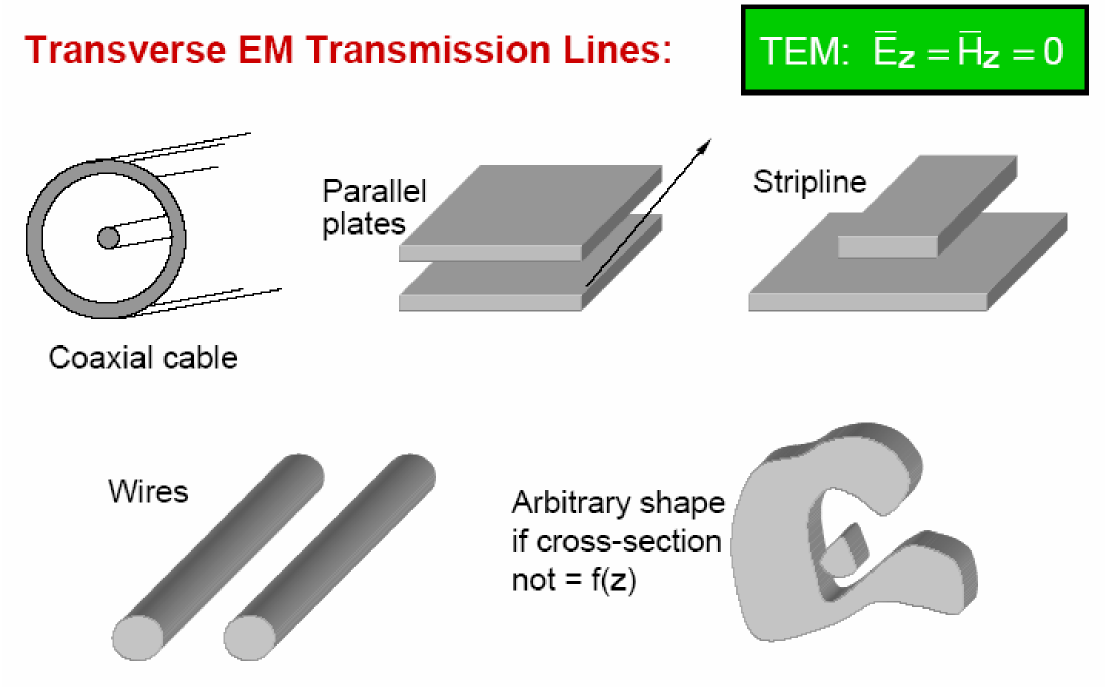
Chapter 5 Transmission Lines

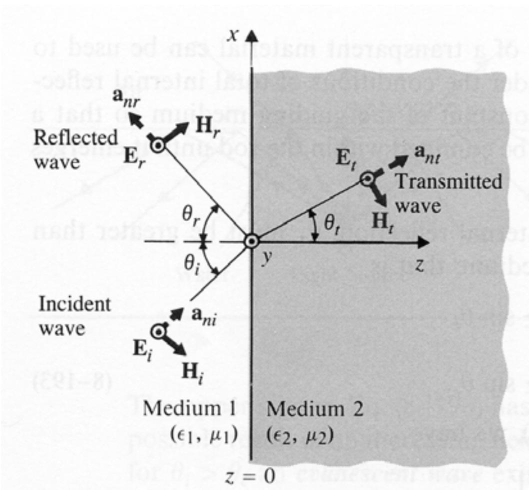


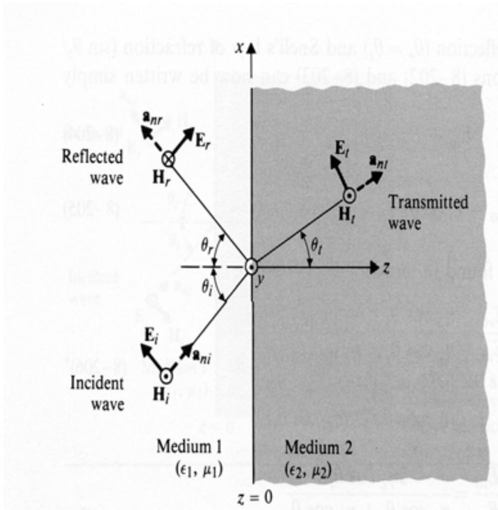
5-1 Characteristics of Transmission Lines

Transmission line: It has **two conductors** carrying current to support an EM wave, which is **TEM** or **quasi-TEM** mode. For the **TEM** mode, $\vec{E} = -Z_{TEM}\hat{a}_n \times \vec{H}$,

$$\vec{H} = \frac{1}{Z_{TEM}}\hat{a}_n \times \vec{E}, \text{ and } Z_{TEM} = \eta = \sqrt{\frac{\mu}{\varepsilon}}.$$

The current and the EM wave have different characteristics. An EM wave propagates into different dielectric media, the partial reflection and the partial transmission will occur. And it obeys the following rules.





Snell's law: $\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2} = \frac{v_{p2}}{v_{p1}} = \frac{\eta_2}{\eta_1} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$ and $\theta_i = \theta_r$

The reflection coefficient: $\Gamma = \dfrac{E_{r0}}{E_{i0}}$ and **the transmission coefficient:** $\tau = \dfrac{E_{t0}}{E_{i0}}$

$$\begin{cases} \Gamma_\perp = \dfrac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = \dfrac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} = \dfrac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)} \\ \tau_\perp = \dfrac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} = \dfrac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} = \dfrac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)} \end{cases}$$

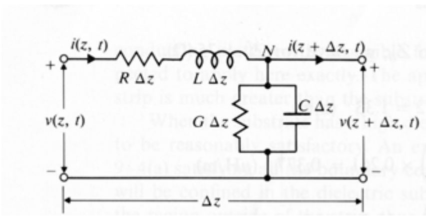
for perpendicular polarization (TE)

$$\begin{cases} \Gamma_\parallel = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \dfrac{n_1/\cos\theta_i - n_2/\cos\theta_t}{n_1/\cos\theta_i + n_2/\cos\theta_t} = \dfrac{\tan(\theta_t-\theta_i)}{\tan(\theta_t+\theta_i)} \\ \tau_\parallel = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \dfrac{2n_1/\cos\theta_t}{n_1/\cos\theta_i + n_2/\cos\theta_t} = \dfrac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \end{cases}$$

for parallel polarization (TM)

In case of normal incidence, $\begin{cases} \Gamma_\perp = \Gamma_{//} = \dfrac{\eta_2-\eta_1}{\eta_2+\eta_1} \\ \tau_\perp = \tau_{//} = \dfrac{2\eta_2}{\eta_2+\eta_1} \end{cases}$, where $\eta_1 = \sqrt{\dfrac{\mu_1}{\varepsilon_1}}$ and $\eta_2 = \sqrt{\dfrac{\mu_2}{\varepsilon_2}}$.

Equivalent-circuit model of transmission line section:



$R(\Omega/m)$, $L(H/m)$, $G(S/m)$, $C(F/m)$

Transmission line equations: In higher-frequency range, the transmission line model is utilized to analyze EM power flow.

$$\begin{cases} -\dfrac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = Ri(z,t) + L\dfrac{\partial i(z,t)}{\partial t} \\ -\dfrac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = Gv(z,t) + C\dfrac{\partial v(z,t)}{\partial t} \end{cases} \Rightarrow \begin{cases} -\dfrac{\partial v}{\partial z} = Ri + L\dfrac{\partial i}{\partial t} \\ -\dfrac{\partial i}{\partial z} = Gv + C\dfrac{\partial v}{\partial t} \end{cases}$$

Set $v(z,t)=Re[V(z)e^{j\omega t}]$, $i(z,t)=Re[I(z)e^{j\omega t}]$

$$\Rightarrow \begin{cases} -\dfrac{dV}{dz} = (R+j\omega L)I(z) \\ -\dfrac{dI}{dz} = (G+j\omega C)V(z) \end{cases} \Rightarrow \begin{cases} \dfrac{d^2V(z)}{dz^2} = (R+j\omega L)(G+j\omega C)V(z) = \gamma^2 V(z) \\ \dfrac{d^2I(z)}{dz^2} = (R+j\omega L)(G+j\omega C)I(z) = \gamma^2 I(z) \end{cases}$$

where $\gamma=\alpha+j\beta=\sqrt{(R+j\omega L)(G+j\omega C)} \Rightarrow V(z)=V_0^+e^{-\gamma z}+V_0^-e^{\gamma z}$, $I(z)=I_0^+e^{-\gamma z}+I_0^-e^{\gamma z}$

Characteristic impedance: $Z_0=\frac{V_0^+}{I_0^+}=-\frac{V_0^-}{I_0^-}=\frac{R+j\omega L}{\gamma}=\frac{\gamma}{G+j\omega C}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$

Note:

1. International Standard Impedance of a Transmission Line is Z_0=50Ω.
2. In transmission-line equivalent-circuit model, $G\neq 1/R$.

Eg. The following characteristics have been measured on a lossy transmission line at 100 MHz: $Z_0=50\Omega$, $\alpha=0.01dB/m=1.15\times10^{-3}Np/m$, $\beta=0.8\pi(rad/m)$. Determine R, L, G, and C for the line.

(Sol.) $50=\sqrt{\frac{R+j2\pi10^8L}{G+j2\pi10^8C}}$, $1.15\times10^{-3}+j0.8\pi=\sqrt{(R+j\omega L)(G+j\omega C)}=50\cdot(G+j2\pi10^8C)$

$$\Rightarrow C=\frac{0.8\pi}{2\pi\times10^8\times50}=80(pF/m),\ \ G=\frac{1.15}{50}\times10^{-3}=2.3\times10^{-5}(S/m),$$

$$R=2500G=0.0575(\Omega/m),\ \ L=2500C=0.2(\mu F/m)$$

Eg. A d-c generator of voltage and internal resistance is connected to a lossy transmission line characterized by a resistance per unit length R and a conductance per unit length G. (a) Write the governing voltage and current transmission-line equations. (b) Find the general solutions for $V(z)$ and $I(z)$.

(Sol.) (a) $\omega=0\Rightarrow\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}=\sqrt{RG}$

$$\frac{d^2V(z)}{dz^2}=RGV(z),\ \ \frac{d^2I(z)}{dz^2}=RGI(z)$$

(b) $V(z)=V_0^+e^{-\sqrt{RG}z}+V_0^-e^{\sqrt{RG}z},\ \ I(z)=I_0^+e^{-\sqrt{RG}z}+I_0^-e^{\sqrt{RG}z}$

Lossless line (R=G=0):

$$\gamma=\alpha+j\beta=j\omega\sqrt{LC}\Rightarrow\alpha=0,\ \ \beta=\omega\sqrt{LC},\ \ v_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}},\ \ Z_0=\sqrt{\frac{L}{C}}=R_0+jX_0\Rightarrow R_0=\sqrt{\frac{L}{C}},\ \ X_0=0$$

Low-loss line (R<<ωL, G<<ωC):

$$\gamma=\alpha+j\beta\approx j\omega\sqrt{LC}\left(1+\frac{1}{2j\omega}\left(\frac{R}{L}+\frac{G}{C}\right)\right)\Rightarrow\alpha=\approx\frac{1}{2}\left(R\sqrt{\frac{C}{L}}+G\sqrt{\frac{L}{C}}\right),\beta=\omega\sqrt{LC},\ \ v_p\approx\frac{1}{\sqrt{LC}}$$

$$Z_0\approx\sqrt{\frac{L}{C}}\left[1+\frac{1}{2j\omega}\left(\frac{R}{L}-\frac{G}{C}\right)\right]$$

Distortionless line (R/L=G/C):

$$\gamma=\alpha+j\beta=\sqrt{\frac{C}{L}}(R+j\omega L)\Rightarrow\alpha=R\sqrt{\frac{C}{L}},\ \ \beta=\omega\sqrt{LC},\ \ v_p=\frac{1}{\sqrt{LC}},\ \ Z_0=\sqrt{\frac{L}{C}}$$

Large-loss line ($\omega L << R$, $\omega C << G$):

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta = \sqrt{RG} \cdot (1 + j\frac{\omega L}{R})^{1/2}(1 + \frac{j\omega C}{G})^{1/2} \approx$$

$$\sqrt{RG}[1 + \frac{j\omega}{2}(\frac{L}{R} + \frac{C}{G})]$$

$$\therefore \alpha \approx \sqrt{RG}, \beta \approx \frac{\omega}{2}(L \cdot \sqrt{\frac{G}{R}} + C \cdot \sqrt{\frac{R}{G}}),\ \ v_p = \frac{1}{2}(L \cdot \sqrt{\frac{G}{R}} + C \cdot \sqrt{\frac{R}{G}})$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R}{G}} \cdot (1 + \frac{j\omega L}{R})^{1/2} \cdot (1 + \frac{j\omega C}{G})^{-1/2} = \sqrt{\frac{R}{G}} \cdot [1 + \frac{j\omega}{2}(\frac{L}{R} - \frac{C}{G})]$$

Eg. A generator with an open-circuit voltage $v_g(t)$=10sin(8000πt) and internal impedance Z_g=40+j30(Ω) is connected to a 50Ω distortionless line. The line has a resistance of 0.5Ω/m, and its lossy dielectric medium has a loss tangent of 0.18%. The line is 50m long and is terminated in a matched load. Find the instantaneous expressions for the voltage and current at an arbitrary location on the line.

