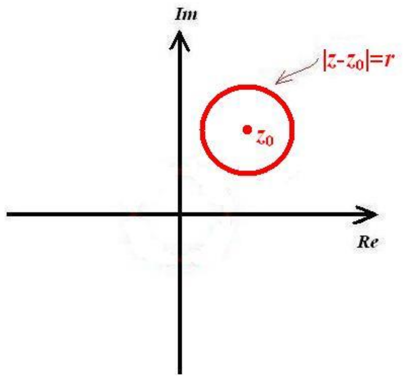
Chapter 2 Integration in the Complex Plane

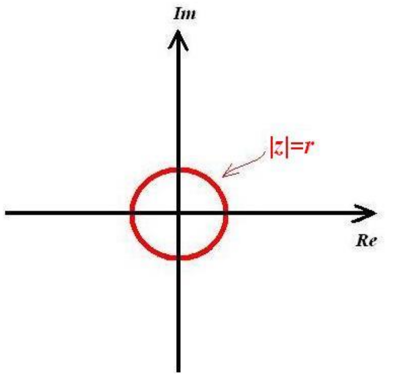
2-1 Complex Line Integrals and Some Integral Theorems

For smooth curve C: $z=z(t)$ for $a \leqq t \leqq b$, then $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$

Special case 1 C: $|z-z_0|=r \Leftrightarrow z(t)=z_0+re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$

Special case 2 C: $|z|=r \Leftrightarrow z(t)=re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$





Eg. Find $\oint_c \frac{1}{z}dz$**,** C**:** $|z|=1$**.**

(Sol.) $|z|=1 \Leftrightarrow z=e^{it}$, $0 \leqq t \leqq 2\pi$, $z'(t)= ie^{it}$, $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$

Eg. Evaluate $\oint_C \frac{dz}{z-3i}$**,** C**:** $|z-3i|=\frac{1}{3}$**.**

(Sol). $z(t)=3i+\frac{1}{3}e^{it}$, $z'(t)=\frac{1}{3}ie^{it}$, $\oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3}e^{it}dt = 2\pi i$

Eg. Evaluate $\oint_c \bar{z}dz$**,** C**:** $|z|=1$**.**

(Sol.) $z(t)=e^{it}$, $\bar{z}=e^{-it}$, $z'(t)= ie^{it}$, $\oint_c \bar{z}dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}dt = 2\pi i$

Eg. Evaluate $\oint_c [z-R_e(z)]dz$**,** C**:** $|z|=2$**.**

(Sol.) $|z|=2 \Leftrightarrow z(t)=2e^{it}$, $z'(t)= i2e^{it}$

$$R_e(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(2e^{it}+2e^{-it}), \quad z-R_e(z) = \frac{1}{2}(z-\bar{z}) = \frac{1}{2}(2e^{it}-2e^{-it})$$

$$\oint_c [z-R_e(z)]dz = \int_0^{2\pi} [z(t)-R_e(z(t))] \cdot z'(t)dt$$

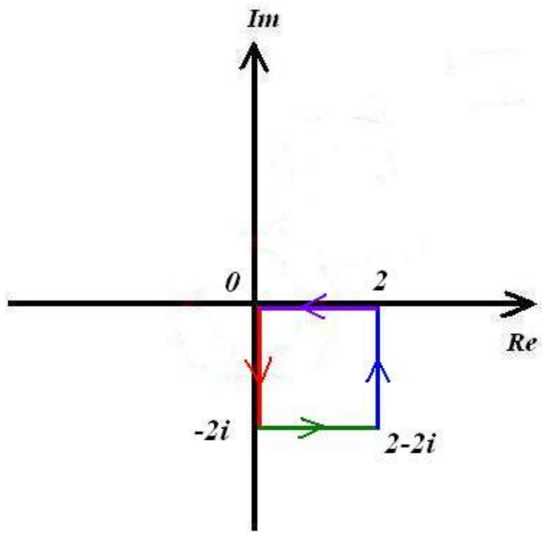
$$= \int_0^{2\pi} \frac{2}{2}(e^{it}-e^{-it}) \cdot 2ie^{it}dt = 2i\int_0^{2\pi} (e^{2it}-1)dt = -4\pi i$$

Eg. Evaluate $\oint_c [z^2 + I_m(z)]dz$, where C is the square with 0, -2*i*, 2-2*i*, and 2.

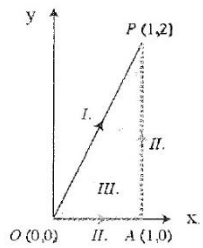
(Sol.) $\oint_c [z^2 + I_m(z)]dz = \int_0^{-2}(-t^2+t)idt + \int_0^2[(t-2i)^2-2]dt + \int_{-2}^0[(2+it)^2+t]idt$

$+\int_2^0(t^2+0)dt = i\left(-\frac{t^3}{3}+\frac{t^2}{2}\right)\Bigg|_0^{-2} + \left(\frac{t^3}{3}-2it^2-6t\right)\Bigg|_0^2$

$+i\left(4t+2it^2-\frac{t^3}{3}+\frac{t^2}{2}\right)\Bigg|_{-2}^0 + \frac{t^3}{3}\Bigg|_2^0 = -4$



