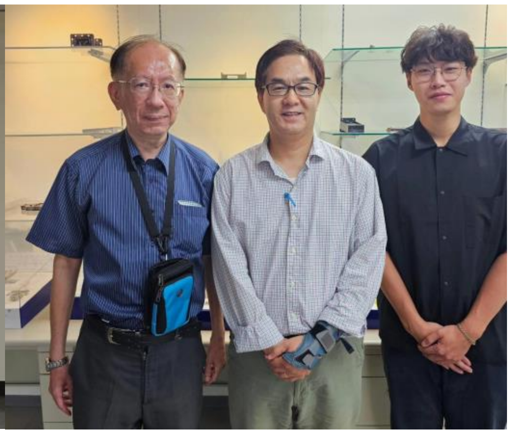

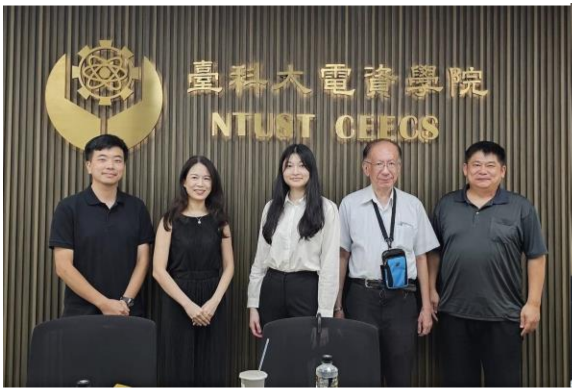
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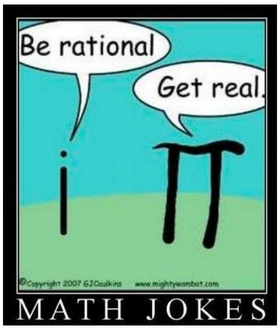
Chapter 1 Complex Numbers, Functions, and Operators

1-1 Complex Numbers and Complex Functions

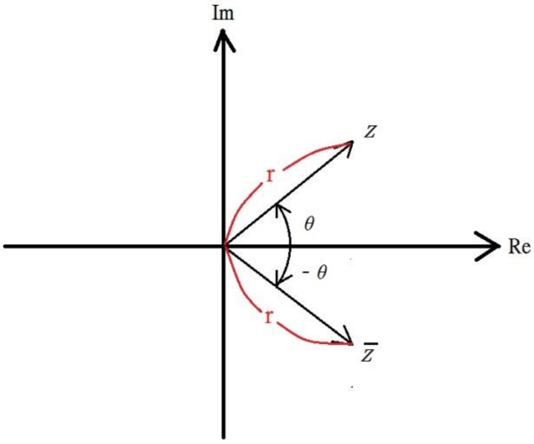


Complex number z**:** $z=a+ib=r\angle\theta=re^{i\theta}=r\cos\theta+ir\sin\theta$, where θ is denoted by $\arg(z)$ in the interval $[-\pi,\pi]$ and $r=|z|=\sqrt{a^2+b^2}$.

Note: $re^{i\theta}=re^{i\theta+i2n\pi}$ and $e^{2n\pi i}=1$

Conjugate complex of z**:** $\bar{z}=a-ib=r\angle-\theta=re^{-i\theta}=r\cos\theta-ir\sin\theta$,

where $r=|\bar{z}|=\sqrt{a^2+b^2}=|z|$ but $\arg(\bar{z})=-\theta=-\arg(z)$



Theorem $z\in R \Leftrightarrow z=\bar{z}$ **,** z **is pure imaginary** $\Leftrightarrow z=-\bar{z}$

Theorem $R_e(z)=\dfrac{z+\bar{z}}{2}=a$, $I_m(z)=\dfrac{z-\bar{z}}{2i}=b$

Theorem $\overline{zw}=\bar{z}\,\bar{w}$, $\overline{z+w}=\bar{z}+\bar{w}$, $|z|^2=z\cdot\bar{z}$