(Sol.) $0.18\% = \frac{\sigma}{\omega\varepsilon} = \frac{G}{\omega C} \Rightarrow C = 2.21 \times 10^{-2} G$, $V_g = 10j$

$\because$ Distortionless, $\therefore$ $\frac{L}{R} = \frac{C}{G} \Rightarrow L = 1.11 \times 10^{-2} H/m$, $\alpha = R\sqrt{\frac{C}{L}} = \frac{R}{Z_0}$=0.01$Np/m$,

$\beta = \omega\sqrt{LC} = \omega L\sqrt{\frac{C}{L}} = \frac{\omega L}{Z_0} = 5.58 rad/m$, $\gamma = \alpha + j\beta = 0.01 + j5.58$

$V_0^+ = \frac{Z_0 V_g}{Z_0 + Z_g} = \frac{5}{3} + j5$, $V_0^- = 0$, $\therefore V(z) = V_0^+ e^{-\gamma z} = (\frac{5}{3} + j5)e^{-(0.01 + j5.58)z}$

$$V(z,t) = \text{Re}[V(z)e^{j8000\pi t}] = \frac{5\sqrt{10}}{3} \cdot e^{-0.01z} \cdot \cos(8000\pi t - 5.58z + 71.6°)$$

$$I(z,t) = \frac{V(z,t)}{Z_0} = \frac{1}{2\sqrt{10}} e^{-0.01z} \cos(8000\pi t - 5.58 + 71.6°)$$

Relationship between transmission-line parameters:

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \approx j\omega\sqrt{LC}\,(1+\frac{G}{j\omega C})^{1/2} = j\omega\sqrt{\mu\varepsilon}\,(1+\frac{\sigma}{j\omega\varepsilon})^{1/2} \Rightarrow G/C=\sigma/\varepsilon$$

and $LC=\mu\varepsilon$

Two-wire line: $I = 2\pi a J_s$, $P_\sigma = \frac{1}{2}I^2(\frac{R_s}{2\pi a}) \Rightarrow R = 2(\frac{R_s}{2\pi a}) = \frac{1}{\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$

Coaxial-cable line: $I = 2\pi a J_{si} = 2\pi b J_{so}$, $P_{\sigma i} = \frac{1}{2}I^2(\frac{R_s}{2\pi a})$, $P_{\sigma o} = \frac{1}{2}I^2(\frac{R_s}{2\pi b})$

$$\Rightarrow R = \frac{R_s}{2\pi}(\frac{1}{a}+\frac{1}{b}) = \frac{1}{2\pi}\sqrt{\frac{\pi f\mu_c}{\sigma_c}}(\frac{1}{a}+\frac{1}{b})$$

Distributed Parameters of Two-Wire and Coaxial Transmission Lines

Parameter	Two-Wire Line	Coaxial Line	Unit
R	$\frac{R_s}{\pi a}$	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	Ω/m
L	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	H/m
G	$\frac{\pi\sigma}{\cosh^{-1}(D/2a)}$	$\frac{2\pi\sigma}{\ln(b/a)}$	S/m
C	$\frac{\pi\epsilon}{\cosh^{-1}(D/2a)}$	$\frac{2\pi\epsilon}{\ln(b/a)}$	F/m

Note: $R_s = \sqrt{\pi f\mu_c/\sigma_c}$; $\cosh^{-1}(D/2a) \cong \ln(D/a)$ if $(D/2a)^2 \gg 1$. Internal inductance is not included.

Eg. It is desired to construct uniform transmission lines using polyethylene (ε_r=2.25) as the dielectric medium. Assume negligible losses. (a) Find the distance of separation for a 300Ω two-wire line, where the radius of the conducting wires is 0.6*mm*; and (b) find the inner radius of the outer conductor for a 75Ω coaxial line, where the radius of the center conductor is 0.6*mm*.

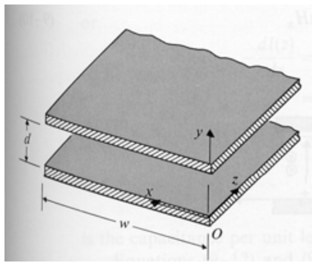
(Sol.) Two-wire line: $C = \frac{\pi\varepsilon}{\cosh^{-1}(D/2a)}$, $L = \frac{\mu}{\pi}\cosh^{-1}(\frac{D}{2a})$, a=0.6*mm*, $\varepsilon=2.25\varepsilon_0$

$$Z_0 = 300 = \sqrt{\frac{L}{C}} = \frac{\cosh^{-1}(\frac{D}{2a})}{\pi}\cdot\sqrt{\frac{4\pi\times10^{-7}}{2.25\times\frac{1}{36\pi}\times10^{-9}}} \Rightarrow D \approx 25.5mm$$

Coaxial line: $C = \frac{2\pi\varepsilon}{\ln(b/a)}$, $L = \frac{\mu}{2\pi}\ln(\frac{b}{a})$

$$a=0.6mm,\ Z_0 = 75 = \sqrt{\frac{L}{C}} = \frac{\ln(\frac{b}{a})}{2\pi}\cdot\sqrt{\frac{4\pi\times10^{-7}}{2.25\times\frac{1}{36\pi}\times10^{-9}}} \Rightarrow b=3.91mm$$

Parallel–plate transmission line:



$$\begin{cases} \vec{E} = \hat{y}E_0 e^{-\gamma z} = \hat{y}E_y \\ \vec{H} = -\hat{x}\dfrac{E_0}{\eta_0}e^{-\gamma z} = \hat{x}H_x \end{cases}, \quad \gamma = j\beta = j\omega\sqrt{\mu\varepsilon}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

At y=0 and y=d, E_x=E_y=0, H_y=0

Distributed Parameters of Parallel-Plate Transmission Line (Width = w, Separation = d)

Parameter	Formula	Unit
R	$\frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$	Ω/m
L	$\mu\frac{d}{w}$	H/m
G	$\sigma\frac{w}{d}$	S/m
C	$\epsilon\frac{w}{d}$	F/m

At y=0, $\hat{a}_n = \hat{y}$,
$$\begin{cases} \hat{y}\cdot\vec{D} = \rho_{sl} \Rightarrow \rho_{sl} = \varepsilon E_y = \varepsilon E_0 e^{-j\beta z} \\ \hat{y}\times\vec{H} = \vec{J}_{sl} \Rightarrow \vec{J}_{sl} = -\hat{z}H_z = \hat{z}\dfrac{E_0}{\eta}e^{-j\beta z} \end{cases}$$

At y=d, $\hat{a}_n = -\hat{y}$,
$$\begin{cases} -\hat{y}\cdot\vec{D} = \rho_{su} \Rightarrow \rho_{su} = -\varepsilon E_y = -\varepsilon E_0 e^{-j\beta z} \\ -\hat{y}\times\vec{H} = \vec{J}_{su} \Rightarrow \vec{J}_{su} = \hat{z}H_x = -\hat{z}\dfrac{E_0}{\eta}e^{-j\beta z} \end{cases}$$

$$\because \nabla\times\vec{E} = -j\omega\mu\vec{H}, \quad \therefore \frac{dE_y}{dz} = j\omega\mu H_x \quad \Rightarrow \frac{d}{dz}\int_0^d E_y dy = j\omega\mu\int_0^d H_x dy$$

$$\Rightarrow -\frac{dV(z)}{dz} = j\omega\mu J_{su}(z)d = j\omega(\mu\frac{d}{\omega})[J_{su}(z)w] = j\omega LI(z) \Rightarrow L = \mu\frac{d}{w} \quad (H/m)$$

$$\because \nabla\times\vec{H} = j\omega\varepsilon\vec{E}, \quad \therefore \frac{dH_x}{dz} = j\omega\varepsilon E_y \quad \Rightarrow \frac{d}{dz}\int_0^w H_x dx = j\omega\varepsilon\int_0^w E_y dx$$

$$\Rightarrow -\frac{dI(z)}{dz} = -j\omega\varepsilon E_y(z)w = j\omega(\varepsilon\frac{w}{d})[-E_y(z)d] = j\omega CV(z) \Rightarrow C = \varepsilon\frac{w}{d} \quad (F/m)$$

$$\begin{cases} -\dfrac{dV}{dz} = j\omega LI \\ -\dfrac{dI}{dz} = j\omega CV \end{cases} \Rightarrow \begin{cases} \dfrac{d^2V(Z)}{dz^2} = -\omega^2 LCV(z) \\ \dfrac{d^2I(z)}{dz^2} = -\omega^2 LCI(z) \end{cases} \Rightarrow \begin{matrix} \beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon}, v_p = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{\mu\varepsilon}} \\ Z_0 = \dfrac{V(z)}{I(z)} = \sqrt{\dfrac{L}{C}} = \dfrac{d}{w}\sqrt{\dfrac{\mu}{\varepsilon}} = \dfrac{d}{w}\eta \end{matrix}$$

Lossy parallel–plate transmission line: $G = \dfrac{\sigma}{\varepsilon}C = \sigma\dfrac{w}{d}$

Surface impedance: $Z_s \equiv \dfrac{E_t}{J_s} = \dfrac{E_z}{H_x} = \eta_c = R_s + jX_s = (1+j)\sqrt{\dfrac{\pi f\mu_c}{\sigma_c}}$

$$\Rightarrow P_\sigma = \frac{1}{2}\text{Re}(|J_{su}|^2 Z_s) = \frac{1}{2}|J_{su}|^2 R_s = \frac{1}{2}I^2(\frac{R_s}{w}) = \frac{1}{2}I^2 R$$

$$R = 2(\frac{R_s}{w}) = \frac{2}{w}\sqrt{\frac{\pi f\mu_c}{\sigma_c}}(\Omega/m)$$

Eg. Consider a transmission line made of two parallel brass strips σ_c=1.6×10^7*S/m* of width 20*mm* and separated by a lossy dielectric slab $\mu=\mu_0$, ε_r=3, σ=10^{-3}*S*/m of thickness 2.5*mm*. The operating frequency is 500*MHz*. (a) Calculate the *R*, *L*, *G*, and *C* per unit length. (b) Find γ and Z_0.

(Sol.) (a) $R=\dfrac{2}{w}\sqrt{\dfrac{\pi f\mu_0}{\sigma_c}}=1.11(\Omega/m)$, $G=\sigma\dfrac{w}{d}=8\times10^{-3}(S/m)$

$L=\mu_0\dfrac{d}{w}=1.57\times10^{-7}(H/m)$, $C=\varepsilon\dfrac{w}{d}=2.12\times10^{-10}(F/m)$

(b) $\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}=18.13\angle-0.41°$, $\omega=2\pi\times500\times10^6$, $Z_0=\sqrt{\dfrac{R+j\omega L}{G+j\omega C}}=27.21\angle0.3°$

Eg. Consider lossless stripline design for a given characteristic impedance. (a) How should the dielectric thickness *d* be changed for a given plate width *w* if the dielectric constant ε_r is doubled? (b) How should *w* be changed for a given *d* if ε_r is doubled? (c) How should *w* be changed for a given ε_r if *d* is doubled?

(Sol.) $Z_0=\sqrt{\dfrac{L}{C}}=\dfrac{d}{w}\sqrt{\dfrac{\mu}{\varepsilon}}$

(a) $\varepsilon\to2\varepsilon\Rightarrow d=\to\sqrt{2}d$, (b) $\varepsilon\to2\varepsilon\Rightarrow w\to\dfrac{w}{\sqrt{2}}$

(c) $d\to2d\Rightarrow w\to2w$

Attenuation constant of transmission line: $\alpha=\dfrac{P_L(z)}{2P(z)}$, where $P_L(z)$ is the time-average power loss in an infinitesimal distance.