Eg. Evaluate $\int_C z^2 dz$, where $z=x+iy$ is a complex number, in each of the following cases. (a) C is the straight line joining the points $O(0,0)$ and $P(1,2)$. (b) C is the straight line from $O(0,0)$ to $A(1,0)$ and then from $A(1,0)$ to $P(1,2)$. (c) C is the parabolic path $y=2x^2$. [2023 台聯大電機聯招]



(Sol.) (I) $\int_C z^2 dz = \int_0^1 (t+2it)^2\cdot(1+2i)dt = (1+2i)^3\cdot\int_0^1 t^2 dt = \frac{-11-2i}{3}$

(II) $\int_C z^2 dz = \int_0^1 t^2 dt + \int_0^2 (1+it)^2\cdot idt = \frac{1}{3}+2i-4-\frac{8i}{3} = \frac{-11-2i}{3}$

(III) $\int_C z^2 dz = \int_0^1 (t+2it^2)^2\cdot(1+4it)dt = \int_0^1 (t^2-20t^4+8it^3-16it^5)dt = \frac{-11-2i}{3}$

Cauchy's integral theorem **Let $f(z)$ be analytic in a simply-connected domain D, C is a simple closed curve in D, then $\oint_c f(z)dz = 0$.**

Eg. Evaluate $\oint_c \frac{1}{z} dz$, C: $|z-2|=1$.

(Sol.) $f(z)=\frac{1}{z}$ is analytic except $z=0$. No poles are within C, ∴ $\oint_c \frac{dz}{z} = 0$

Eg. Evaluate $\oint_{|z|=1} \frac{dz}{z^2-4}$. 【1991 交大土木所】

(Sol.) $f(z)=\frac{1}{z^2-4}$ is analytic except $z=\pm 2$. No poles are within C, ∴ $\oint_{|z|=1} \frac{dz}{z^2-4}=0$

Eg. Evaluate $\oint_C \frac{z}{\sin(z)(z-2i)^3} dz$, C: $|z-8i|=1$.

(Sol.) $f(z) = \frac{z}{\sin(z)(z-2i)^3}$ is analytic except $z=2i$, $n\pi$ ($n=0, \pm 1, \pm 2, \ldots$)

No poles are within C, ∴ $\oint f(z)dz = 0$

Cauchy's integral formulae **Let $f(z)$ be analytic in a simply-connected region D, and let C be a simple curve enclosing z_0 in D, then $\oint_c \frac{f(z)}{z-z_0} dz = 2\pi i \cdot f(z_0)$ and**

$$\oint_c \frac{f(z)dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{(n-1)}(z_0).$$

Eg. Evaluate $\oint_{|z|=3} \frac{e^{2z}dz}{z-2}$. [1991 交大土木所]

(Sol.) $\oint_{|z|=3} \frac{e^{2z}}{z-2} = 2\pi i \cdot e^{2\cdot 2} = 2\pi i e^4$

Eg. Evaluate $\oint_c \frac{e^{iz}}{z^3} dz$, C: $|z|=3$.

(Sol.) $\oint_c \frac{e^{iz}}{z^3} dz = i2\pi \cdot \frac{(e^{iz})''}{(3-1)!}\Big|_{z_0=0} = -i\pi$

Eg. Given $\int_C \frac{f(z)}{z} dz = 2\pi i$ and $\int_C \frac{f(z)}{(z-1)^2} dz = 4\pi i$. If let $f(z)=a+bz$, find a and b.

[2005成大電研]

(Sol.) $2\pi i = 2\pi i \cdot f(0) = 2\pi ai$ and $4\pi i = 2\pi i \cdot f'(0) = 2\pi bi$, ∴ $a=1$ and $b=2$.

Eg. Evaluate $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz$ **and** $\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz$ **if** $C:\left|z-\frac{\pi}{6}\right|=\delta>0$**.**

(Sol.) Let $f(z)=\sin^6(z)$ and $n=3$, $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz = 2\pi i\cdot\sin^6\left(\frac{\pi}{6}\right)=\frac{\pi i}{32}$,

$$\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz=\frac{2\pi i}{2!}\cdot[\sin^6(z)]''\Big|_{z_0=\frac{\pi}{6}}=\frac{21\pi i}{16}$$

Eg. Let z_0 be within C, find $\oint_c \frac{dz}{z-z_0}$ **and** $\oint_c \frac{dz}{(z-z_0)^n}$**, $n\geqq 2$.**

(Sol.) Let $f(z)=1$, $f^{(n-1)}(z_0)=0 \Rightarrow \oint \frac{dz}{z-z_0}=2\pi i$ and $\oint_c \frac{dz}{(z-z_0)^n}=0$.

