Chapter 1 Complex Numbers, Functions, and Operators
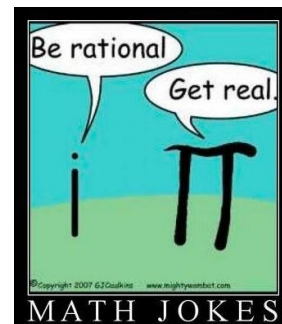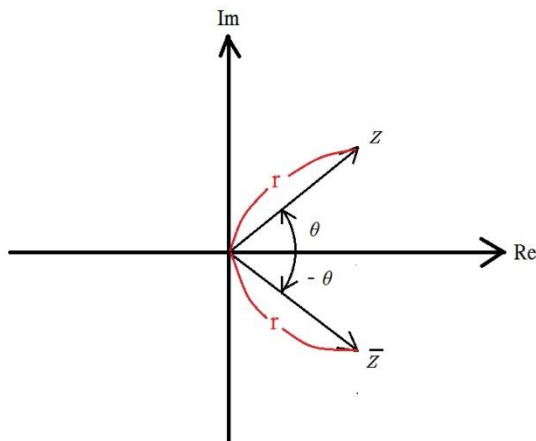
1-1 Complex Numbers and Complex Functions

Complex number z: $z=a+ib=r\angle\theta=re^{i\theta}=r\cos\theta+ir\sin\theta$, where θ is denoted by arg(z) in the interval $[-\pi,\pi]$ and $r=|z|=\sqrt{a^2+b^2}$.

Note: $re^{i\theta}=re^{i\theta+i2n\pi}$ and $e^{2n\pi i}=1$

Conjugate complex of z: $\bar{z}=a-ib=r\angle-\theta=re^{-i\theta}=r\cos\theta-ir\sin\theta$,

where $r=|\bar{z}|=\sqrt{a^2+b^2}=|z|$ but arg($\bar{z}$)= $-\theta$= -arg(z)

Theorem $z \in R \Leftrightarrow z=\bar{z}$, z **is pure imaginary** $\Leftrightarrow z=-\bar{z}$

Theorem $R_e(z)=\dfrac{z+\bar{z}}{2}=a$, $I_m(z)=\dfrac{z-\bar{z}}{2i}=b$

Theorem $\overline{zw}=\bar{z}\,\bar{w}$, $\overline{z+w}=\bar{z}+\bar{w}$, $|z|^2=z\cdot\bar{z}$

(Proof) $z=a+bi$, $w=c+di$, $\overline{z+w}=(a+c)-(b+d)i=\bar{z}+\bar{w}$

$\overline{zw}=\overline{(a+bi)(c+di)}=\overline{(ac-bd)+(ad+bc)i}=(ac-bd)-(ad+bc)i$

$\bar{z}\cdot\bar{w}=\overline{a+bi}\cdot\overline{c+di}=(a-bi)(c-di)=(ac-bd)-(ad+bc)i,$

$\therefore \overline{zw}=\bar{z}\,\bar{w}$

$|z|^2=a^2+b^2=z\cdot\bar{z}$ but $z^2=(a+bi)^2=a^2-b^2+2abi\neq|z|^2$

Theorem $|zw|=|z||w|$, $\mathbf{arg}(zw)=\mathbf{arg}(z)+\mathbf{arg}(w)$, $|z/w|=|z|/|w|$ **if** $w\neq0$, $\mathbf{arg}(z/w)=\mathbf{arg}(z)-\mathbf{arg}(w)$

(Proof) $z=a+bi=r_1e^{i\theta}$, $w=c+di=r_2e^{i\varphi}$, $|zw|=r_1r_2=|z||w|$, arg(zw)=$\theta+\varphi$=arg(z)+arg(w)

arg(z/w)=$\theta-\varphi$=arg(z)-arg(w)