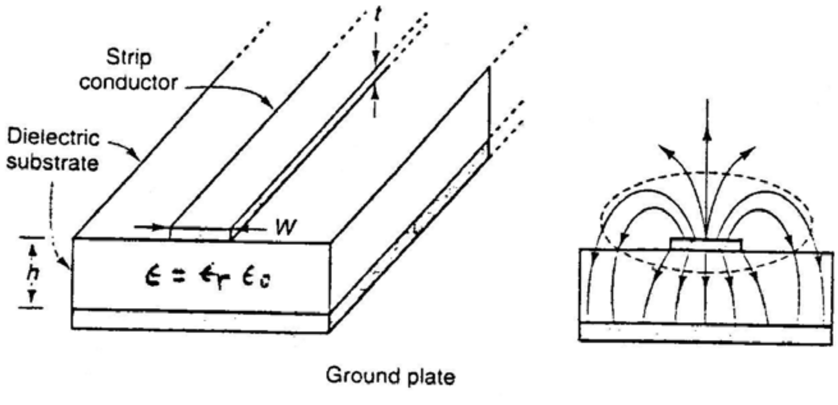
$\gamma=\alpha+j\beta\Rightarrow\alpha=\mathrm{Re}(\gamma)=\mathrm{Re}[\sqrt{(R+j\omega L)(G+j\omega C)}]$

Suppose no reflection, $V(z)=V_0e^{-(\alpha+j\beta)z}$, $I(z)=\dfrac{V_0}{Z_0}e^{-(\alpha+j\beta)z}$

$$\Rightarrow P(z)=\frac{1}{2}\mathrm{Re}[V(z)I^*(z)]=\frac{V_0^{\,2}}{2|Z_0|^2}\cdot R_0e^{-2\alpha z}\quad\propto\quad e^{-2\alpha z}$$

$$\Rightarrow-\frac{\partial P(z)}{\partial z}=P_L(z)=2\alpha P(z)\Rightarrow\alpha=\frac{P_L(z)}{2P(z)}$$

Microstrip lines: are usually used in the *mm* wave range.



(a) (b)

$$v_p = \frac{c}{\sqrt{\varepsilon_{ff}}}, \quad Z_o = \frac{1}{v_p C} = \sqrt{\frac{L}{C}}, \quad \lambda = \frac{v_p}{f} = \frac{\lambda_O}{\sqrt{\varepsilon_{ff}}}$$

Assuming the quasi-TEM mode:

***Case 1*: *t*/*h* < 0.005, *t* is negligible.**

Given h, W, and ε_r, obtain Z_0 as follows:

For $W/h \le 1$: $Z_o = \dfrac{60}{\sqrt{\varepsilon_{ff}}} \ln\left(8\dfrac{h}{W} + 0.25\dfrac{W}{h}\right)$,

where $\varepsilon_{ff} = \dfrac{\varepsilon_r + 1}{2} + \dfrac{\varepsilon_r - 1}{2}\left[\left(1 + 12\dfrac{h}{W}\right)^{-1/2} + 0.04\left(1 - \dfrac{W}{h}\right)^2\right]$

For $W/h \ge 1$: $Z_o = \dfrac{120\pi / \sqrt{\varepsilon_{ff}}}{W/h + 1.393 + 0.667 \ln\left(W/h + 1.444\right)}$,

where $\varepsilon_{ff} = \dfrac{\varepsilon_r + 1}{2} + \dfrac{\varepsilon_r - 1}{2}\left(1 + 12\dfrac{h}{W}\right)^{-1/2}$

Given Z_0, h, and ε_r, obtain W as follows:

For $W/h \le 2$: $W = \dfrac{8he^A}{e^{2A} - 2}$, where $A = \dfrac{Z_0}{60}\sqrt{\dfrac{\varepsilon_r + 1}{2}} + \dfrac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \dfrac{0.11}{\varepsilon_r}\right)$

For $W/h > 2$: $W = \dfrac{2h}{\pi}\left\{B - 1 - \ln(2B - 1) + \dfrac{\varepsilon_r - 1}{2\varepsilon_r}\left[\ln(B - 1) + 0.39 - \dfrac{0.61}{\varepsilon_r}\right]\right\}$, where

$$B = \frac{377\pi}{2Z_0\sqrt{\varepsilon_r}}$$

***Case 2*:** $t/h > 0.005$. In this case, we obtain W_{eff} firstly.

For $W/h \geq 1/2\pi$: $\frac{W_{eff}}{h} = \frac{W}{h} + \frac{t}{\pi h}\left(1 + \ln\frac{2h}{t}\right)$

For $W/h \leq 1/2\pi$: $\frac{W_{eff}}{h} = \frac{W}{h} + \frac{t}{\pi h}\left(1 + \ln\frac{4\pi W}{t}\right)$

And then we substitute W_{eff} into W in the expressions in Case 1.

Assuming not the quasi-TEM mode:

$Z_O(f) = \frac{377h}{W_{eff}(f)\sqrt{\varepsilon_{ff}}}$, where $W_{eff}(f) = W + \frac{W_{eff}(0) - W}{1 + \left(f/f_p\right)^2}$, $f_p = \frac{Z_O}{8\pi h}$ (h in cm)

and $W_{eff}(0) = \frac{377h}{Z_O(0)\sqrt{\varepsilon_{ff}(0)}}$, $G=0.6+0.009Z_0$, $\varepsilon_{ff}(f) = \varepsilon_r - \frac{\varepsilon_r - \varepsilon_{ff}}{1 + G\left(f/f_p\right)^2}$ (f in GHz)

The frequency below which dispersion may be neglected is given by

$f_0(GHz) = 0.3\sqrt{\frac{Z_O}{h\sqrt{\varepsilon_r - 1}}}$, where h must be expressed in cm.

Attenuation constant: $\alpha = \alpha_d + \alpha_c$

For a dielectric with low losses: $\alpha_d = 27.3\frac{\varepsilon_r}{\sqrt{\varepsilon_{ff}}}\frac{\varepsilon_{ff} - 1}{\varepsilon_r - 1}\frac{\tan\delta}{\lambda_O}$ $\left(\frac{dB}{cm}\right)$

For a dielectric with high losses: $\alpha_d = 4.34\frac{\varepsilon_{ff} - 1}{\sqrt{\varepsilon_{ff}}(\varepsilon_r - 1)}\left(\frac{\mu_O}{\varepsilon_O}\right)^{1/2}\sigma$ $\left(\frac{dB}{cm}\right)$

For $W/h \to \infty$: $\alpha_c = \frac{8.68}{Z_O W}R_s$, where $R_s = \sqrt{\frac{\pi f\mu_O}{\sigma}}$

For $W/h \leq 1/2\pi$: $\alpha_c = \frac{8.68R_sP}{2\pi Z_O h}\left[1 + \frac{h}{W_{eff}} + \frac{h}{\pi W_{eff}}\left(\ln\frac{4\pi W}{t} + \frac{t}{W}\right)\right]$

For $1/2\pi < W/h \leq 2$: $\alpha_c = \frac{8.68R_s}{2\pi Z_O h}PQ$, where $P = 1 - \left(\frac{W_{eff}}{4h}\right)^2$

and $Q = 1 + \frac{h}{W_{eff}} + \frac{h}{\pi W_{eff}}\left(\ln\frac{2h}{t} - \frac{t}{h}\right)$

For $W/h \geq 2$:

$$\alpha_c = \frac{8.68 R_s Q}{Z_O h}\left\{\frac{W_{eff}}{h} + \frac{2}{\pi}\ln\left[2\pi e\left(\frac{W_{eff}}{2h} + 0.94\right)\right]\right\}^{-2}\left[\frac{W_{eff}}{h} + \frac{W_{eff}/\pi h}{\left(W_{eff}/2h\right) + 0.94}\right]$$

Eg. A high-frequency test circuit with microstrip lines.

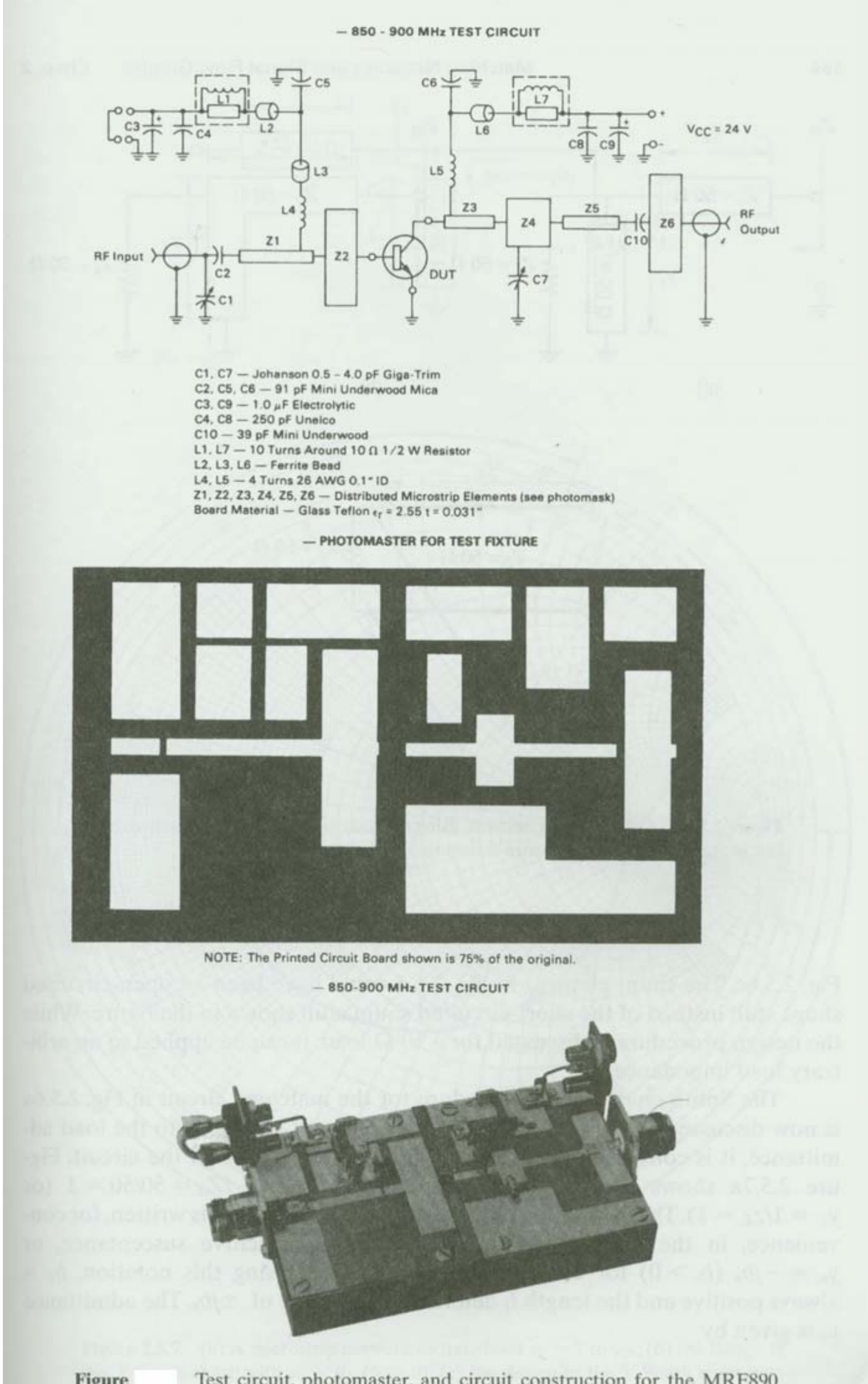
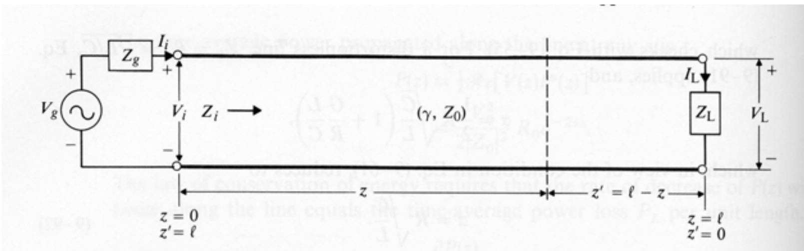


Figure Test circuit, photomaster, and circuit construction for the MRF890 transistor. (From *Motorola RF Device Data,* Vol. 1, 6th edition; copyright of Motorola, used by permission.)