Eg. Evaluate $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz$**, C is a closed curve not passing 1.**

(Sol.) If C does not enclose 1, $\frac{2\sin(z^2)}{(z-1)^4}$ is analytic within C, $\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz=0$

If C encloses 1, let $f(z)=2\sin(z^2)\Rightarrow \frac{2\sin(z^2)}{(z-1)^4}=\frac{f(z)}{(z-1)^4}$, $n=4$, $n-1=3$

$$f^{(3)}(z)=-24z\sin(z^2)-16z^3\cos(z^2)$$

$$\therefore \oint_c \frac{2\sin(z^2)}{(z-1)^4}dz=\frac{2\pi i}{3!}[-24\sin(1)-16\cos(1)]=\frac{\pi i}{3}[-24\sin(1)-16\cos(1)]$$

Gaussian mean-value theorem **Let $f(z)$ be analytic within $|z-a|=r$, then** $\int_0^{2\pi} f(a+re^{i\theta})d\theta$ **$=2\pi\cdot f(a)$.**

Eg. Evaluate $\int_0^{2\pi}\sin^2[\frac{\pi}{6}+2e^{i\theta}]d\theta$**.**

(Sol.) $a=\frac{\pi}{6}$, $r=2$, and analytic within $|z-\frac{\pi}{6}|=2$,

$\therefore$ $\int_0^{2\pi}\sin^2[\frac{\pi}{6}+2e^{i\theta}]d\theta=2\pi\cdot\sin^2(\frac{\pi}{6})=\pi/2$.

Argument principle Let $f(z)$ be a meromorphic function inside and on some closed contour C, and $f(z)$ has no zeros or poles on C, then $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = N-P$, where N and P are the numbers of zeros and poles of $f(z)$ within C, respectively.

Eg. Evaluate $\oint \frac{f'(z)}{f(z)}dz$ for C: $|z|=\pi$ if (a) $f(z)=\sin(\pi z)$, (b) $f(z)=\cos(\pi z)$, and (c) $f(z)=\tan(\pi z)$..

(Sol.) $\oint \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$

(a) There are 7 zeros: ±3, ±2, ±-1, 0 but no poles within C,

$\therefore \oint \frac{f'(z)}{f(z)}dz = 2\pi i(7-0) = 14\pi i$

(b) There are 6 zeros: ±2.5, ±1.5, ±0.5 but no poles within C,

$\therefore \oint \frac{f'(z)}{f(z)}dz = 2\pi i(6-0) = 12\pi i$

(b) There are 7 zeros: ±3, ±2, ±-1, 0 but 6 poles: ±2.5, ±1.5, ±0.5 within C,

$\therefore \oint \frac{f'(z)}{f(z)}dz = 2\pi i(7-6) = 2\pi i$

Eg. Evaluate $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz$ if $f(z)=\frac{(z^2+1)^2}{(z^2+2z+2)^3}$ and C: $|z|=4$.

(Sol.) $N=4$ and $P=6$, $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = 4-6 = -2$.

Poisson's integral formulae for a circle Let $f(z)$ be analytic within $|z|=R$ and $f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$, then

$f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(Re^{i\phi})}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$, $u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$,

and $v(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)v(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$ for $z=re^{i\theta}$ within $|z|=R$.

Eg. Show that $\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi = 2\pi$.

(Proof) Let $f(re^{i\theta})=1=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$,

$\therefore \int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi = 2\pi$.

Eg. Evaluate $\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi$ **and** $\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi$.

(Sol.) Let R=2, r=1 and $f(re^{i\theta})=e^{e^{i\theta}}=e^{\cos\theta+i\sin\theta}=e^{\cos\theta}[\cos(\sin\theta)+i\sin(\sin\theta)]$,

$\therefore\ \frac{1}{2\pi}\int_0^{2\pi} \frac{3e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=e^{\cos\theta}\cos(\sin\theta)$

and $\frac{1}{2\pi}\int_0^{2\pi} \frac{3e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=e^{\cos\theta}\sin(\sin\theta)$.

$\Rightarrow \int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=\frac{2\pi}{3}e^{\cos\theta}\cos(\sin\theta)$

and $\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=\frac{2\pi}{3}e^{\cos\theta}\sin(\sin\theta)$

Poisson's integral formulae on the upper half plane **Let $f(z)$ be analytic for $y \geqq 0$ on the z-plane and $f(z)=u(x,y)+iv(x,y)$, then**

$f(z)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)^2+y^2}dt$, $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(t,0)}{(t-x)^2+y^2}dt$, **and** $v(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{v(t,0)}{(t-x)^2+y^2}dt$.

2-2 Laurent's Theorem & the Residue Theorem

(*principal part*)

If $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-3}}{(z-z_0)^3}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}$

$+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$

(*analytic part*)

Laurent's theorem $a_n=\frac{1}{2\pi i}\oint_c\frac{f(z)}{(z-z_0)^{n+1}}dz$

Residue: $a_{-1}=\frac{1}{2\pi i}\oint_c f(z)dz$

Residue theorem Let $f(z)$ be analytic in D except z_1, z_2, …, z_n and C encloses z_1, z_2, …, z_n within D. Then we have $\oint_c f(z)dz=2\pi i\cdot\sum_{j=1}^{n}\operatorname{Res}_{z_j}(f)$ **and**

$\operatorname{Res}_{z_j}(f)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$, **where m is the order of a pole $z=z_j$.**

In case of m=1, $\operatorname{Res}_{z_j}(f)=\lim_{z\to z_j}[(z-z_j)\cdot f(z)]$.

Eg. Find the residues of $f(z)=\frac{z}{z-1}$.

(Sol.) At z=1, m=1, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{z-1}]=1$

Eg. Find the residues of $f(z)=\frac{z}{(z-1)(z+1)^2}$.