Theorem $w^2=z=a+ib=r\angle\theta$

$\Leftrightarrow w=\sqrt{r}\angle(\dfrac{\theta}{2}+n\pi)=\sqrt{r}\cos(\dfrac{\theta}{2}+n\pi)+i\sqrt{r}\sin(\dfrac{\theta}{2}+n\pi)$

Theorem $z \cdot w = R_e(\bar{z}w) = \frac{1}{2}(\bar{z}w + z\bar{w}) = |z||w|\cos\phi$

$$z \times w = I_m(\bar{z}w) = \frac{1}{2i}(\bar{z}w - z\bar{w}) = |z||w|\sin\phi$$

Analytic function $f(z)$: $f(z)$ and $f'(z)$ are continuous and bounded at a certain point.

Theorem For an analytic complex function $f(z)=f(x+iy)=u(x,y)+iv(x,y)$, we have
$$f'(z) = \frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Cauchy-Riemann Equations to test whether $f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic:

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ is analytic $\Leftrightarrow$ $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

Eg. Is $f(z)=1/z$ is an analytic function?

(Sol.) $f(z)=\dfrac{1}{z} \to \infty$ as $z \to 0$.

If $z \neq 0$, $f(z)=\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x}{x^2+y^2} - \dfrac{iy}{x^2+y^2} = u(x,y)+iv(x,y)$

$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{-x^2+y^2}{(x^2+y^2)^2} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2} = -\dfrac{\partial v}{\partial x} \end{cases}$, Cauchy-Riemann Equations are fulfilled.

$\therefore$ If $z=0$, $f(z)=1/z$ is not analytic. But it is analytic except $z=0$.

Eg. Is $f(x,y)=x+y+i(-x+y)$ an analytical function?

(Sol.) $u(x,y)=x+y$, $v(x,y)=-x+y \Rightarrow \dfrac{\partial u}{\partial x} = 1 = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = 1 = -\dfrac{\partial v}{\partial x}$, $\therefore f(x,y)$ is analytic.

Eg. Is $f(x,y)=x^2+iy^2$ an analytic function?

(Sol.) $\begin{array}{l} u(x,y)=x^2 \\ v(x,y)=y^2 \end{array} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = 2x \neq \dfrac{\partial v}{\partial y} = 2y \\ \dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore f(x,y)$ is not analytic.

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Eg. Is $f(x,y)=x^2-iy^2$ an analytic function? [2009 台灣聯合大學系統碩士班聯招]

(Sol.) $\begin{aligned} u(x,y)&=x^2 \\ v(x,y)&=-y^2 \end{aligned} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x \neq \dfrac{\partial v}{\partial y}=-2y \\[2mm] \dfrac{\partial u}{\partial y}=0=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is not analytic.

Eg. Is $f(x,y)=x^2-y^2-2y+i(2xy+2x)$ an analytic function?

(Sol.) $\begin{cases} u(x,y)=x^2-y^2-2y \\ v(x,y)=2xy+2x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y}=-2y-2=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $f(x,y)$ is analytic.

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=x^2-y^2$. Find out its imaginary part $v(x,y)$.

(Sol.) $u(x,y)=x^2-y^2 \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y}=-2y=-\dfrac{\partial v}{\partial x} \end{cases} \Rightarrow v(x,y)=2xy+c.$

Eg. An analytic function $f(z)=u(x,y)+iv(x,y)$ has a real part $u(x,y)=3xy^2-x^3$. Find out its imaginary part $v(x,y)=?$ [2005 台科大電子所]

(Sol.) $u(x,y)=3xy^2-x^3 \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=3y^2-3x^2=\dfrac{\partial v}{\partial y} \\[2mm] \dfrac{\partial u}{\partial y}=6xy=-\dfrac{\partial v}{\partial x} \end{cases}$, $\therefore$ $v(x,y)=y^3-3x^2y+c.$

Eg. Show that $f(z)=z^3$ is an analytic function.

(Proof) $f(z)=f(x+iy)=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)=u(x,y)+iv(x,y)$

$\dfrac{\partial u}{\partial x}=3x^2-3y^2=\dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y}=-6xy=-\dfrac{\partial v}{\partial x}$, Cauchy-Riemann Equations are fulfilled.