5-2 Wave Characteristics of Finite Transmission Line



Eg. Show that the input impedance is $Z_i = (Z)_{\substack{z=0 \\ z'=\ell}} = Z_0 \dfrac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}$.

(Proof) $\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \ldots(1) \\ I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \ldots\ldots(2) \end{cases}$, $Z_0 = \dfrac{V_0^+}{I_0^+} = -\dfrac{V_0^-}{I_0^-}$

Let $z=l$, $V(l)=V_L$, $I(l)=I_L$

$$\Rightarrow \begin{cases} V_L = V_0^+ e^{-\gamma\ell} + V_0^- e^{\gamma\ell} \\ I_L = \dfrac{V_0^+}{Z_0} e^{-\gamma\ell} - \dfrac{V_0^-}{Z_0} e^{\gamma\ell} \end{cases} \Rightarrow \begin{cases} V_0^+ = \dfrac{1}{2}(V_L + I_L Z_0) e^{\gamma\ell} \\ V_0^- = \dfrac{1}{2}(V_L - I_L Z_0) e^{-\gamma\ell} \end{cases}$$

$$\Rightarrow \begin{cases} V(z) = \dfrac{I_L}{2}[(Z_L + Z_0) e^{\gamma(\ell - z)} + (Z_L - Z_0) e^{-\gamma(\ell - z)}] \\ I(z) = \dfrac{I_L}{2Z_0}[(Z_L + Z_0) e^{\gamma(\ell - z)} - (Z_L - Z_0) e^{-\gamma(\ell - z)}] \end{cases}$$

$$\Rightarrow \begin{cases} V(z') = \dfrac{I_L}{2}[(Z_L + Z_0) e^{\gamma z'} + (Z_L - Z_0) e^{-\gamma z'}] \\ I(z') = \dfrac{I_L}{2Z_0}[(Z_L + Z_0) e^{\gamma z'} - (Z_L - Z_0) e^{-\gamma z'}] \end{cases} \Rightarrow \begin{cases} V(z') = I_L (Z_L \cosh \gamma z' + Z_0 \sinh \gamma z') \\ I(z') = \dfrac{I_L}{Z_0}(Z_L \sinh \gamma z' + Z_0 \cosh \gamma z') \end{cases}$$

$$\Rightarrow Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'},\ Z_i = (Z)_{\substack{z=0 \\ z'=\ell}} = Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}$$

Lossless case (α=0, γ=$j\beta$, Z_0=R_0, tanh(γl)=$j$tanβl): $Z_i = R_0 \cdot \dfrac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l}$

Note: In the high-frequency circuit, the input current $I_i = \dfrac{V_g}{Z_g + Z_i} \neq \dfrac{V_g}{Z_g + Z_L}$: the value in the low-frequency case. And the high-frequency I_i is dependent on the length $\boldsymbol{l}$, the characteristic impedance $\boldsymbol{Z_0}$, the propagation constant $\boldsymbol{\gamma}$ of the transmission line, and the load impedance $\boldsymbol{Z_L}$. But the low-frequency I_i is only dependent on $\boldsymbol{Z_0}$ and $\boldsymbol{Z_L}$.

Eg. A 2*m* lossless air-spaced transmission line having a characteristic impedance 50Ω is terminated with an impedance 40+*j*30(Ω) at an operating frequency of 200*MHz*. Find the input impedance.

(Sol.) $\beta = \frac{\omega}{v_p} = \frac{4}{3}\pi$, $R_0 = 50\Omega$, $Z_L = 40 + j30$, $\ell = 2m$

$$Z_i = 50\frac{(40+j30) + j50\cdot\tan(\frac{4\pi}{3}\cdot 2)}{50 + j(40+j30)\cdot\tan(\frac{4\pi}{3}\cdot 2)} = 26.3 - j9.87$$

Eg. A transmission line of characteristic impedance 50Ω is to be matched to a load Z_L=40+*j*10(Ω) through a length *l'* of another transmission line of characteristic impedance R_0'. Find the required *l'* and R_0' for matching.

(Sol.) $50 = R_0' \cdot \frac{40 + j10 + jR_0' \cdot \tan\beta\ell'}{R_0' + j(40+j10)\cdot\tan\beta\ell'} \Rightarrow R_0' = \sqrt{1500} \approx 38.7(\Omega),\ \ \ell' \approx 0.105\lambda$

Eg. Prove that a maximum power is transferred from a voltage source with an internal impedance Z_g to a load impedance Z_L over a lossless transmission line when Z_i=Z_g*, where Z_i is the impedance looking into the loaded line. What is the maximum power transfer efficiency?

(Proof) $I_i = \frac{V}{Z_i + Z_g}$, $V_i = \frac{Z_i}{Z_i + Z_g}V$

$$(Power)_{out} = \frac{1}{2}\mathrm{Re}[V_i I_i^*] = \frac{R_i|V|^2}{2[(R_i + R_g)^2 + (X_i + X_g)^2]}$$

When $R_i = R_g$ and $X_i = -X_g$, $(Power)_{out} \to Max$, $\therefore\ Z_i = Z_g^*$

In this case, $(Power)_{out} = \frac{|V|^2}{4R_g}$, $P_s = \frac{1}{2}\mathrm{Re}[VI_i^*] = \frac{|V|^2}{2R_g}$, $e = \frac{(Power)_{out}}{P_s} = \frac{1}{2}$

Transmission lines as circuit elements:

Consider a general case: $Z_i = Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}$

1. *Open-circuit termination* ($Z_L \to \infty$): $Z_i = Z_{io} = Z_0 \coth(\gamma l)$
2. *Short-circuit termination* ($Z_L = 0$): $Z_i = Z_{is} = Z_0 \tanh(\gamma l)$

$$\therefore\ Z_0 = \sqrt{Z_{i0} \cdot Z_{is}},\ \gamma = \frac{1}{\ell} \tanh^{-1} \sqrt{\frac{Z_{is}}{Z_{i0}}}$$

3. *Quarter-wave section* in a lossless case ($l = \lambda/4$, $\beta l = \pi/2$): $Z_i = \frac{{R_0}^2}{Z_L}$

4. *Half-wave section* in a lossless case ($l = \lambda/2$, $\beta l = \pi$): $Z_i = Z_L$

Eg. The open-circuit and short-circuit impedances measured at the input terminals of an air-spaced transmission line 4*m* long are 250∠-50°Ω and 360∠20°Ω, respectively. (a) Determine Z_0, α, and β of the line. (b) Determine *R*, *L*, *G*, and *C*.

(Sol.) (a) $Z_0 = \sqrt{250 e^{-j50°} \cdot 360 e^{j20°}} = 289.8 - j77.6$,

$$\gamma = \frac{1}{4} \tanh^{-1} \sqrt{\frac{360\angle 20°}{250\angle -50°}} = 0.139 + j0.235 = \alpha + j\beta$$

(b) $R + j\omega L = Z_0 \cdot \gamma = 58.5 + j57.3$, $L = \frac{57.3}{\omega} = \frac{57.3}{c\beta} = 0.812(\mu H/m)$

$$G + j\omega C = \frac{\gamma}{Z_0} = 24.5 \times 10^{-5} + j8.76 \times 10^{-4},\ C = \frac{8.76 \times 10^{-4}}{c\beta} = 12.4(pF/m)$$

Eg. Measurements on a 0.6*m* lossless coaxial cable at 100*kHz* show a capacitance of 54*pF* when the cable is open-circuited and an inductance of 0.30*μH* when it is short-circuited. Determine Z_0 and the dielectric constant of its insulating medium.

(Sol.) (a) $C = \frac{54 \times 10^{-12}}{0.6} = 9 \times 10^{-11}(F/m)$, $L = \frac{0.3 \times 10^{-6}}{0.6} = 5 \times 10^{-7}(H/m)$

Lossless $\Rightarrow Z_0 = R_0 = \sqrt{\frac{L}{C}}$ =74.5Ω, $\mu\varepsilon = \mu_0 \mu_r \varepsilon_0 \varepsilon_r = LC \Rightarrow \varepsilon_r = 4.05$

General expressions for *V*(*z*) and *I*(*z*) on the transmission lines:

Let $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$, $z' = l - z$

$$\begin{cases} V(z') = \frac{I_L}{2}(Z_L + Z_0)\cdot e^{\gamma z'}\cdot[1+\Gamma e^{-2\gamma z'}] \\ I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)\cdot e^{\gamma z'}\cdot[1-\Gamma e^{-2\gamma z'}] \end{cases}$$

$$\Rightarrow \begin{cases} V(z') = \frac{I_L}{2}(Z_L + Z_0)\cdot e^{\gamma z'}\cdot[1+|\Gamma| e^{j(\theta_\Gamma - 2\gamma z')}] \\ I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)\cdot e^{\gamma z'}\cdot[1-|\Gamma| e^{j(\theta_\Gamma - 2\gamma z')}] \end{cases}$$

For a lossless line, $V(z) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z}[1+\Gamma e^{-j2\beta(\ell - z)}]$

Eg. A 100*MHz* generator with $V_g = 10\angle 0°$ (*V*) and internal resistance 50Ω is connected to a lossless 50Ω air line that is 3.6*m* long and terminated in a 25+*j*25(Ω) load. Find (a) *V*(*z*) at a location *z* from the generator, (b) V_i at the input terminals and V_L at the load, (c) the voltage standing-wave radio on the line, and (d) the average power delivered to the load.

(Sol.) $V_g = 10\angle 0°(V)$, $Z_g = 50(\Omega)$, $f = 10^8(Hz)$, $Z_0 = 50(\Omega)$,

$Z_L = 25 + j25 = 35.36\angle 45°(\Omega)$,

$\ell = 3.6(m)$, $\beta = \frac{\omega}{c} = \frac{2\pi 10^8}{3\times 10^8} = \frac{2\pi}{3}(rad/m)$, $\beta\ell = 2.4\pi(rad/m)$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(25+j25)-50}{(25+j25)+50} = 0.447\angle 0.648\pi, \quad \Gamma_g = 0$$

(a) $V(z) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z}[1+\Gamma e^{-j2\beta(\ell - z)}] = 5[e^{-j2\pi z/3} + 0.447 e^{j(2z/3 - 0.152)\pi}]$

(b) $V_i = V(0) = 5(1 + 0.447 e^{-j0.152\pi}) = 7.06\angle -8.43°(V)$

(c) $V_L = V(3.6) = 5[e^{-j0.4\pi} + 0.447 e^{j0.248\pi}] = 4.47\angle -45.5°(V)$

(d) $S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.447}{1-0.447} = 2.62$, $P_{av} = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}(\frac{4.47}{35.36})^2 \times 25 = 0.200(W)$

Eg. A sinusoidal voltage generator V_g=110sin(ω*t*) and internal impedance Z_g=50Ω is connected to a quarter-wave lossless line having a characteristic impedance Z_0=50Ω that is terminated in a purely reactive load Z_L=*j*50Ω. (a) Obtain the voltage and current phasor expressions *V*(*z*') and *I*(*z*'). (b) Write the instantaneous voltage and current expressions *V*(*z*',*t*) and *I*(*z*',*t*).

(Sol.) (a) $V_g = 110j,\ \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{50j - 50}{50j + 50} = j,\ \Gamma_g = \dfrac{Z_g - Z_0}{Z_g + Z_0} = 0,\ \ell = \dfrac{\lambda}{4}$

(c) $V(z') = \dfrac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z'} \cdot (1 + \Gamma e^{-2j\beta z'}) = j55(e^{-j\beta z'} - je^{j\beta z'})$,

$$I(z') = \frac{V_g}{Z_0 + Z_g} e^{-j\beta z'} \cdot (1 - \Gamma e^{-2j\beta z'}) = -j1.1(e^{-j\beta z'} + je^{j\beta z'})$$

$$P = V(z'=0,t)I(z'=0,t) = 60.5\cos(2\omega t),\ P_{av} = \frac{1}{T}\int_0^T P dt = 0$$

(b) $V(z',t) = \mathrm{Im}[V(z')e^{j\omega t}] = 55[\sin(\omega t - \beta z') - \cos(\omega t + \beta z')]$

$I(z',t) = \mathrm{Im}[I(z')e^{j\omega t}] = -1.1[\sin(\omega t - \beta z') + \cos(\omega t + \beta z')]$

Eg. A sinusoidal voltage generator with V_g=0.1∠0° (*V*) and internal impedance Z_g=Z_0 is connected to a lossless transmission line having a characteristic impedance Z_0=50Ω. The line is *l* meters long and is terminated in a load resistance Z_L=25Ω. Find (a) V_i, I_i, V_L錯誤! 尚未定義書籤。, and I_L; (b) the standing-wave radio on the line; and (c) the average power delivered to the load.