(Proof) $z=a+bi$, $w=c+di$, $\overline{z+w}=(a+c)-(b+d)i=\bar{z}+\bar{w}$

$$\overline{zw}=\overline{(a+bi)(c+di)}=\overline{(ac-bd)+(ad+bc)i}=(ac-bd)-(ad+bc)i$$

$$\bar{z}\cdot\bar{w}=\overline{a+bi}\cdot\overline{c+di}=(a-bi)(c-di)=(ac-bd)-(ad+bc)i,$$

$\therefore\ \overline{zw}=\bar{z}\,\bar{w}$

$|z|^2=a^2+b^2=z\cdot\bar{z}$ but $z^2=(a+bi)^2=a^2-b^2+2abi\neq|z|^2$

Theorem $|zw|=|z||w|$, $\arg(zw)=\arg(z)+\arg(w)$, $|z/w|=|z|/|w|$ **if** $w\neq0$, $\arg(z/w)=\arg(z)-\arg(w)$

(Proof) $z=a+bi=r_1e^{i\theta}$, $w=c+di=r_2e^{i\varphi}$, $|zw|=r_1r_2=|z||w|$, $\arg(zw)=\theta+\varphi=\arg(z)+\arg(w)$

$\arg(z/w)=\theta-\varphi=\arg(z)-\arg(w)$

Theorem $w^2=z=a+ib=r\angle\theta$

$$\Leftrightarrow\ w=\sqrt{r}\angle(\frac{\theta}{2}+n\pi)=\sqrt{r}\cos(\frac{\theta}{2}+n\pi)+i\sqrt{r}\sin(\frac{\theta}{2}+n\pi)$$

Theorem $z \cdot w = R_e(\bar{z}w) = \frac{1}{2}(\bar{z}w + z\bar{w}) = |z||w|\cos\phi$

$$z \times w = I_m(\bar{z}w) = \frac{1}{2i}(\bar{z}w - z\bar{w}) = |z||w|\sin\phi$$

Analytic function *f*(*z*): $f(z)$ and $f'(z)$ are continuous and bounded at a certain point.

Theorem For an analytic complex function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$, we have $f'(z) = \frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

Cauchy-Riemann Equations to test whether $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic:

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ **is analytic** $\Leftrightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

Eg. Is $f(z)=1/z$ is an analytic function?

(Sol.) $f(z)=\frac{1}{z} \to \infty$ as $z\to 0$.

If $z\neq 0$, $f(z)=\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = u(x,y)+iv(x,y)$

$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x} \end{cases}$, Cauchy-Riemann Equations are fulfilled.

$\therefore$ If $z=0$, $f(z)=1/z$ is not analytic. But it is analytic except $z=0$.

Eg. Is $f(x,y)=x+y+i(-x+y)$ an analytical function?

(Sol.) $u(x,y)=x+y$, $v(x,y)=-x+y \Rightarrow \frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = 1 = -\frac{\partial v}{\partial x}$, $\therefore$ $f(x,y)$ is analytic.

Eg. Is $f(x,y)=x^2+iy^2$ an analytic function?

(Sol.) $\begin{matrix} u(x,y) = x^2 \\ v(x,y) = y^2 \end{matrix} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y} = 2y \\ \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is not analytic.

Eg. Is $f(x,y)=x^2-iy^2$ an analytic function? [2009 台灣聯合大學系統碩士班聯招]

(Sol.) $\begin{matrix} u(x,y)=x^2 \\ v(x,y)=-y^2 \end{matrix} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x \neq \dfrac{\partial v}{\partial y}=-2y \\ \dfrac{\partial u}{\partial y}=0=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is not analytic.

Eg. Is $f(x,y)=x^2-y^2-2y+i(2xy+2x)$ an analytic function?

(Sol.) $\begin{cases} u(x,y)=x^2-y^2-2y \\ v(x,y)=2xy+2x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}=-2y-2=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is analytic.

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=x^2-y^2$. Find out its imaginary part $v(x,y)$.