Eg. Show that $f(z)=|z|^2=z\bar{z}$ is not analytic at any point $z\neq0$.

(Proof) $f(z)=f(x+iy)=x^2+y^2=u(x,y)+iv(x,y) \Rightarrow v(x,y)=0$

$\dfrac{\partial u}{\partial x}=2x\neq\dfrac{\partial v}{\partial y}$ if $x\neq0$, $\dfrac{\partial u}{\partial y}=2y\neq-\dfrac{\partial v}{\partial x}$ if $y\neq0 \Rightarrow f(z)$ is not analytic at $z\neq0$

Eg. Is $f(z)= \bar{z}$ an analytic function?

(Sol.) $f(z) = f(x+iy) = x - iy = u(x,y) + iv(x,y)$, $\begin{cases} \dfrac{\partial u}{\partial x} = 1 \neq \dfrac{\partial v}{\partial y} = -1 \\ \dfrac{\partial u}{\partial y} = 0 = -\dfrac{\partial v}{\partial x} \end{cases}$,

$\therefore$ $f(z)$ is not analytic.

Check $f'(i) = \lim\limits_{z \to i} \dfrac{f(z) - f(i)}{z - i} = \lim\limits_{z \to i} \dfrac{\bar{z} - (-i)}{z - i} = \lim\limits_{z \to i} \dfrac{\bar{z} + i}{z - i}$

(a) if $z = \alpha i$, α is real, $\dfrac{\bar{z} + i}{z - i} = \dfrac{-\alpha i + i}{\alpha i - i} = \dfrac{1 - \alpha}{\alpha - 1} = -1$, (b) if $z = \beta + i$, β is real,

$\dfrac{\bar{z} + i}{z - i} = \dfrac{\beta - i + i}{\beta + i - i} = 1 \Rightarrow f'(i)$ does not exist! $\therefore$ $f(z) = \bar{z}$ is not analytic

1-2 Elementary Complex Functions

For $z = x + iy$, where $x, y \in R$

Exponential function: $e^z = e^x[\cos(y) + i\sin(y)]$

Theorem $|e^z| = e^x$.

Theorem $e^z = e^w \Leftrightarrow z = w + 2n\pi i$, where n is an integer.

Eg. Find e^z for (a) $z = -\dfrac{i\pi}{4}$, (b) $z = 3 + \dfrac{\pi}{2}i$. [1990 清大材料所]

(Sol.) (a) $e^{\frac{-i\pi}{4}} = \cos(\dfrac{\pi}{4}) - i\sin(\dfrac{\pi}{4}) = \dfrac{1 - i}{\sqrt{2}}$

(b) $e^{3 + \frac{i\pi}{2}} = e^3 \cdot [\cos(\dfrac{\pi}{2}) + i\sin(\dfrac{\pi}{2})] = ie^3$

Eg. Solve $e^{4z} = 1 - i\sqrt{3}$. [1991 台大機研]

(Sol.) $e^{4z} = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}} = e^{\ln 2 - i\frac{\pi}{3} + i2n\pi} \Rightarrow 4z = \ln 2 + i\left(2n\pi - \dfrac{\pi}{3}\right)$

$\Rightarrow z = \dfrac{1}{4}\left[\ln 2 + i\left(2n\pi - \dfrac{\pi}{3}\right)\right]$

Eg. $(-i)^i = ?$ and $(-1)^{2i} = ?$

(Sol.) $-1 = e^{i\pi}$, $ln(-1) = i\pi$ and $-i = e^{-i\pi/2}$, $ln(-i) = -i\pi/2$,

$(-i)^i = e^{i\ln(-i)} = e^{i(-i\pi/2)} = e^{\frac{\pi}{2}}$

$(-1)^{2i} = e^{2i\ln(-1)} = e^{2i(i\pi)} = e^{-2\pi}$

Eg. Compute $(1+i)^{1-i}$=? [2004 清大電研]