(Sol.) At z=1, m=1, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{(z-1)(z+1)^2}]=\frac{1}{4}$

At z=-1, m=2, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{z}{(z+1)^2(z-1)}]=\left.\frac{(z-1)-z}{(z-1)^2}\right|_{z=-1}=-\frac{1}{4}$

Eg. Find the residues of $f(z)=\frac{1}{(z-1)^2(z+1)^2}$.

(Sol.)

At z=1, m=2, $\operatorname{Res}_{1}(f)=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z+1)}{(z+1)^4}\right|_{z=1}=-\frac{1}{4}$

At z=-1, m=2, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z-1)}{(z-1)^4}\right|_{z=-1}=\frac{1}{4}$

Eg. Find the residues of $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$.

(Sol.) $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}=\dfrac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}$

At $z=-1$, $m=2$,

$$\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}]=\frac{(2z-2)(z^2+4)-(z^2-2z)(2z)}{(z^2+4)^2}\bigg|_{z=-1}$$

$$=-\frac{14}{25}$$

At $z=2i$, $m=1$, $\operatorname{Res}_{2i}(f)=\lim_{z\to 2i}[(z-2i)\cdot\dfrac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}]=\dfrac{7+i}{25}$

At $z=-2i$, $m=1$, $\operatorname{Res}_{-2i}(f)=\lim_{z\to -2i}[(z+2i)\cdot\dfrac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}]=\dfrac{7-i}{25}$

Eg. Find the residues of $f(z)=\dfrac{\cot(z)\coth(z)}{z^3}$.

(Sol.) It is difficult to compute the residue at 0 by $\operatorname{Res}_{z_j}(f)=\dfrac{1}{(m-1)!}\lim_{z\to z_j}\dfrac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$.

We utilize $f(z)=\displaystyle\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$

$$f(z)=\frac{\cot(z)\coth(z)}{z^3}=\frac{\cos(z)\cosh(z)}{z^3\sin(z)\sinh(z)}$$

$$=\frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+-\cdots\right)}\frac{\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{\left(z+\frac{z^3}{3!}+\cdots\right)}=\frac{\left(1-\frac{z^4}{6}+\cdots\right)}{\left(z^5-\frac{z^9}{90}+-\cdots\right)}==\frac{1}{z^5}-\frac{7}{45}\cdot\frac{1}{z}+\cdots$$

$\therefore$ $\operatorname{Res}_0(f)=-\dfrac{7}{45}$

Eg. Evaluate $\oint_c\dfrac{z^2+1}{z^2-1}dz$, C: $|z-1|=1$. [1991 交大科管所]

(Sol.) There is only one pole 1 within C.

$\operatorname{Res}_1(f)=\lim_{z\to 1}[(z-1)\cdot\dfrac{z^2+1}{z^2-1}]=\lim_{z\to 1}\dfrac{z^2+1}{z+1}=1$, $\therefore$ $\oint_c\dfrac{z^2+1}{z^2-1}dz=2\pi i\cdot 1=2\pi i$

Eg. Evaluate $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ for (a) C: $|z|=\frac{1}{3}$, (b) C: $|z-1|=\frac{1}{3}$, (c) C: $|z|=2$. [1991 台大機研]

(Sol.)

(a) There is only one pole 0 within C.

$\operatorname{Re}_0 s(f)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}[z^2\cdot\frac{\cos(z)}{z^2(z-1)}]=\left.\frac{-(z-1)\sin(z)-\cos(z)}{(z-1)^2}\right|_{z=0}$ =-1, $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =-2πi

(b) There is only one pole 1 within C.

$\operatorname{Re}_1 s(f)=\lim_{z\to 1}[(z-1)\cdot\frac{\cos(z)}{z^2(z-1)}]$=cos(1), $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =2πi · cos(1)

(c) There are two poles 0 and 1 within C. $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =2πi · [-1+cos(1)]

Eg. Find the residue of $f(z)=\frac{\sin(z)}{(z-i)^3}$ and evaluate $\oint_c \frac{\sin(z)}{(z-i)^3}dz$, C: $|z-i|=2$.

(Sol.) m=3, $\operatorname{Re}_i s(f)=\frac{1}{(3-1)!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\cdot\frac{\sin(z)}{(z-i)^3}]=\frac{-1}{2}\sin(i)=-\frac{1}{2}i\sinh(1)$

$\therefore\ \oint \frac{\sin(z)}{(z-i)^2}dz=2\pi i\cdot\left(-\frac{1}{2}i\sinh(1)\right)=\pi\sinh(1)$

Eg. Evaluate $\oint_c \tan z dz$, C: $|z|=2$. [1993 中山電研]

(Sol.) There are two poles ±π/2 within C.

$$\operatorname{Re}_{\frac{\pi}{2}} s(f)=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to\frac{\pi}{2}}\frac{z\sin(z)-\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$=\lim_{z\to\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)-\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$$\operatorname{Re}_{-\frac{\pi}{2}} s(f)=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to-\frac{\pi}{2}}\frac{z\sin(z)+\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$=\lim_{z\to-\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)+\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$\oint_c \tan z dz$=2πi · [(-1)+(-1)]= -4πi

Eg. Evaluate $\oint_c \frac{\sin(z)}{z^2(z^2+4)}dz$**, *C* is any piecewise-smooth curve enclosing 0, 2*i*, and –2*i*.**