(Sol.) (a) $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = -\dfrac{1}{3},\ \Gamma_g = \dfrac{Z_g - Z_0}{Z_g + Z_0} = 0,\ V_i = V(z'=\ell) = \dfrac{0.1}{2}(1 - \dfrac{1}{3}e^{-2j\beta\ell})$

$$I_i = I(z'=\ell) = \frac{0.1}{100} \times (1 + \frac{1}{3}e^{-2j\beta\ell})$$

$$V_L = V(z'=0) = \frac{0.1}{2}e^{-j\beta\ell}(1 - \frac{1}{3}) = \frac{1}{30}e^{-j\beta\ell}$$

$$I_L = I(z'=0) = \frac{0.1}{100}e^{-j\beta\ell}(1 + \frac{1}{3}) = \frac{1}{750}e^{-\beta\ell}$$

(b) $S = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} = 2$, (c) $P_{av} = \dfrac{1}{2}\mathrm{Re}[V_L I_L^*] = 2.22 \times 10^{-5} W$

Eg. Consider a lossless transmission line of characteristic impedance R_0. A time-harmonic voltage source of an amplitude V_g and an internal impedance R_g=R_0 is connected to the input terminals of the line, which is terminated with a load impedance Z_L=R_L+jX_L. Let P_{inc} be the average incident power associated with the wave traveling in the +z direction. (a) Find the expression for P_{inc} in terms of V_g and R_0. (b) Find the expression for the average power P_L delivered to the load in terms of V_g and the reflection coefficient Γ. (c) Express the ratio P_L/P_{inc} in terms of the standing-wave ratio S.

(Sol.) $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$, $I(z) = \frac{1}{R_0}(V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z})$, $V_{inc}(z=0) = V_0^+ = \frac{V_g}{2}$, $I_{inc}(z=0) = I_0^+ = \frac{V_g}{2R_0}$

(a) $P_{inc} = \frac{1}{2}\mathrm{Re}[V_0^+ (I_0^+)^*] = \frac{\left|V_0^+\right|^2}{2R_0} = \frac{V_g^{\ 2}}{8R_0}$

(b) $P_L = \frac{1}{2}\mathrm{Re}[V(z)I^*(z)] = \frac{1}{2R_0}\mathrm{Re}[(V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z})(V_0^{+*} e^{j\beta z} + V_0^{-*} e^{-j\beta z})] = \frac{1}{2R_0}\{\left|V_0^+\right|^2 - \left|V_0^-\right|^2\}$

$= \frac{\left|V_0^+\right|^2}{2R_0}\{1-|\Gamma|^2\} = \frac{V_g^{\ 2}}{8R_0}\{1-|\Gamma|^2\}$

(c) $\frac{P_L}{P_{inc}} = 1-|\Gamma|^2 = 1-(\frac{S-1}{S+1})^2 = \frac{4S}{(S+1)^2}$

Case 1 For a pure resistive load: $Z_L = R_L$

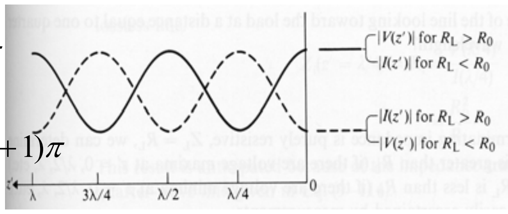
$$\Rightarrow \begin{cases} V(z') = V_L \cdot \cos\beta z' + jI_L R_0 \cdot \sin\beta z' \\ I(z') = I_L \cdot \cos\beta z' + j\dfrac{V_L}{R_0} \cdot \sin\beta z' \end{cases} \Rightarrow \begin{cases} |V(z')| = V_L \cdot \sqrt{\cos^2\beta z' + (R_0/R_L)^2 \sin^2\beta z'} \\ |I(z')| = I_L \cdot \sqrt{\cos^2\beta z' + (R_L/R_0)^2 \sin^2\beta z'} \end{cases}$$

$S = \dfrac{1+|\Gamma|}{1-|\Gamma|}$, $|\Gamma| = \dfrac{S-1}{S+1} \Rightarrow$ 1. $\Gamma=0 \Leftrightarrow S=1$ when $Z_L = Z_0$ (matched load)

2. $\Gamma=-1 \Leftrightarrow S=\infty$ when $Z_L=0$ (short-circuit), 3. $\Gamma=1 \Leftrightarrow S=-\infty$ when $Z_L=\infty$ (open-circuit)

$|V_{max}| \& |I_{min}|$ occurs at $\theta_\Gamma - 2\beta z'_{max} = -2n\pi$

$|V_{min}| \& |I_{max}|$ occurs at $\theta_\Gamma - 2\beta z'_{min} = -(2n+1)\pi$



If $R_L > R_0 \Rightarrow \Gamma > 0 \Rightarrow \theta_\Gamma = 0$, $z'_{max} = \dfrac{n\lambda}{2}$, $n = 0,1,2,3....$

If $R_L < R_0 \Rightarrow \Gamma < 0 \Rightarrow \theta_\Gamma = -\pi$, $z'_{min} = \dfrac{n\lambda}{2}$

If $R_L = \infty \Rightarrow z'_{max} = \dfrac{n\lambda}{2}$

Eg. The standing-wave radio S on a transmission line is an easily measurable quality. Show how the value of a terminating resistance on a lossless line of known characteristic impedance R_0 can be determined by measuring S.

(Sol.) If $R_L > R_0$, $\theta_\Gamma = 0$, $|V_{max}|$ occurs at $\beta z' = 0$ and $|V_{min}|$ occurs at $\beta z' = \pi/2$.

$|V_{max}| = V_L$, $|V_{min}| = V_L \dfrac{R_0}{R_L}$, $|I_{min}| = I_L$, $|I_{max}| = I_L \dfrac{R_L}{R_0}$, $\dfrac{|V_{max}|}{|V_{min}|} = \dfrac{|I_{max}|}{|I_{min}|} = S = \dfrac{R_L}{R_0}$ or

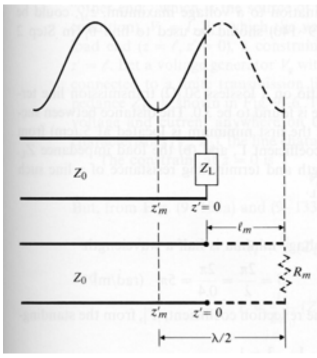
$R_L = SR_0$.

If $R_L < R_0$, $\theta_\Gamma = -\pi$, $|V_{min}|$ occurs at $\beta z' = 0$, and $|V_{max}|$ occurs at $\beta z' = \pi/2$.

$|V_{min}| = V_L$, $|V_{max}| = V_L \dfrac{R_0}{R_L}$, $|I_{max}| = I_L$, $|I_{min}| = I_L \dfrac{R_L}{R_0}$. $\dfrac{|V_{max}|}{|V_{min}|} = \dfrac{|I_{max}|}{|I_{min}|} = S = \dfrac{R_0}{R_L}$ or

$R_L = \dfrac{R_0}{S}$

***Case 2 For a lossless transmission line,* and *arbitrary load*:**



$Z_L = R_0 \cdot \dfrac{R_m + jR_0 \tan \beta\ell_m}{R_0 + jR_m \tan \beta\ell_m}$, $z_m' + l_m = \lambda/2$

Find Z_L=?

1. $|\Gamma| = \dfrac{S-1}{S+1}$, 2. At $\theta_\Gamma = 2\beta z_m' - \pi$, $V(z')$ is a minimum.

3. $Z_L = R_L + jX_L = R_0 \cdot \dfrac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}} = R_0 \cdot \dfrac{1+\Gamma}{1-\Gamma}$

Eg. Consider a lossless transmission line. (a) Determine the line's characteristic resistance so that it will have a minimum possible standing-wave ratio for a load impedance 40+*j*30(Ω). (b) Find this minimum standing-wave radio and the corresponding voltage reflection coefficient. (c) Find the location of the voltage minimum nearest to the load.

(Sol.)

$$|\Gamma| = \left|\frac{Z_L - R_0}{Z_L + R_0}\right| = \left|\frac{40 - R_0 + j30}{40 + R_0 + j30}\right| = \left[\frac{(40-R_0)^2 + 30^2}{(40+R_0)^2 + 30^2}\right]^{1/2},\ S = \frac{1+|\Gamma|}{1-|\Gamma|},\ \frac{dS}{dR_0} = 0 \Rightarrow R_0 = 50\Omega \Rightarrow |\Gamma| = \frac{1}{3}$$

$$\Rightarrow S = 2,\ \Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{-10 + j30}{90 + j30} = \frac{1}{3}\angle -90° = \frac{1}{3}\angle -\frac{\pi}{2},\ \theta_\Gamma = -\frac{\pi}{2}$$

$$z_{\min}' = \frac{1}{2\beta}\left(\pi - \frac{\pi}{2}\right) = \frac{\lambda}{8},\ \ell_m = \frac{\lambda}{2} - \frac{\lambda}{8} = \frac{3\lambda}{8}$$

Eg. *SWR* on a lossless 50Ω terminated line terminated in an unknown load impedance is 3. The distance between successive minimum is 20*cm*. And the first minimum is located at 5*cm* from the load. Determine Γ, Z_L, l_m, and R_m.

(Sol.) $\dfrac{\lambda}{2} = 0.2 \Rightarrow \lambda = 0.4m,\ \beta = \dfrac{2\pi}{\lambda} = 5\pi$

$$|\Gamma| = \frac{3-1}{3+1} = 0.5,\ z'_m = 0.05 \Rightarrow \ell_m = \frac{\lambda}{2} - z_m' = 0.15m$$

$$\theta_\Gamma = 2\beta z_m' - \pi = -0.5\pi,\ \Gamma = |\Gamma|e^{j\theta_\Gamma} = 0.5e^{-j0.5\pi} = -\frac{j}{2}$$

$$R_0 = 50,\ Z_L = 50 \cdot \frac{1 + (\frac{-j}{2})}{1 - (-\frac{j}{2})} = 30 - j40 = 50 \cdot \frac{R_m + j50 \tan \beta\ell_m}{50 + jR_m \tan \beta\ell_m}$$

$$\Rightarrow R_m = \frac{50}{3} = 16.7(\Omega)$$

Eg. A lossy transmission line with characteristic impedance Z_0 is terminated in an arbitrary load impedance Z_L. (a) Express the standing-wave radio S on the line in terms of Z_0 and Z_L. (b) Find the impedance looking toward the load at the location of a voltage maximum. (c) Find the impedance looking toward the load at a location of a voltage minimum.