(Sol.) $1+i=\sqrt{2}e^{\frac{i\pi}{4}}$, $\ln(1+i)=\frac{\ln(2)}{2}+\frac{i\pi}{4}$, $(1+i)^{1-i}=e^{(1-i)\ln(1+i)}=e^{(1-i)[\frac{\ln(2)}{2}+\frac{i\pi}{4}]}$

$=e^{[\frac{\ln(2)}{2}+\frac{\pi}{4}]+i[\frac{\pi}{4}-\frac{\ln(2)}{2}]}=\sqrt{2}e^{\frac{\pi}{4}+i[\frac{\pi}{4}-\frac{\ln(2)}{2}]}=\sqrt{2}e^{\frac{\pi}{4}}\cdot\{\cos[\frac{\pi}{4}-\frac{\ln(2)}{2}]+i\sin[\frac{\pi}{4}-\frac{\ln(2)}{2}]\}$

Eg. Compute $(1-2i)^9 e^{3i}$. [2005 台科大電子所]

(Sol.) $1-2i=\sqrt{5}\,e^{-i\tan^{-1}(2)}$, $(1-2i)^9 e^{3i}=(\sqrt{5})^9\,e^{3i-9i\tan^{-1}(2)}$.

Eg. Compute $(i)^{1-2i}$=? [2017 台聯大電研工數 A]

(Sol.) $(i)^{1-2i}=e^{(1-2i)\ln(i)}$, $i=e^{\frac{i\pi}{2}}$, $\ln(i)=\frac{i\pi}{2}$, $(1-2i)\cdot\ln(i)=\pi+\frac{i\pi}{2}$, $\therefore (i)^{1-2i}=e^{(1-2i)\cdot\ln(i)}=e^{\pi+\frac{i\pi}{2}}=ie^{\pi}$

Eg. Compute $(3+4i)^{1/3}$ =? [2016 台聯大電研工數 A]

(Sol.) $(3+4i)^{1/3}=e^{\frac{\ln(3+4i)}{3}}$, $3+4i=5e^{i\cdot\tan^{-1}(\frac{4}{3})}$, $\ln(3+4i)=\ln(5)+i\cdot\tan^{-1}(\frac{4}{3})$

$(3+4i)^{1/3}=e^{\frac{\ln(3+4i)}{3}}=e^{\frac{\ln(5)}{3}}\cdot e^{\frac{i\cdot\tan^{-1}(\frac{4}{3})}{3}}=e^{\frac{\ln(5)}{3}}\cdot[\cos(\frac{\tan^{-1}(\frac{4}{3})}{3})+i\sin(\frac{\tan^{-1}(\frac{4}{3})}{3})]$

Trigonometric functions: $\sin(z)=\dfrac{e^{iz}-e^{-iz}}{2i}=\dfrac{1}{2i}[e^{i(x+iy)}-e^{-i(x+iy)}]$

$$=\sin(x)\cosh(y)+i\cos(x)\sinh(y)$$

$\cos(z)=\dfrac{e^{iz}+e^{-iz}}{2}=\dfrac{1}{2}[e^{i(x+iy)}+e^{-i(x+iy)}]=\cos(x)\cosh(y)-i\sin(x)\sinh(y)$

Theorem $\sin(iy)=i\sinh(y)$ and $\cos(iy)=\cosh(y)$.