(Sol.) $u(x,y)=x^2-y^2 \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}=-2y=-\dfrac{\partial v}{\partial x} \end{cases} \Rightarrow v(x,y)=2xy+c.$

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=3xy^2-x^3$. Find out its imaginary part $v(x,y)$=? [2005 台科大電子所]

(Sol.) $u(x,y)=3xy^2-x^3 \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=3y^2-3x^2=\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}=6xy=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $v(x,y)=y^3-3x^2y+c$.

Eg. Show that $f(z)=z^3$ is an analytic function.

(Proof) $f(z)=f(x+iy)=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)=u(x,y)+iv(x,y)$

$\dfrac{\partial u}{\partial x}=3x^2-3y^2=\dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y}=-6xy=-\dfrac{\partial v}{\partial x}$, Cauchy-Riemann Equations are fulfilled.

Eg. Show that $f(z)=|z|^2=z\bar{z}$ is not analytic at any point $z\neq 0$.

(Proof) $f(z)=f(x+iy)=x^2+y^2=u(x,y)+iv(x,y) \Rightarrow v(x,y)=0$

$\dfrac{\partial u}{\partial x}=2x\neq\dfrac{\partial v}{\partial y}$ if $x\neq 0$, $\dfrac{\partial u}{\partial y}=2y\neq-\dfrac{\partial v}{\partial x}$ if $y\neq 0 \Rightarrow f(z)$ is not analytic at $z\neq 0$

Eg. Is $f(z)=\bar{z}$ an analytic function?

(Sol.) $f(z)=f(x+iy)=x-iy=u(x,y)+iv(x,y)$, $\begin{cases} \dfrac{\partial u}{\partial x}=1\neq\dfrac{\partial v}{\partial y}=-1 \\ \dfrac{\partial u}{\partial y}=0=-\dfrac{\partial v}{\partial x} \end{cases}$,

$\therefore$ $f(z)$ is not analytic.

Check $f'(i)=\lim\limits_{z\to i}\dfrac{f(z)-f(i)}{z-i}=\lim\limits_{z\to i}\dfrac{\bar{z}-(-i)}{z-i}=\lim\limits_{z\to i}\dfrac{\bar{z}+i}{z-i}$

(a) if $z=\alpha i$, α is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{-\alpha i+i}{\alpha i-i}=\dfrac{1-\alpha}{\alpha-1}=-1$, (b) if $z=\beta+i$, β is real, $\dfrac{\bar{z}+i}{z-i}=\dfrac{\beta-i+i}{\beta+i-i}=1\Rightarrow f'(i)$ does not exist! $\therefore$ $f(z)=\bar{z}$ is not analytic

1-2 Elementary Complex Functions

For $z=x+iy$, where $x,y\in R$

Exponential function: $e^z=e^x[\cos(y)+i\sin(y)]$

Theorem $|e^z|=e^x$.

Theorem $e^z=e^w\Leftrightarrow z=w+2n\pi i$, where n is an integer.

Eg. Find e^z for (a) $z=-\dfrac{i\pi}{4}$, (b) $z=3+\dfrac{\pi}{2}i$. [1990 清大材料所]

(Sol.) (a) $e^{\frac{-i\pi}{4}}=\cos(\dfrac{\pi}{4})-i\sin(\dfrac{\pi}{4})=\dfrac{1-i}{\sqrt{2}}$

(b) $e^{3+\frac{i\pi}{2}}=e^3\cdot[\cos(\dfrac{\pi}{2})+i\sin(\dfrac{\pi}{2})]=ie^3$

Eg. Solve $e^{4z}=1-i\sqrt{3}$. [1991 台大機研]

(Sol.) $e^{4z}=1-i\sqrt{3}=2e^{-i\frac{\pi}{3}}=e^{\ln 2-i\frac{\pi}{3}+i2n\pi}\Rightarrow 4z=\ln 2+i\left(2n\pi-\dfrac{\pi}{3}\right)$

$$\Rightarrow z=\frac{1}{4}\left[\ln 2+i\left(2n\pi-\frac{\pi}{3}\right)\right]$$

Eg. $(-i)^i$=? and $(-1)^{2i}$=?