(Sol.) $f(z)=\frac{\sin(z)/z}{z(z+2i)(z-2i)}$, $\because \lim_{z\to 0}[\sin(z)/z]=1$, $\therefore$ *f*(*z*) has a removable singularity at 0$\Rightarrow m$ of the pole *z*=0 in *f*(*z*) is 1. $\operatorname{Re}\underset{0}{s}(f)=\lim_{z\to 0} z\cdot\frac{\sin(z)}{z^2(z^2+4)}=\frac{1}{4}$

$$\operatorname{Re}\underset{2i}{s}(f)=\lim_{z\to 2i}(z-2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\operatorname{Re}\underset{-2i}{s}(f)=\lim_{z\to -2i}(z+2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\oint_c f(z)dz=2\pi i\left[\frac{1}{4}-\frac{1}{16}\sinh(2)-\frac{1}{16}\sinh(2)\right]=\frac{\pi i}{2}-\frac{\pi i}{4}\sinh(2)$$

Eg. Evaluate $\oint_c \frac{z^3e^{\frac{1}{z}}}{z^3+1}dz$**, *C*: |*z*|=3. [2013 中山電研]**

(Sol.) Let $t=1/z$, $z=1/t$, $dz=-dt/t^2$, *C*: |*t*|=1/3

$$\oint_c \frac{z^3e^{\frac{1}{z}}}{z^3+1}dz=\oint_c\frac{(\frac{1}{t})^3e^t}{(\frac{1}{t})^3+1}\cdot(\frac{-dt}{t^2})=\oint_c\frac{-e^t}{t^2(t^3+1)}dt$$. There is only one pole 0 within *C*.

$$\operatorname{Re}\underset{0}{s}(f)=\frac{1}{1!}\lim_{t\to 0}\frac{d}{dt}[t^2\cdot\frac{-e^t}{t^2(t^3+1)}]=-1,\ \therefore\ \oint_c\frac{z^3e^{\frac{1}{z}}}{z^3+1}dz=2\pi i\cdot(-1)=-2\pi i$$

Eg. Evaluate $\oint_c \frac{e^{-z}}{\cos(z)}dz$**, *C*: |*z*|=2. [2015 台聯大系統]**

(Sol.) cos(*z*) has two zeros at $z=\pm\pi/2$ within |*z*|=2.

$$\operatorname{Re}\underset{\frac{\pi}{2}}{s}(f)=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{e^{-z}}{\cos(z)}]=\lim_{z\to\frac{\pi}{2}}\frac{ze^{-z}-\frac{\pi}{2}e^{-z}}{\cos(z)}=\lim_{z\to\frac{\pi}{2}}\frac{e^{-z}-ze^{-z}+\frac{\pi}{2}e^{-z}}{-\sin(z)}=-e^{-\frac{\pi}{2}}$$

$$\operatorname{Re}\underset{-\frac{\pi}{2}}{s}(f)=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{e^{-z}}{\cos(z)}]=\lim_{z\to-\frac{\pi}{2}}\frac{ze^{-z}+\frac{\pi}{2}e^{-z}}{\cos(z)}=\lim_{z\to-\frac{\pi}{2}}\frac{e^{-z}-ze^{-z}-\frac{\pi}{2}e^{-z}}{-\sin(z)}=e^{\frac{\pi}{2}}$$

$$\therefore\ \oint_c\frac{e^{-z}}{\cos(z)}dz=2\pi i\cdot(e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}})$$

Eg. Evaluate $\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz$**, *C*: |*z*|=0.5. [2014 中央電研固態組、生醫電子組]**

(Sol.) Let $t=1/z$, $z=1/t$, $dz=-dt/t^2$, C: $|t|=2$

$$\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz=\oint_c \frac{e^t}{(\frac{1}{t})+1}\cdot(\frac{-dt}{t^2})=\oint_c \frac{-e^t}{t(t+1)}dt$$

. There are two poles 0 and -1 within *C*.

$$\operatorname{Res}_0 s(f)=\lim_{t\to 0}[t\cdot\frac{-e^t}{t(t+1)}]=-1,\quad \operatorname{Res}_{-1} s(f)=\lim_{t\to -1}[(t+1)\cdot\frac{-e^t}{t(t+1)}]=e^{-1},$$

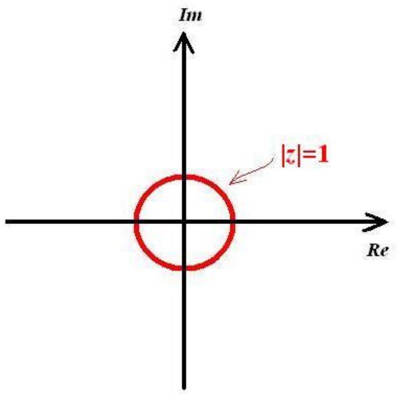
$$\therefore \oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz=2\pi i\cdot(-1+e^{-1})$$

Eg. Evaluate $\oint_C \frac{\ln(z+1)}{z^2+1}dz$**, *C*: |*z*-*i*|=1.4. [2024 台聯大電機聯招]**

(Sol.) $\oint_C \frac{\ln(z+1)}{z^2+1}dz=2\pi i\cdot(z-i)\cdot\frac{\ln(z+1)}{(z+i)(z-i)}\Big|_{z=i}=\pi\cdot\ln(\sqrt{2}\cdot e^{\frac{i\pi}{4}})=\pi\cdot(\frac{\ln(2)}{2}+i\frac{\pi}{4})$

$=\frac{\pi\cdot\ln(2)}{2}+i\frac{\pi^2}{4}$

2-3 Evaluation of Real Integrals



Case 1 $\int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta$

Choose C: $|z|=1$, $z=e^{i\theta}$

$\Rightarrow \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$,

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $d\theta = \frac{dz}{iz}$

$$\Rightarrow I = \int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta = \oint_c K\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\cdot\frac{dz}{iz}$$

Eg. Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$.