Hyperbolic functions: $\sinh(z)=\dfrac{e^z-e^{-z}}{2}$, $\cosh(z)=\dfrac{e^z+e^{-z}}{2}$

Logarithm function: $\log(z)=\log|z|+i\arg(z)+2n\pi i=\log|z|+i\arg(z)$

$$=\log\sqrt{x^2+y^2}+i\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right)$$

Inverse trigonometric functions:

$$\sin^{-1}z=k\pi+(-1)^k\cdot\left[\frac{\pi}{2}-i\,Log(z+\sqrt{1-z^2})\right],\ \ k=0,\pm1,\pm2,\cdots$$

$$\cos^{-1}z=2k\pi+i(-1)^k\cdot Log(z+\sqrt{1-z^2}),\ \ k=0,\pm1,\pm2\cdots,\ \ n=1,2$$

$$\tan^{-1}z=k\pi+\frac{1}{2i}Log\left(\frac{1+iz}{1-iz}\right),\ \ k=0,\pm1,\pm2\cdots$$

$$\sinh^{-1}z=k\pi i-(-1)^k\cdot\left\{Log\left[iz+\sqrt{1+z^2}\right]+\frac{\pi}{2}i\right\}$$

$$\cosh^{-1} z = 2k\pi i + (-1)^n \cdot Log\left\lfloor z + \sqrt{1-z^2}\right\rfloor$$

Eg. Show that tan⁻¹z= $\dfrac{i}{2}\ln[\dfrac{i+z}{i-z}]$ **. [1991 清大動機研]**

(Sol.) Let $w=\tan^{-1}z$,

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{e^{iw}-e^{-iw}}{2i}\cdot\frac{2}{e^{iw}+e^{-iw}} = \frac{1}{i}\cdot\frac{e^{2iw}-1}{e^{2iw}+1} \Rightarrow e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$\Rightarrow 2iw = \ell n\frac{i-z}{i+z} \Rightarrow w = \tan^{-1}z = \frac{1}{2i}\ln[\frac{i-z}{i+z}] = -\frac{i}{2}\ln[\frac{i-z}{i+z}] = \frac{i}{2}\ln[\frac{i+z}{i-z}]$$

Eg. Find all roots of the equation sin(z)=cosh(4). [2015 中央電研固態組、生醫電子組]

(Sol.) $\sin(z) = \dfrac{e^{iz}-e^{-iz}}{2i} = \cosh(4) = \dfrac{e^4+e^{-4}}{2}$, $e^{iz}-e^{-iz} = i\cdot(e^4+e^{-4})$

$1°$ $e^{iz} = ie^{4+i2n\pi} = e^{\frac{i\pi}{2}+i2n\pi+4} \Rightarrow z = \frac{\pi}{2}+2n\pi-4i$

$2°$ $e^{-iz} = -ie^{-4-i2n\pi} = e^{\frac{-i\pi}{2}-i2n\pi-4} \Rightarrow z = \frac{\pi}{2}+2n\pi-4i$

$\therefore z = \frac{\pi}{2}+2n\pi-4i$

1-3 Complex Operators $\nabla \equiv \dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}$, $\overline{\nabla} \equiv \dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}$**, and** $\nabla^2 = 4\dfrac{\partial^2}{\partial z\partial\overline{z}}$

Theorem Let $F(x,y)=G(z,\overline{z})$ **be a complex scalar function, and** $A(x,y)=P(x,y)+iQ(x,y)=B(z,\overline{z})$ **be a complex vector function. Then we have** $\nabla F = 2\dfrac{\partial G}{\partial\overline{z}}$, $\nabla\cdot A = 2\text{Re}\left(\dfrac{\partial B}{\partial z}\right)$, $\nabla\times A = 2I_m\left(\dfrac{\partial B}{\partial z}\right)$**, and** $\nabla^2 = 4\dfrac{\partial^2}{\partial z\partial\overline{z}}$**.**

(Proof) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial x}+\dfrac{\partial}{\partial\overline{z}}\dfrac{\partial\overline{z}}{\partial x} = \dfrac{\partial}{\partial z}+\dfrac{\partial}{\partial\overline{z}}$, $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z}\dfrac{\partial z}{\partial y}+\dfrac{\partial}{\partial\overline{z}}\dfrac{\partial\overline{z}}{\partial y} = i\dfrac{\partial}{\partial z}-i\dfrac{\partial}{\partial\overline{z}}$