(Sol.) $-1=e^{i\pi}$, $\ln(-1)=i\pi$ and $-i=e^{-i\pi/2}$, $\ln(-i)=-i\pi/2$,

$(-i)^i=e^{i\ln(-i)}=e^{i(-i\pi/2)}=e^{\frac{\pi}{2}}$

$(-1)^{2i}=e^{2i\ln(-1)}=e^{2i(i\pi)}=e^{-2\pi}$

Eg. Compute $(1+i)^{1-i}$=? [2004 清大電研]

(Sol.) $1+i=\sqrt{2}e^{\frac{i\pi}{4}}$, $\ln(1+i)=\frac{\ln(2)}{2}+\frac{i\pi}{4}$, $(1+i)^{1-i}=e^{(1-i)\ln(1+i)}=e^{(1-i)[\frac{\ln(2)}{2}+\frac{i\pi}{4}]}$

$=e^{[\frac{\ln(2)}{2}+\frac{\pi}{4}]+i[\frac{\pi}{4}-\frac{\ln(2)}{2}]}=\sqrt{2}e^{\frac{\pi}{4}+i[\frac{\pi}{4}-\frac{\ln(2)}{2}]}=\sqrt{2}e^{\frac{\pi}{4}}\cdot\{\cos\left[\frac{\pi}{4}-\frac{\ln(2)}{2}\right]+i\sin[\frac{\pi}{4}-\frac{\ln(2)}{2}]\}$

Eg. Compute $(1-2i)^9e^{3i}$. [2005 台科大電子所]

(Sol.) $1-2i=\sqrt{5}\,e^{-i\tan^{-1}(2)}$, $(1-2i)^9e^{3i}=(\sqrt{5})^9\,e^{3i-9i\tan^{-1}(2)}$.

Eg. Compute $(i)^{1-2i}$=? [2017 台聯大電研工數 A]

(Sol.) $(i)^{1-2i}=e^{(1-2i)ln(i)}$, $i=e^{\frac{i\pi}{2}}$, $ln(i)=\frac{i\pi}{2}$, $(1-2i)\cdot ln(i)=\pi+\frac{i\pi}{2}$, $\therefore$ $(i)^{1-2i}=e^{(1-2i)\cdot ln(i)}=e^{\pi+\frac{i\pi}{2}}=ie^{\pi}$

Eg. Compute $(3+4i)^{1/3}$ =? [2016 台聯大電研工數 A]

(Sol.) $(3+4i)^{1/3}=e^{\frac{\ln(3+4i)}{3}}$, $3+4i=5e^{i\cdot tan^{-1}(\frac{4}{3})}$, $ln(3+4i)=ln(5)+i\cdot tan^{-1}(\frac{4}{3})$

$(3+4i)^{1/3}=e^{\frac{\ln(3+4i)}{3}}=e^{\frac{\ln(5)}{3}}\cdot e^{\frac{i\cdot tan^{-1}(\frac{4}{3})}{3}}=e^{\frac{\ln(5)}{3}}\cdot[\cos\left(\frac{tan^{-1}(\frac{4}{3})}{3}\right)+i\sin(\frac{tan^{-1}(\frac{4}{3})}{3})]$

Trigonometric functions: $\sin(z)=\dfrac{e^{iz}-e^{-iz}}{2i}=\dfrac{1}{2i}[e^{i(x+iy)}-e^{-i(x+iy)}]$

$=\sin(x)\cosh(y)+i\cos(x)\sinh(y)$

$\cos(z)=\dfrac{e^{iz}+e^{-iz}}{2}=\dfrac{1}{2}[e^{i(x+iy)}+e^{-i(x+iy)}]=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$

Theorem $\sin(iy)=i\sinh(y)$ and $\cos(iy)=\cosh(y)$.