(Sol.) $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \oint_c \frac{dz/iz}{5-3\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_c \frac{2idz}{3z^2-10z+3} = \oint_c \frac{2idz}{3(z-\frac{1}{3})(z-3)}$

There is only one pole $\frac{1}{3}$ within $|z|=1$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = 2\pi i\cdot \operatorname*{Res}_{\frac{1}{3}}(f) = 2\pi i\cdot\lim_{z\to\frac{1}{3}}(z-\frac{1}{3})\cdot\frac{2i}{3(z-\frac{1}{3})(z-3)} = 2\pi i\cdot\frac{2i}{-8} = \frac{\pi}{2}.$$

Eg. Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$.

(Sol.)

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{dz/iz}{3-2\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \oint_c \frac{2dz}{(1-2i)z^2+6iz-1-2i}$$

$$= \oint_c \frac{2dz}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}.$$

There is only one pole $\frac{2-i}{5}$ within $|z|=1$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

$$= 2\pi i\cdot \operatorname*{Res}_{\frac{2-i}{5}}(f) = 2\pi i\cdot \lim_{z\to\frac{2-i}{5}}\{(z-\frac{2-i}{5})\cdot\frac{2}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}\} = 2\pi i\cdot(\frac{1}{2i}) = \pi.$$

Eg. Show that $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$ **if $a>|b|$.**

(Proof) $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\oint_c\frac{dz/iz}{a+b\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)}=\oint_c\frac{2dz}{bz^2+2iaz-b}=\oint_c\frac{2dz}{b(z-z_1)(z-z_2)}$

Poles: $z_1=\frac{i}{b}\left(-a+\sqrt{a^2-b^2}\right)$ is within C: $|z|=1$, but $z_2=\frac{i}{b}\left(-a-\sqrt{a^2-b^2}\right)$ is not.

$\operatorname{Res}_{z_1}(f)=\lim_{z\to z_1}(z-z_1)\cdot\frac{2}{b(z-z_1)(z-z_2)}=\frac{2}{b}\cdot\frac{1}{z_1-z_2}=\frac{2}{b}\cdot\frac{b}{2i\sqrt{a^2-b^2}}=\frac{1}{i\sqrt{a^2-b^2}}$

$\therefore\ \int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=2\pi i\cdot\frac{1}{i\sqrt{a^2-b^2}}=\frac{2\pi}{\sqrt{a^2-b^2}}$

Eg. Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{1+\sin^2\theta}$**. [2024 台聯大電研]**

(Sol.) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{1+\sin^2\theta}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{1+\frac{1-\cos(2\theta)}{2}}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2d\theta}{3-\cos(2\theta)}=\int_{-\pi}^{\pi}\frac{d\phi}{3-\cos(\phi)}$

$=\int_{-\pi}^{0}\frac{d\phi}{3-\cos(\phi)}+\int_0^{\pi}\frac{d\phi}{3-\cos(\phi)}=\int_{\pi}^{2\pi}\frac{d\phi}{3-\cos(\phi)}+\int_0^{\pi}\frac{d\phi}{3-\cos(\phi)}=\int_0^{2\pi}\frac{d\phi}{3-\cos(\phi)}=\frac{\pi}{\sqrt{2}}$

Eg. Evaluate $\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}$**. [2011 成大電研]**

(Sol.) $\because\ \cos^2(t)=\frac{\cos(2t)+1}{2}$ and $\sin^2(t)=\frac{1-\cos(2t)}{2}$,

$\therefore\ a^2\cos^2(t)+b^2\sin^2(t)=\frac{(a^2+b^2)+(a^2-b^2)\cos(2t)}{2}$.

Let $\theta=2t$, $\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}=\int_0^{2\pi}\frac{2dt}{(a^2+b^2)+(a^2-b^2)\cos(2t)}$

$=\int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$\because$ The above integrand has a period of 2π,

$\therefore\ \int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}=2\int_0^{2\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$\because\ a^2+b^2>a^2-b^2$, using $\int_0^{2\pi}\frac{d\theta}{c+d\sin\theta}=\int_0^{2\pi}\frac{d\theta}{c+d\cos\theta}=\frac{2\pi}{\sqrt{c^2-d^2}}$ if $c>|d|$.

Set $c=a^2+b^2$ and $d=a^2-b^2$, we have

$\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}=\int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$=2\int_0^{2\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}=2\cdot\frac{2\pi}{\sqrt{(a^2+b^2)^2-(a^2-b^2)^2}}=\frac{2\pi}{ab}$

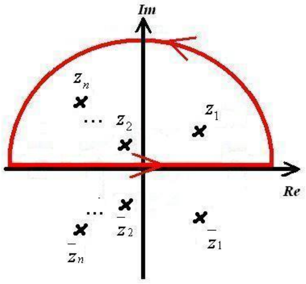
Eg. Show that $\int_0^{2\pi} e^{\cos\theta}\cdot\cos(\theta+\sin\theta)d\theta=\int_0^{2\pi} e^{\cos\theta}\cdot\sin(\theta+\sin\theta)d\theta=0$.