(Sol.) (a) $|\Gamma| = \left|\dfrac{Z_L - Z_0}{Z_L + Z_0}\right| e^{-2\alpha z'}$, $S = \dfrac{1+|\Gamma|}{1-|\Gamma|} = \dfrac{|Z_L + Z_0| + |Z_L - Z_0| e^{-2\alpha z'}}{|Z_L + Z_0| - |Z_L - Z_0| e^{-2\alpha z'}}$

(b) $\theta_\Gamma - 2\beta z_{\max}' = -2n\pi \Rightarrow e^{j\theta_\Gamma} \cdot e^{-2\gamma z_{\max}'} = e^{-2\alpha z_{\max}'}$, $\therefore Z(z_{\max}') = Z_0 \cdot \dfrac{1+|\Gamma| e^{-2\alpha z_{\max}'}}{1-|\Gamma| e^{-2\alpha z_{\max}'}} = \dfrac{Z_0}{S(z_{\max}')}$

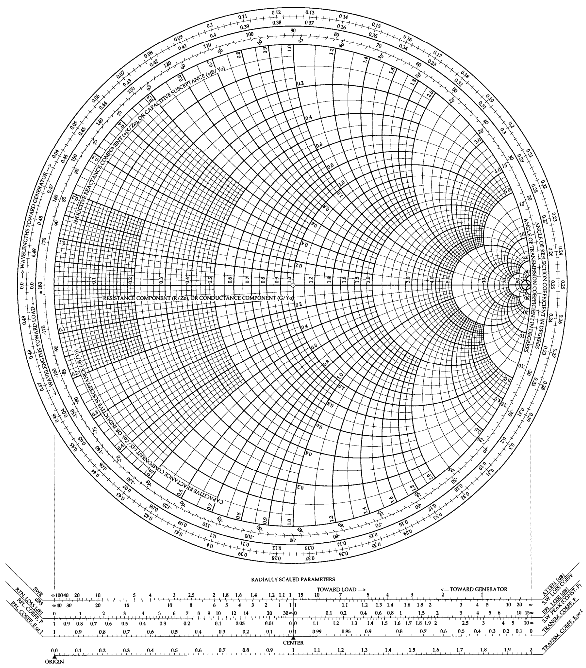
(c) $\theta_\Gamma - 2\beta z_{\min}' = -(2n+1)\pi \Rightarrow e^{j\theta_\Gamma} \cdot e^{-2\gamma z_{\min}'} = -e^{-2\alpha z_{\min}'}$

$$Z(z_{\min}') = Z_0 \cdot \frac{1-|\Gamma| e^{-2\alpha z_{\min}'}}{1+|\Gamma| e^{-2\alpha z_{\min}'}} = \frac{Z_0}{S(z_{\min}')}$$

5-3 Introduction to Smith Chart

The Complete Smith Chart

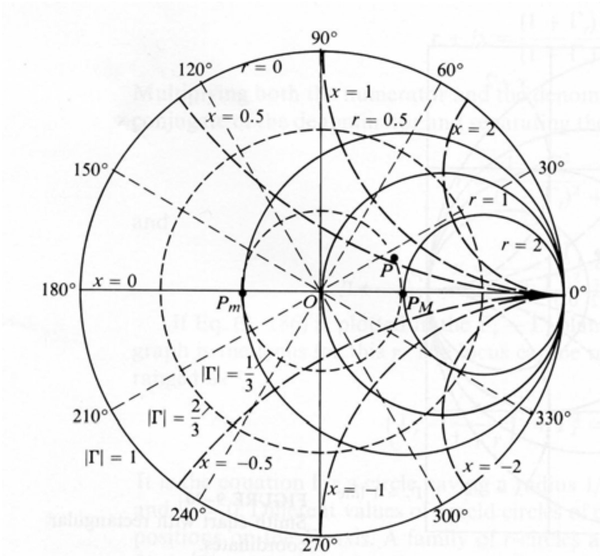
Black Magic Design



$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_\Gamma} = \frac{Z_L / R_0 - 1}{Z_L / R_0 + 1} = \frac{z_L - 1}{z_L + 1} = \Gamma_r + j\Gamma_i \Rightarrow z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma| e^{j\theta_\Gamma}}{1-|\Gamma| e^{j\theta_\Gamma}} = r + jx$$

$$\Rightarrow r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}, \quad x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\Rightarrow \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 : r\text{-circle}, \quad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 : x\text{-circle}$$



Several salient properties of the *r*-circles:

1. The centers of all *r*-circles lie on the Γ_r-axis.
2. The *r*=0 circle, having a unity radius and centered at the origin, is the largest.
3. The *r*-circles become progressively smaller as *r* increases from 0 toward ∞, ending at the (Γ_r=1, Γ_i=0) point for open-circuit.
4. All *r*-circles pass through the (Γ_r=1, Γ_i=0) point.

Salient properties of the *x*-circles:

1. The centers of all *x*-circles lie on the Γ_r=1 line, those for *x*>0 (inductive reactance) lie above the Γ_r–axis, and those for *x*<0 (capacitive reactance) lie below the Γ_r–axis.
2. The *x*=0 circle becomes the Γ_r–axis.
3. The *x*-circle becomes progressively smaller as |*x*| increases from 0 toward ∞, ending at the (Γ_r=1, Γ_i=0) point for open-circuit.
4. All *x*-circles pass through the (Γ_r=1, Γ_i=0) point.

Summary

1. All |Γ|–circles are centered at the origin, and their radii vary uniformly from 0 to 1.
2. The angle, measured from the positive real axis, of the line drawn from the origin through the point representing z_L equals θ_Γ.
3. The value of the *r*-circle passing through the intersection of the |Γ|–circle and the positive-real axis equals the standing-wave radio *S*.

Application of Smith Chart in lossless transmission line:

$$Z_i(z') = \frac{V(z')}{I(z')} = z_0[\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}],\quad z_i(z') = \frac{Z_i(z)}{Z_0} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} \quad \text{when}$$

$$\phi = \theta_\Gamma - 2\beta z'$$

keep $|\Gamma|$ constant and subtract (rotate in the clockwise direction) an angle $= 2\beta z' = \frac{4\pi z'}{\lambda}$ from θ_Γ. This will locate the point for $|\Gamma|e^{j\varphi}$, which determine Z_i.

Increasing z' ⇔ wavelength toward generator in the clockwise direction

A change of half a wavelength in the line length $\Delta z' = \frac{\lambda}{2}$ ⇔ A change of $2\beta(\Delta z') = 2\pi$ in φ.

Eg. Use the Smith chart to find the input impedance of a section of a 50Ω lossless transmission line that is 0.1 wavelength long and is terminated in a short-circuit.

(Sol.) Given $z_L = 0$, $R_0 = 50(\Omega)$, $z' = 0.1\lambda$

1. Enter the Smith chart at the intersection of r=0 and x=0 (point P_{sc} on the extreme left of chart; see Fig.)
2. Move along the perimeter of the chart $(|\Gamma = 1)|$ by 0.1 "wavelengths toward generator" in a clockwise direction to P_1.

At P_1, read r=0 and $x \cong 0.725$, or $z_i = j0.725$, $Z_i = 50(j0.725) = j36.3(\Omega)$.

Eg. A lossless transmission line of length 0.434λ and characteristic impedance 100Ω is terminated in an impedance 260+*j*180(Ω). Find (a) the voltage reflection coefficient, (b) the standing-wave radio, (c) the input impedance, and (d) the location of a voltage maximum on the line.

(Sol.) (a) Given l=0.434λ, R_0=100Ω, Z_L=260+j180

1. Enter the Smith chart at $z_L = Z_L/R_0 = 2.6 + j1.8$ (point P_2 in Fig.)
2. With the center at the origin, draw a circle of radius $\overline{OP_2} = |\Gamma| = 0.60$. ($\overline{OP_{sc}} = 1$)
3. Draw the straight line OP_2 and extend it to P_2' on the periphery. Read 0.22 on "wavelengths toward generator" scale. $\theta_\Gamma = 21°$, $\Gamma = |\Gamma|e^{j\theta_\Gamma} = 0.60\angle 21°$.

(b) The $|\Gamma| = 0.60$ circle intersects with the positive-real axis OP_{oc} at r=S=4.

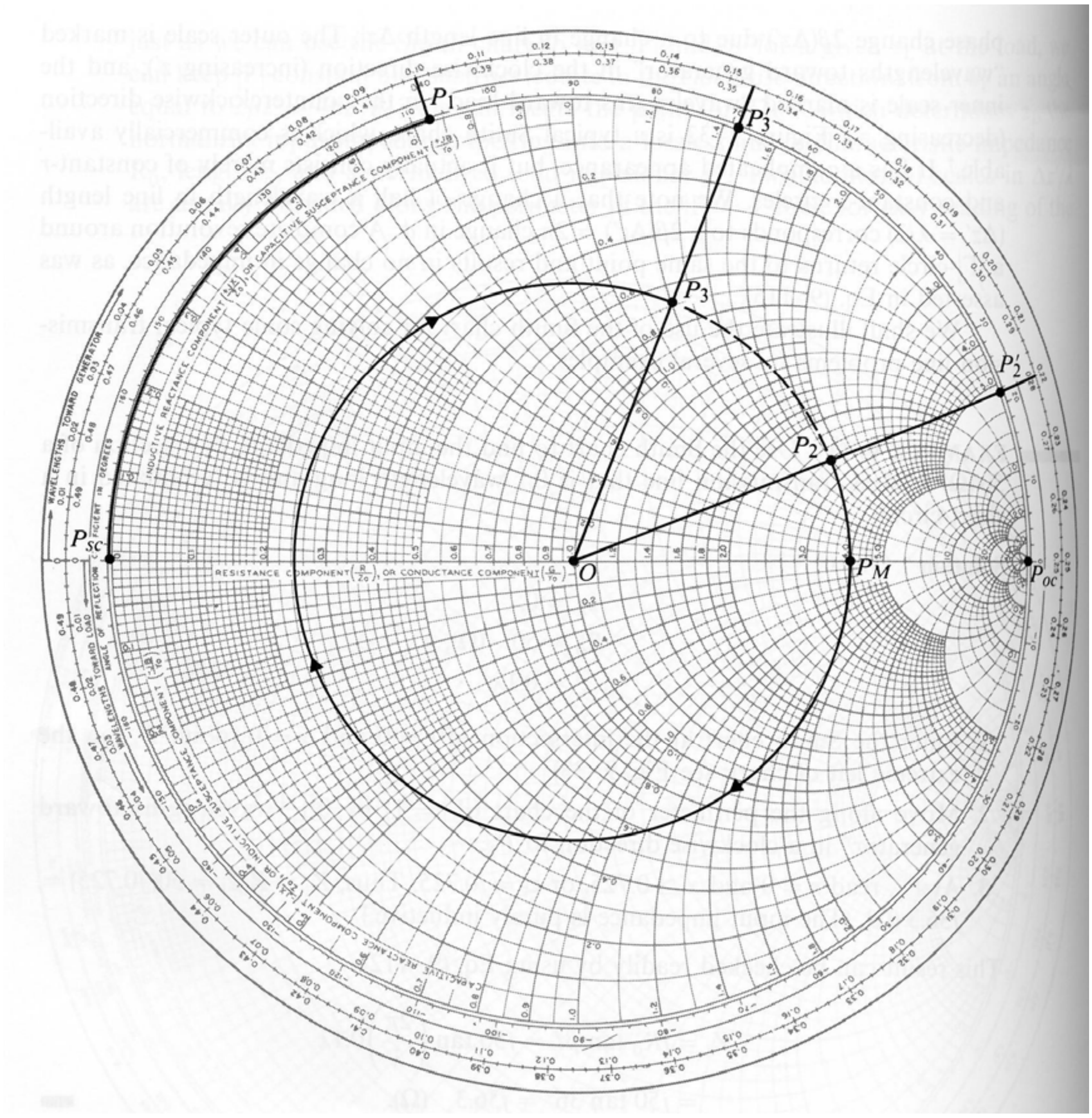
(c) To find the input impedance:

1. Move $P_2{}'$ at 0.220 by a total of 0.434 "wavelengths toward generator," first to 0.500 and then further to 0.154 to $P_3{}'$.

2. Join O and $P_3{}'$ by a straight line which intersects the $|\Gamma| = 0.60$ circle at P_3.

3. Read r=0.69 and x=1.2 at P_3. $Z_i = R_0 z_i = 100(0.69 + j1.2) = 69 + j120(\Omega)$.

(d) In going from P_2 to P_3, the $|\Gamma| = 0.60$ circle intersects the positive-real axis OP_{oc} at P_{M}, where the voltage is a maximum. Thus a voltage maximum appears at (0.250-0.220)λ or 0.030λ from the load.



Application of Smith Chart in lossy transmission line

$$z_i = \frac{1+\Gamma e^{-2\alpha z'}\cdot e^{-2j\beta z'}}{1-\Gamma e^{-2\alpha z'}\cdot e^{-2j\beta z'}} = \frac{1+|\Gamma| e^{-2\alpha z'}\cdot e^{j\phi}}{1-|\Gamma| e^{-2\alpha z'}\cdot e^{j\phi}}$$

∴ We can not simply move close the |Γ|-circle; auxiliary calculation is necessary for the $e^{-2\alpha z'}$ factor.

Eg. The input impedance of a short-circuited lossy transmission line of length 2*m* and characteristic impedance 75Ω (approximately real) is 45+*j*225(Ω). (a) Find *α* and *β* of the line. (b) Determine the input impedance if the short-circuit is replaced by a load impedance Z_L= 67.5-*j*45(Ω).

(Sol.) (a) Enter $z_{i1} = (45 + j225)/75 = 0.60 + j3.0$ in the chart as P_1 in Fig.

Draw a straight line from the origin *O* through P_1 to P_1'.

