dfrac{\mu}{\mu_0}-1)nI=\hat{a}_\phi(\dfrac{\mu}{\mu_0}-1)nI$

Eg. A ferromagnetic sphere of radius b is uniformly magnetized with a magnetization $\vec{M}=\hat{z}M_0$. (a) Determine the equivalent magnetization current densities $\vec{J}_m$ and $\vec{J}_{ms}$. (b) Determine the magnetic flux density at the center of the sphere. [台大電研]

(Sol.) (a) $\vec{J}_m=\nabla'\times\vec{M}=0$, $\vec{J}_{ms}=(\hat{a}_R\cos\theta-\hat{a}_\theta\sin\theta)M_o\times\hat{a}_R=\hat{a}_\phi M_o\sin\theta$

(b) $d\vec{B}=\hat{z}\dfrac{\mu_o(J_{ms}b\,\mathrm{d}\theta)(b\sin\theta)^2}{2(b^2)^{3/2}}=\hat{z}\dfrac{\mu_o M_o}{2}\sin^3\theta$, $\vec{B}=\hat{z}\dfrac{\mu_o M_o}{2}\int_0^\pi\sin^3\theta\,\mathrm{d}\theta=\hat{z}\dfrac{2}{3}\mu_0 M_0$.

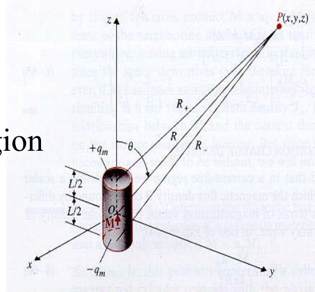
Eg. Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius b, length L and axial magnetization $\vec{M}=\hat{z}M_0$. [台大物研]



(Sol.) $\vec{J}_m=\nabla'\times\vec{M}=0$, $\vec{J}_{ms}=\vec{M}\times\hat{a}_n=(\hat{z}M_0)\times\hat{a}_r=\hat{a}_\phi M_0$, $dI=J_{ms}dz'$

$$\vec{B}=\hat{z}\int_0^L\frac{\mu_0 M_0 b^2 dz'}{2\left[(z-z')^2+b^2\right]^{3/2}}=\hat{z}\frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2+b^2}}-\frac{z-L}{\sqrt{(z-L)^2+b^2}}\right]$$

Eg. A cylindrical bar magnet of radius b and length L has a uniform magnetization $\vec{M} = \hat{z}M_0$ along its axis. Use the equivalent magnetization charge density concept to determine the magnetic flux density at an arbitrary distant point. [交大電子所]



(Sol.) $\rho_{ms} = \vec{M} \cdot \hat{a}_n = \begin{cases} M_0, & top \\ -M_0, & bottom \\ 0, & sidewall \end{cases}$, ρ_m=0 in the interior region

$$q_m = \pi b^2 \rho_{ms} = \pi b^2 M_0 \quad \Rightarrow V_m(x,y,z) = \frac{q_m}{4\pi}\left(\frac{1}{R+} - \frac{1}{R-}\right) \quad (A)$$

$$= \frac{q_m L\cos\theta}{4\pi R^2} = \frac{(\pi b^2 M_o)L\cos\theta}{4\pi R^2} = \frac{M_T \cos\theta}{4\pi R^2}, \text{ where } M_T = \pi b^2 L M_0$$

$$\vec{B} = -\mu_o \nabla V_m = \frac{\mu_o M_T}{4\pi R^3}(\hat{a}_R 2\cos\theta + a_\theta \sin\theta) \quad (T)$$

Consider an infinitely long solenoid with n turns per unit length around to create a magnetic field; a voltage $V_1 = -n\, d\Phi/dt$ is induced unit length, which opposes the current change. Power P_1=-$V_1 I$ per unit length must be supplied to overcome this induced voltage in order to increase the current to I. The work per unit volume required to produce a final magnetic flux density B_f is $W_1 = \int_0^{B_f} HdB$.