Hyperbolic functions: $\sinh(z)=\dfrac{e^{z}-e^{-z}}{2}$, $\cosh(z)=\dfrac{e^{z}+e^{-z}}{2}$

Logarithm function: $\log(z)=\log|z|+i\arg(z)+2n\pi i=\log|z|+i\arg(z)$

$=\log\sqrt{x^2+y^2}+i\tan^{-1}\left(\dfrac{y}{x}\right)=\dfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\dfrac{y}{x}\right)$

Inverse trigonometric functions:

$$\sin^{-1}z=k\pi+(-1)^k\cdot\left[\frac{\pi}{2}-iLog(z+\sqrt{1-z^2})\right],\quad k=0,\pm1,\pm2,\cdots$$

$$\cos^{-1}z=2k\pi+i(-1)^k\cdot Log(z+\sqrt{1-z^2}),\quad k=0,\pm1,\pm2\cdots,\quad n=1,2$$

$$\tan^{-1}z=k\pi+\frac{1}{2i}Log\left(\frac{1+iz}{1-iz}\right),\quad k=0,\pm1,\pm2\cdots$$

$$\sinh^{-1}z=k\pi i-(-1)^k\cdot\left\{Log\left[iz+\sqrt{1+z^2}\right]+\frac{\pi}{2}i\right\}$$

$$\cosh^{-1} z = 2k\pi i + (-1)^n \cdot Log\left[z + \sqrt{1-z^2}\right]$$

Eg. Show that $\tan^{-1}z = \frac{i}{2}\ln[\frac{i+z}{i-z}]$**. [1991 清大動機研]**

(Sol.) Let $w = \tan^{-1}z$,

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{e^{iw}-e^{-iw}}{2i} \cdot \frac{2}{e^{iw}+e^{-iw}} = \frac{1}{i} \cdot \frac{e^{2iw}-1}{e^{2iw}+1} \Rightarrow e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$\Rightarrow 2iw = \ell n \frac{i-z}{i+z} \Rightarrow w = \tan^{-1} z = \frac{1}{2i}\ln[\frac{i-z}{i+z}] = -\frac{i}{2}\ln[\frac{i-z}{i+z}] = \frac{i}{2}\ln[\frac{i+z}{i-z}]$$

Eg. Find all roots of the equation sin(z)=cosh(4). [2015 中央電研固態組、生醫電子組]

(Sol.) $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \cosh(4) = \frac{e^4+e^{-4}}{2}$, $e^{iz}-e^{-iz} = i\cdot(e^4+e^{-4})$

1° $\quad e^{iz} = ie^{4+i2n\pi} = e^{\frac{i\pi}{2}+i2n\pi+4} \Rightarrow z = \frac{\pi}{2} + 2n\pi - 4i$

2° $\quad e^{-iz} = -ie^{-4-i2n\pi} = e^{\frac{-i\pi}{2}-i2n\pi-4} \Rightarrow z = \frac{\pi}{2} + 2n\pi - 4i$

$\therefore \quad z = \frac{\pi}{2} + 2n\pi - 4i$

1-3 Complex Operators $\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$**,** $\overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$**, and** $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \overline{z}}$

Theorem **Let** $F(x,y)$**=** $G(z,\overline{z})$ **be a complex scalar function, and** $A(x,y)=P(x,y)+iQ(x,y)$**=** $B(z,\overline{z})$ **be a complex vector function. Then we have** $\nabla F = 2\frac{\partial G}{\partial \overline{z}}$**,** $\nabla \cdot A = 2\mathrm{Re}\left(\frac{\partial B}{\partial z}\right)$**,** $\nabla \times A = 2I_m\left(\frac{\partial B}{\partial z}\right)$**, and** $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \overline{z}}$**.**