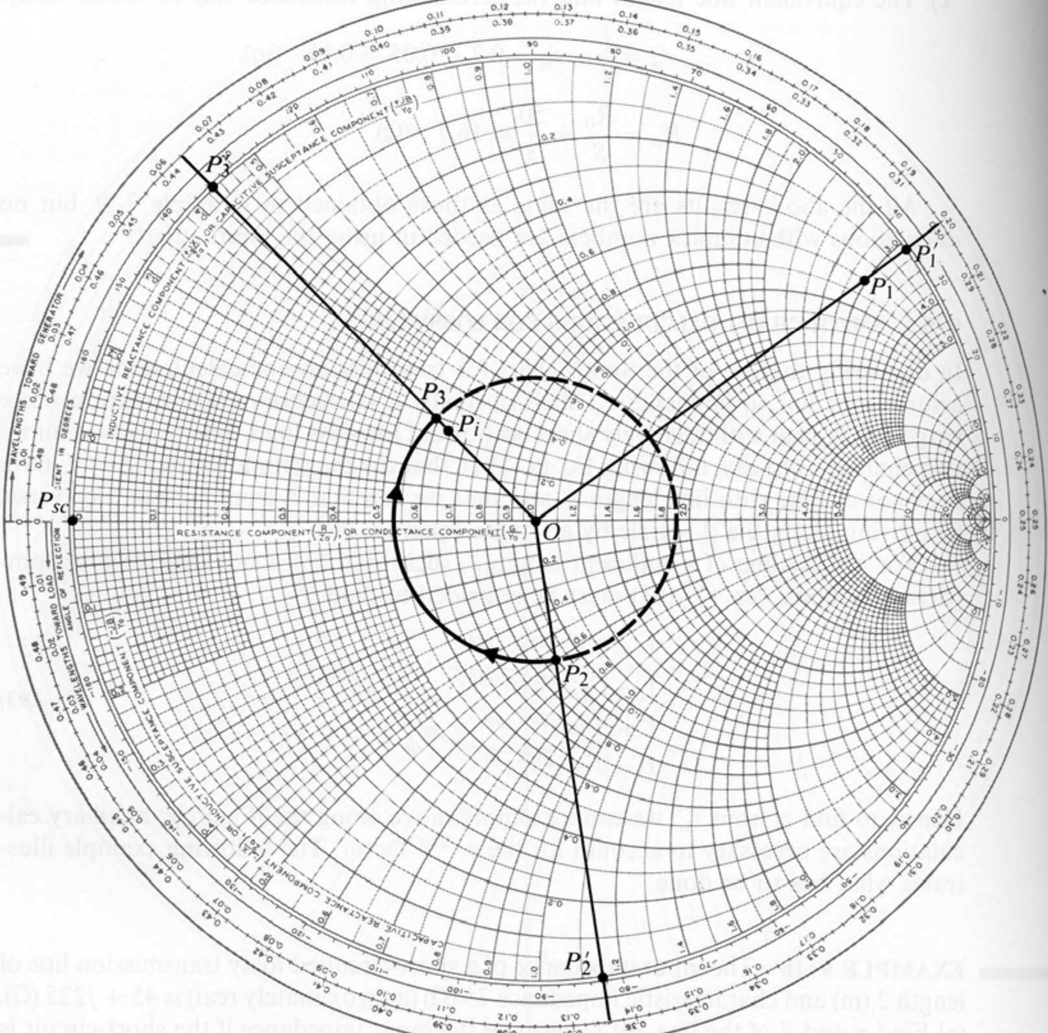
Measure $\overline{OP_1}/\overline{OP_1'} = 0.89 = e^{-2\alpha\ell}$, $\alpha = \frac{1}{2\ell}\ln(\frac{1}{0.89}) = \frac{1}{4}\ln(1.124) = 0.029(Np/m)$

Record that the arc $P_{sc}P_1'$ is 0.20 "wavelengths toward generator". $\ell/\lambda = 0.20$,

$2\beta\ell = 4\pi\ell/\lambda = 0.8\pi$. $\beta = \frac{0.8\pi}{2\ell} = \frac{0.8\pi}{4} = 0.2\pi(rad/m)$.

(b) To find the input impedance for:

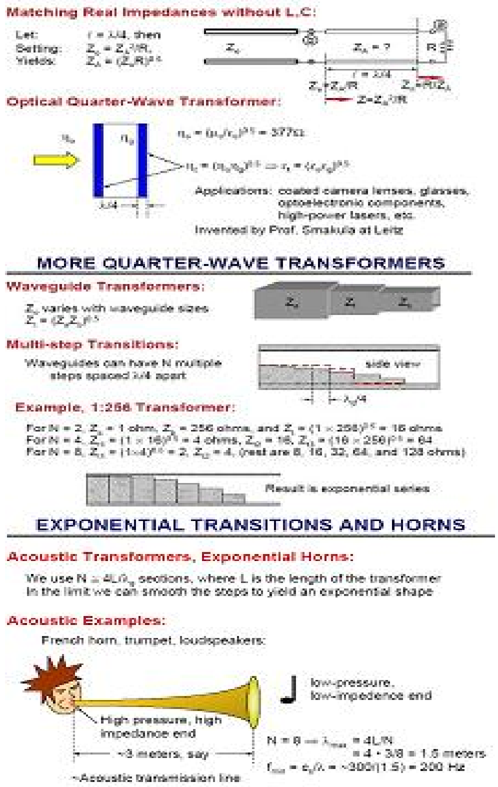
1. Enter $z_L = Z_L/Z_0 = (67.5 - j45)/75 = 0.9 - j0.6$ on the Smith chart as P_2.
2. Draw a straight line from *O* through P_2 to P_2' where the "wavelengths toward generator" reading is 0.364.
3. Draw a $|\Gamma|$–circle centered at *O* with radius $\overline{OP_2}$.
4. Move P_2' along the perimeter by 0.2 "wavelengths toward generator" to P_3' at 0.364+0.20=0.564 or 0.064.
5. Joint P_3' and *O* by a straight line, intersecting the $|\Gamma|$–circle at P_3.
6. Mark on line OP_3 a point P_i such that $\overline{OP_i}/\overline{OP_3} = e^{-2\alpha\ell} = 0.89$.
7. At P_i, read $z_i = 0.64 + j0.27$. $Z_i = 75(0.64 + j0.27) = 48.0 + j20.3(\Omega)$



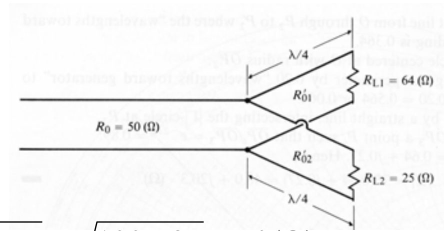
P3'
P1'
P1
P3
Pi
Psc
O
P2
P2'
WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
RESISTANCE COMPONENT (R/Z0), OR CONDUCTANCE COMPONENT (G/Y0)
INDUCTIVE REACTANCE COMPONENT
CAPACITIVE REACTANCE COMPONENT

5-4 Transmission-line Impedance Matching

Impedance matching by λ/4-transformer: $R_0' = \sqrt{R_0 R_L}$



Eg. A signal generator is to feed equal power through a lossless air transmission line of characteristic impedance 50Ω to two separate resistive loads, 64Ω and 25Ω. Quarter-wave transformers are used to match the loads to the 50Ω line. (a) Determine the required characteristic impedances of the quarter-wave lines. (b) Find the standing-wave radios on the matching line sections.



(Sol.) (a) $R_{i1} = R_{i2} = 2R_0 = 100(\Omega)$.

$$R'_{01} = \sqrt{R_{i1} R_{L1}} = \sqrt{100 \times 64} = 80(\Omega), \quad R'_{02} = \sqrt{R_{i2} R_{L2}} = \sqrt{100 \times 25} = 50(\Omega)$$

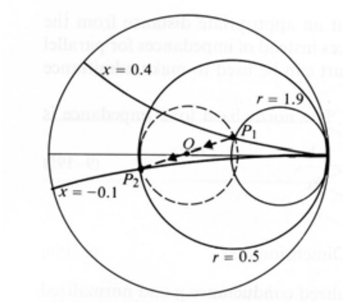
(b) Matching section No. 1:

$$\Gamma_1 = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64 - 80}{64 + 80} = -0.11, \quad S_1 = \frac{1 + |\Gamma_1|}{1 - |\Gamma_1|} = \frac{1 + 0.11}{1 - 0.11} = 1.25$$

Matching section No. 2:

$$\Gamma_2 = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33, \quad S_2 = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$

Application of Smith Chart in obtaining admittance:



$Y_L = 1/Z_L$, $z_L = \dfrac{Z_L}{R_0} = \dfrac{1}{R_0 Y_L} = \dfrac{1}{y_L}$, where $y_L = Y_L / Y_0 = Y_0 / G_0 = R_0 Y_L = y + jb$

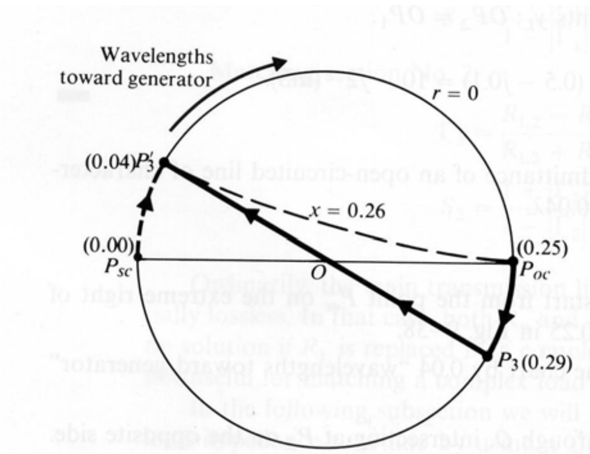
Eg. Find the input admittance of an open-circuited line of characteristic impedance 300Ω and length 0.04λ.

(Sol.) 1. For an open-circuited line we start from the point P_{oc} on the extreme right of the impedance Smith chart, at 0.25 in Fig.

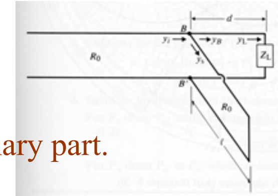
2. Move along the perimeter of the chart by 0.04 “wavelengths toward generator” to P_3 (at 0.29).

3. Draw a straight line from P_3 through O, intersecting at P_3' on the opposite side.

4. Read at P_3': $y_i = 0 + j0.26$, $Y_i = \dfrac{1}{300}(0 + j0.26) = j0.87mS$.



Application of Smith Chart in single-stub matching:



$$Y_i = Y_B + Y_S = Y_0 = \frac{1}{R_0} \Rightarrow 1 = y_B + y_S$$, where $y_B = R_0 Y_B$, $y_s = R_0 Y_s$

$\because$ $1 + jb_s = y_B$, $\therefore$ $y_s = -jb_s$ and l_B is required to cancel the imaginary part.

Using the Smith chart as an admittance chart, we proceed as y_L follows for single-stub matching:

1. Enter the point representing the normalized load admittance.
2. Draw the $|\Gamma|$-circle for y_L, which will intersect the g=1 circle at two points. At these points, $y_{B1} = 1 + jb_{B1}$ and $y_{B2} = 1 + jb_{B2}$. Both are possible solutions.
3. Determine load-section lengths d_1 and d_2 from the angles between the point representing y_L and the points representing y_{B1} and y_{B2}.

Determine stub length l_{B1} and l_{B2} from the angles between the short-circuit point on the extreme right of the chart to the points representing $-jb_{B1}$ and $-jb_{B2}$, respectively.