$\therefore \nabla \equiv \dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial\overline{z}}$, $\overline{\nabla} \equiv \dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial z} \Rightarrow \nabla F(x,y) = 2\dfrac{\partial}{\partial\overline{z}}\{G(z,\overline{z})\}$

$\nabla\cdot A = R_e(\overline{\nabla}A) = R_e\left(2\dfrac{\partial}{\partial z}B\right) = 2R_e\left(\dfrac{\partial B}{\partial z}\right) = R_e\left\{\left(\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}\right)(P+iQ)\right\} = \dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}$

$\nabla\times A = I_m(\overline{\nabla}A) = I_m\left(2\dfrac{\partial}{\partial z}B\right) = 2I_m\left(\dfrac{\partial B}{\partial z}\right) = I_m\left\{\left(\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}\right)(P+iQ)\right\} = \dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}$

$\nabla^2 = \nabla\cdot\nabla = R_e(\overline{\nabla}\nabla) = R_e\left(2\dfrac{\partial}{\partial z}2\dfrac{\partial}{\partial\overline{z}}\right) = 4\dfrac{\partial^2}{\partial z\partial\overline{z}}$

Eg. Solve $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \dfrac{8}{x^2 + y^2}$.

(Sol.) $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\dfrac{\partial^2 U}{\partial z \partial \bar{z}} = \dfrac{8}{x^2 + y^2} = \dfrac{8}{z\bar{z}}$

$\Rightarrow U(z,\bar{z}) = 2\ln|z| \cdot \ln|\bar{z}| + F_1(z) + F_2(\bar{z})$

$U(x,y) = 2\left[\ell n\sqrt{x^2+y^2} + i\tan^{-1}\left(\dfrac{y}{x}\right)\right] \cdot \left[\ell n\sqrt{x^2+y^2} - i\tan^{-1}\left(\dfrac{y}{x}\right)\right]$

$+ F_1(x+iy) + F_2(x-iy) = 2\left[\ell n\sqrt{x^2+y^2}\right]^2 + 2\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]^2 + F_1(x+iy) + F_2(x-iy)$

Eg. Solve $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = x^2 - y^2$.

(Sol.) $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\dfrac{\partial^2 U}{\partial z \partial \bar{z}} = x^2 - y^2 = (\dfrac{z+\bar{z}}{2})^2 - (\dfrac{z-\bar{z}}{2i})^2 = \dfrac{z^2 + \bar{z}^2}{2}$

$\Rightarrow U(z,\bar{z}) = \dfrac{z^3\bar{z} + z\bar{z}^3}{24} + F_1(z) + F_2(\bar{z})$

$U(x,y) = \dfrac{x^4 - y^4}{12} + F_1(x+iy) + F_2(x-iy)$

Green's theorem $\displaystyle\oint_c P(x,y)dx + Q(x,y)dy = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)dxdy$

General Green's theorem $\displaystyle\oint_c P(z,\bar{z})dz + Q(z,\bar{z})d\bar{z} = 2i\iint_R \left(\dfrac{\partial P}{\partial \bar{z}} - \dfrac{\partial Q}{\partial z}\right)dA$

Eg. Show that $A = \dfrac{1}{4i}\displaystyle\oint_c \bar{z}dz - zd\bar{z}$.

(Proof) $\dfrac{1}{4i}\displaystyle\oint_c \bar{z}dz - zd\bar{z} = \dfrac{1}{4i} \cdot 2i\iint_R [1-(-1)]dxdy = \iint_R dxdy = A$

Eg. Show that the area of the circle with radius r is πr^2.

(Proof) $A = \dfrac{1}{4i}\displaystyle\oint_c \bar{z}dz - zd\bar{z} = \dfrac{1}{4i}\oint_c [re^{-i\theta}d(re^{i\theta}) - re^{i\theta}d(re^{-i\theta})]$

$= \dfrac{1}{4i} \cdot \displaystyle\int_0^{2\pi} [ir^2 d\theta - (-i)r^2 d\theta] = \pi r^2$

$|z|=r$