(Proof) $\frac{\partial}{\partial x} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$, $\quad \frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}}$

$$\therefore \quad \nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \overline{z}}, \quad \overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z} \Rightarrow \nabla F(x,y) = 2\frac{\partial}{\partial \overline{z}}\{G(z,\overline{z})\}$$

$$\nabla \cdot A = R_e(\overline{\nabla} A) = R_e\left(2\frac{\partial}{\partial z}B\right) = 2R_e\left(\frac{\partial B}{\partial z}\right) = R_e\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P+iQ)\right\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\nabla \times A = I_m(\overline{\nabla} A) = I_m\left(2\frac{\partial}{\partial z}B\right) = 2I_m\left(\frac{\partial B}{\partial z}\right) = I_m\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P+iQ)\right\} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\nabla^2 = \nabla \cdot \nabla = R_e(\overline{\nabla}\nabla) = R_e\left(2\frac{\partial}{\partial z}2\frac{\partial}{\partial \overline{z}}\right) = 4\frac{\partial^2}{\partial z \partial \overline{z}}$$

Eg. Solve $\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}=\dfrac{8}{x^2+y^2}$.

(Sol.) $\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}=\nabla^2 U=4\dfrac{\partial^2 U}{\partial z\partial \bar{z}}=\dfrac{8}{x^2+y^2}=\dfrac{8}{z\bar{z}}$

$\Rightarrow U(z,\bar{z})=2\ln|z|\cdot\ln|\bar{z}|+F_1(z)+F_2(\bar{z})$

$U(x,y)==2\left[\ell n\sqrt{x^2+y^2}+i\tan^{-1}\left(\dfrac{y}{x}\right)\right]\cdot\left[\ell n\sqrt{x^2+y^2}-i\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

$+F_1(x+iy)+F_2(x-iy)=2\left[\ell n\sqrt{x^2+y^2}\right]^2+2\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]^2+F_1(x+iy)+F_2(x-iy)$

Eg. Solve $\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}=x^2-y^2$.

(Sol.) $\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}=\nabla^2 U=4\dfrac{\partial^2 U}{\partial z\partial \bar{z}}=x^2-y^2=(\dfrac{z+\bar{z}}{2})^2-(\dfrac{z-\bar{z}}{2i})^2=\dfrac{z^2+\bar{z}^2}{2}$

$\Rightarrow U(z,\bar{z})=\dfrac{z^3\bar{z}+z\bar{z}^3}{24}+F_1(z)+F_2(\bar{z})$

$U(x,y)=\dfrac{x^4-y^4}{12}+F_1(x+iy)+F_2(x-iy)$

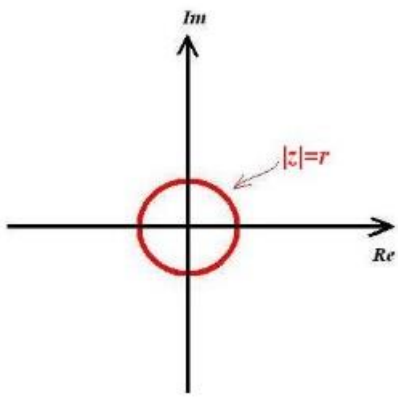
Green's theorem $\oint_c P(x,y)dx+Q(x,y)dy=\iint_R\left(\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}\right)dxdy$

General Green's theorem $\oint_c P(z,\bar{z})dz+Q(z,\bar{z})d\bar{z}=2i\iint_R\left(\dfrac{\partial P}{\partial \bar{z}}-\dfrac{\partial Q}{\partial z}\right)dA$

Eg. Show that $A=\dfrac{1}{4i}\oint_c \bar{z}dz-zd\bar{z}$.

(Proof) $\dfrac{1}{4i}\oint_c \bar{z}dz-zd\bar{z}=\dfrac{1}{4i}\cdot 2i\iint_R[1-(-1)]dxdy=\iint_R dxdy=A$

Eg. Show that the area of the circle with radius *r* is πr^2.



(Proof) $A=\dfrac{1}{4i}\oint_c \bar{z}dz-zd\bar{z}=\dfrac{1}{4i}\cdot\oint_c[re^{-i\theta}d(re^{i\theta})-re^{i\theta}d(re^{-i\theta})]$

$=\dfrac{1}{4i}\cdot\int_0^{2\pi}[ir^2d\theta-(-i)r^2d\theta]=\pi r^2$