(Proof) Let C: $|z|=1 \Rightarrow z(\theta)=e^{i\theta}=\cos\theta+i\sin\theta$, $\therefore$ $f(z)=e^z$ is analytic within C.

$$\oint_c e^z dz=0=\int_0^{2\pi} e^{\cos\theta+i\sin\theta}de^{i\theta}=\int_0^{2\pi} ie^{\cos\theta}\, e^{i(\theta+\sin\theta)}d\theta$$

$$=i\int_0^{2\pi} e^{\cos\theta}\{[\cos(\theta+\sin\theta)+i[\sin(\theta+\sin\theta)]\}d\theta$$

$$=\int_0^{2\pi} e^{\cos\theta}\cdot\{-\sin(\theta+\sin\theta)+i\cos(\theta+\sin\theta)\}d\theta,\ \therefore\ Re(\cdots)=Im(\cdots)=0$$



Case 2 $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$

Choose C as a semi-circle with infinite radius enclosing the upper half-plane. Poles: z_1, z_2, …, z_n are in the upper half-plane, $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n$ are in the lower half-plane. Assume *deg*(q) ≥ *deg*(p)+2, then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \oint_c f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j}(f)$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{xdx}{(x^2-2x+2)^2}$ **[1991中山電研]**

(Sol.) Poles: 1+i (upper half-plane), 1-i (lower half-plane)

$$\operatorname{Res}_{1+i}\left[\frac{z}{(z^2-2z+2)^2}\right] = \frac{1}{(2-1)!} \lim_{z\to z_j} \frac{d^{2-1}}{dz^{2-1}} \left\{[z-(1+i)]^2 \cdot \frac{z}{[z-(1+i)]^2 \cdot [z-(1-i)]^2}\right\}$$

$$= \lim_{z\to z_j} \frac{d}{dz}\left\{\frac{z}{[z-(1-i)]^2}\right\} = \left.\frac{[z-(1-i)]^2 - 2z[z-(1-i)]}{[z-(1-i)]^4}\right|_{1+i} = \frac{-i}{4},$$

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2-2x+2)^2} = 2\pi i \cdot \operatorname{Res}_{1+i}\left[\frac{z}{(z^2-2z+1)^2}\right] = 2\pi i \cdot \left(\frac{-i}{4}\right) = \pi/2.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+64}dx$.

(Sol.) (a) Poles: 8i (upper half-plane), -8i (lower half-plane)

$$\operatorname{Res}_{8i}(f) = \lim_{z\to 8i}(z-8i) \cdot \frac{1}{(z+8i)(z-8i)} = \frac{1}{16i}, \quad \int_{-\infty}^{\infty} \frac{1}{x^2+64}dx = 2\pi i \cdot \frac{1}{16i} = \frac{\pi}{8}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$. **[2023 台聯大研究所電機工數 A].**

(Sol.) Poles: $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$ (upper half-plane), $e^{\frac{5\pi i}{4}}$, $e^{\frac{7\pi i}{4}}$ (lower half-plane)

$$\operatorname{Res}_{e^{i\pi/4}}(f) = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{z-e^{\frac{i\pi}{4}}}{1+z^4}\right] = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{1}{4z^3}\right] = \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4}e^{-3\pi i/4} = \frac{1}{4}\left[-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right],$$

$$\operatorname{Res}_{e^{i3\pi/4}}(f) = \frac{1}{4\left(e^{i3\pi/4}\right)^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}\left[\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right], \quad \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i\left[\frac{1}{4}\cdot(-i\sqrt{2})\right] = \frac{\pi\sqrt{2}}{2}.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}dx$.

(Sol.) $\oint\frac{e^{iz}}{z^2+1}=2\pi i\cdot\operatorname{Re}\underset{i}{s}(\frac{e^{iz}}{z^2+1})=2\pi i\cdot\lim_{z\to i}(z-i)\cdot\frac{e^{iz}}{(z-i)(z+i)}=2\pi i\cdot\frac{e^{-1}}{2i}=\pi e^{-1}$

$\therefore\ \int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}dx=\pi e^{-1}$ and $\int_{-\infty}^{\infty}\frac{\sin(x)}{x^2+1}dx=0$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx$ **and** $\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx$.

(Sol.) Poles: 2i, 3i (upper half-plane), -2i, -3i (lower half-plane)

$f(z)=\frac{e^{iz}}{(z^2+4)(z^2+9)}$, $\operatorname{Re}\underset{2i}{s}(f)=\frac{e^{-2}}{20i}$, $\operatorname{Re}\underset{3i}{s}(f)=\frac{-e^{-3}}{30i}$

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x^2+9)}dx=2\pi i\left(\frac{e^{-2}}{20i}-\frac{e^{-3}}{30i}\right)=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right)$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right),\ \int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx=0.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx$.