Eg. Single-Stub Matching:

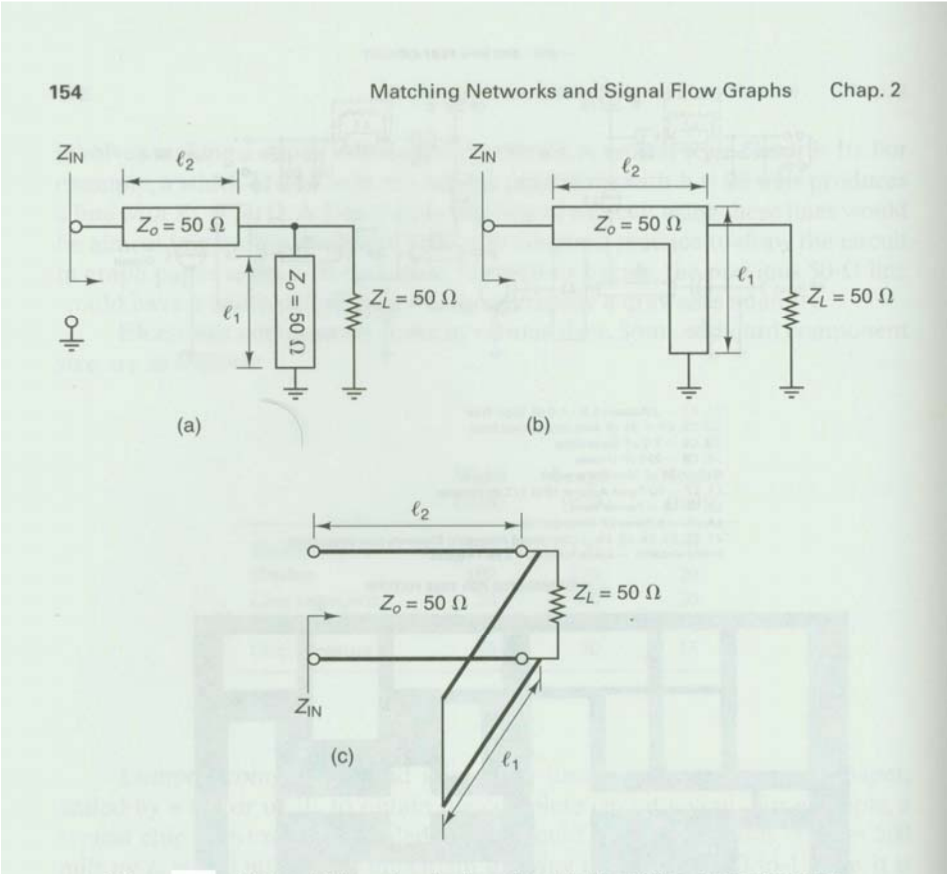


Figure (a) A matching network using microstrip lines; (b) an alternative drawing; (c) schematic using two-wire transmission lines.

Eg. A 50Ω transmission line is connected to a load impedance Z_L= 35-j47.5(Ω). Find the position and length of a short-circuited stub required to match the line.

(Sol.) Given $R_0 = 50(\Omega)$, $Z_L = 35 - j47.5(\Omega)$, $z_L = Z_L / R_0 = 0.70 - j0.95$

1. Enter z_L on the Smith chart as P_1. Draw a $|\Gamma|$–circle centered at O with radius $\overline{OP_1}$.

2. Draw a straight line from P_1 through O to P'_2 on the perimeter, intersecting the $|\Gamma|$–circle at P_2, which represents y_L. Note 0.109 at P'_2 on the "wavelengths toward generator" scale.

3. Two points of intersection of the $|\Gamma|$–circle with the g=1 circle.

At P_3: $y_{B1} = 1 + j1.2 = 1 + jb_{B1}$. At P_4: $y_{B2} = 1 - j1.2 = 1 + jb_{B2}$;

4. Solutions for the position of the stubs:

For P_3 (from P'_2 to P'_3): $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$

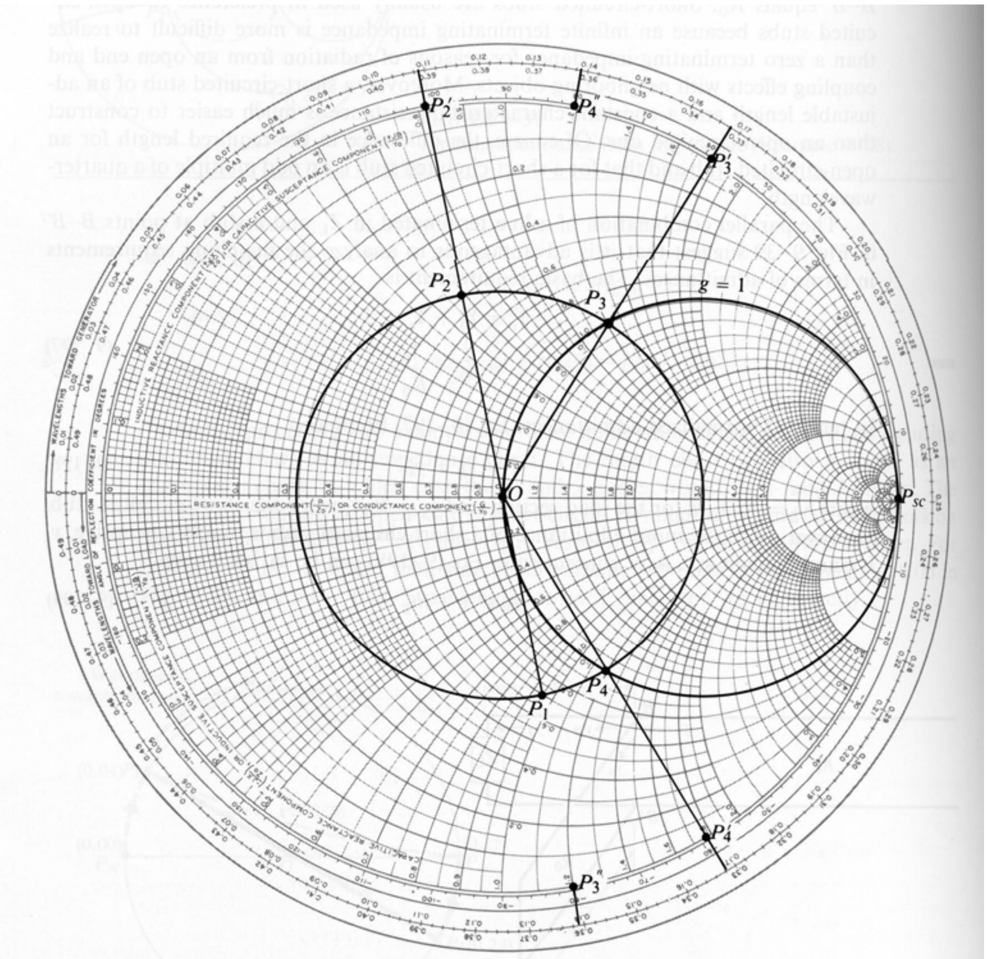
For P_4 (from P'_2 to P'_4): $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$

For P_3 (from P_{sc} to P''_3, which represents $-jb_{B1} = -j1.2$):

$\ell_{B1} = (0.361 - 0.250)\lambda = 0.111\lambda$

For P_4 (from P_{sc} to P''_4, which represents $-jb_{B2} = j1.2$):

$\ell_{B2} = (0.139 + 0.250)\lambda = 0.389\lambda$



5-5 Introduction to *S*-parameters

***S*-parameters:** $[S]=\begin{bmatrix} S_{11} \; S_{12} \\ S_{21} \; S_{22} \end{bmatrix}$ for analyzing the high-frequency circuits.

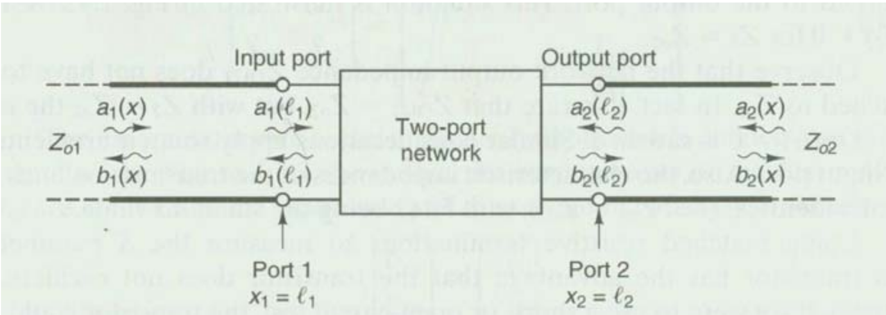


Figure Incident and reflected waves in a two-port network.

Define $a(x)=\dfrac{1}{2\sqrt{Z_0}}[V(x)+Z_0 I(x)]$, $b(x)=\dfrac{1}{2\sqrt{Z_0}}[V(x)-Z_0 I(x)]$

$b_1(l_1)= S_{11}a_1(l_1)+ S_{12}a_2(l_2)$, $b_2(l_2)=S_{21}a_1(l_1)+S_{22}a_2(l_2)$

$\Rightarrow \begin{bmatrix} b_1(l_1) \\ b_2(l_2) \end{bmatrix} = \begin{bmatrix} S_{11} \; S_{12} \\ S_{21} \; S_{22} \end{bmatrix} \cdot \begin{bmatrix} a_1(l_1) \\ a_2(l_2) \end{bmatrix}$, where $S_{11} = \dfrac{b_1(l_1)}{a_1(l_1)}\Big|_{a_2(l_2)=0}$, $S_{21} = \dfrac{b_2(l_2)}{a_1(l_1)}\Big|_{a_2(l_2)=0}$,

$S_{22} = \dfrac{b_2(l_2)}{a_2(l_2)}\Big|_{a_1(l_1)=0}$, and $S_{12} = \dfrac{b_1(l_1)}{a_2(l_2)}\Big|_{a_1(l_1)=0}$.

New *S*-parameters obtained by shifting reference planes:

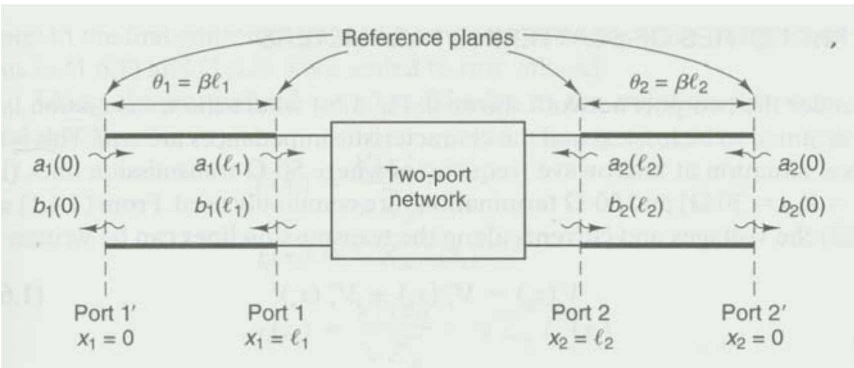


Figure Model for shifting reference planes.

$b_1(l_1)=b_1(0)e^{j\theta_1}$, $a_1(l_1)=a_1(0)e^{-j\theta_1}$, $b_2(l_2)=b_2(0)e^{j\theta_2}$, $a_2(l_2)=a_2(0)e^{-j\theta_2}$

$$\Rightarrow \begin{bmatrix} b_1(0) \\ b_2(0) \end{bmatrix} = \begin{bmatrix} S_{11}e^{-j2\theta_1} & S_{12}e^{-j(\theta_1+\theta_2)} \\ S_{21}e^{-j(\theta_1+\theta_2)} & S_{22}e^{-j2\theta_2} \end{bmatrix} \cdot \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix}, \text{ where}$$

$$\begin{bmatrix} S'_{11} \; S'_{12} \\ S'_{21} \; S'_{22} \end{bmatrix} = \begin{bmatrix} S_{11}e^{-j2\theta_1} & S_{12}e^{-j(\theta_1+\theta_2)} \\ S_{21}e^{-j(\theta_1+\theta_2)} & S_{22}e^{-j2\theta_2} \end{bmatrix} \text{ and } \begin{bmatrix} S_{11} \; S_{12} \\ S_{21} \; S_{22} \end{bmatrix} = \begin{bmatrix} S'_{11}\, e^{j2\theta_1} & S'_{12}\, e^{j(\theta_1+\theta_2)} \\ S'_{21}\, e^{j(\theta_1+\theta_2)} & S'_{22}\, e^{j2\theta_2} \end{bmatrix}$$

***T*-parameters:** $\begin{bmatrix} a_1(l_1) \\ b_1(l_1) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \cdot \begin{bmatrix} b_2(l_2) \\ a_2(l_2) \end{bmatrix}$, where $\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{S_{21}} & -\dfrac{S_{22}}{S_{21}} \\ \dfrac{S_{11}}{S_{21}} & S_{12} - \dfrac{S_{11}S_{22}}{S_{21}} \end{bmatrix}$

and $\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{T_{21}}{T_{11}} & T_{22} - \dfrac{T_{21}T_{12}}{T_{11}} \\ \dfrac{1}{T_{11}} & -\dfrac{T_{12}}{T_{11}} \end{bmatrix}$