(Sol.) $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx$. Poles: i (upper half-plane), -i (lower half-plane). $f(z)=\frac{z^2}{(z^2+1)^2}=\frac{z^2}{(z+i)^2(z-i)^2}$, m of $(z-i)^2$ is 2.

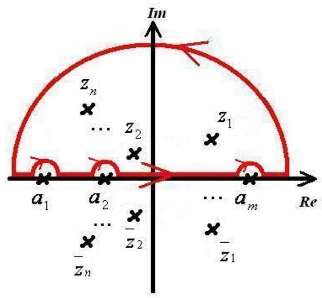
$$\lim_{z\to i}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-i)^2\cdot\frac{z^2}{(z+i)^2(z-i)^2}\right]=\frac{2z(z+i)^2-2z^2(z+i)}{(z+i)^4}\bigg|_{z=i}=\frac{-8i+4i}{16}=-\frac{i}{4}$$

$$\therefore\ \int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{2\pi i}{2}\left(-\frac{i}{4}\right)=\frac{\pi}{4}.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x\sin(x)}{x^2+4}dx$. **[1991 交大電信所]** (Ans.) $\frac{\pi e^{-2}}{2}$

Eg. Evaluate $\int_{0}^{\infty}\frac{\cos(ax)}{x^2+b^2}dx$, $\begin{matrix}a\ge 0\\ b>0\end{matrix}$. **[1991 台大機研]**

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{x^3\sin(ax)}{x^4+4}dx$, a>0. **[2015 中央電研固態組、生醫電子組]**



Case 3 $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$.

Some poles of $q(z)$ are located on the real axis.

Choose C as a semi-circle with infinite radius enclosing the upper half-plane, but excluding the poles on the real axis. Let z_k $(1 \le k \le n)$ be the pole on the upper half-plane and a_j $(1 \le j \le m)$ be the simple pole on the real axis. Then we have $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \cdot \sum_{k=1}^{n} \operatorname{Res}_{z_k}(f) + \pi i \cdot \sum_{j=1}^{m} \operatorname{Res}_{a_j}(f)$.

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x(x^2-4x+5)}dx$. **[2003交大電信所]**

(Sol.) $f(z) = \frac{1}{z(z^2-4z+5)} = \frac{1}{z[z-(2+i)][z-(2-i)]}$ has 3 poles:

0 (on the real axis), $2+i$ (upper half-plane), $2-i$ (lower half-plane)

$$\int_{-\infty}^{\infty} \frac{1}{x(x^2-4x+5)}dx = 2\pi i \cdot \operatorname{Res}_{2+i}(f) + \pi i \cdot \operatorname{Res}_{0}(f)$$

$$= 2\pi i \cdot \lim_{z \to 2+i} [[z-(2+i)] \cdot \frac{1}{z[z-(2+i)][z-(2-i)]}] + \pi i \cdot \lim_{z \to 0} [z \cdot \frac{1}{z(z^2-4z+5)}]$$

$$= \frac{2\pi i}{(2+i) \cdot 2i} + \frac{\pi i}{5} = \frac{\pi(2-i)}{5} + \frac{\pi i}{5} = \frac{2\pi}{5}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^3+1}$. **[2012台聯大系統類似題]**

(Sol.) Poles: $e^{\frac{\pi i}{3}}$ (upper half-plane), -1 (on the real axis), $e^{\frac{5\pi i}{3}}$ (lower half-plane)

$$\operatorname{Res}_{e^{\frac{i\pi}{3}}}(f) = \lim_{z \to e^{\frac{i\pi}{3}}} [\frac{z - e^{\frac{i\pi}{3}}}{z^3+1}] = \lim_{z \to e^{\frac{i\pi}{3}}} [\frac{1}{3z^2}] = \frac{1}{3(e^{\pi i/3})^2} = \frac{1}{3}e^{-2\pi i/3} = \frac{1}{3}\left[-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right],$$

$$\operatorname{Res}_{-1}(f) = \frac{1}{3(-1)^2} = \frac{1}{3}, \quad \int_{-\infty}^{\infty} \frac{dx}{x^3+1} = 2\pi i \cdot \left[\frac{1}{6}(-1-i\sqrt{3})\right] + \pi i \cdot \left[\frac{1}{3}\right] = \frac{\pi\sqrt{3}}{3}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{\cos(\frac{\pi x}{2})}{x-2}dx$ **and** $\int_{-\infty}^{\infty} \frac{\sin(\frac{\pi x}{2})}{x-2}dx$.

(Sol.) Pole: 2 (on the real axis), $\oint_c \frac{e^{i\frac{\pi z}{2}}}{z-2}dz = \pi i \cdot \operatorname{Res}_2[\frac{e^{i\frac{\pi z}{2}}}{z-2}] = \pi i \cdot e^{i\frac{\pi}{2} \cdot 2} = \pi i \cdot e^{i\pi} = -\pi i$,

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(\frac{\pi x}{2})}{x-2}dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{2}x)}{x-2}dx = -\pi.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{\sin(x)}{(x+1)(x-1)} dx$

(Sol.) Poles: 1 and -1 (on the real axis),

$$\operatorname{Re}_{-1} s[\frac{e^{iz}}{(z+1)(z-1)}] = \lim_{z\to -1}[\frac{e^{iz}}{(z+1)(z-1)}\cdot(z+1)] = \lim_{z\to -1}[\frac{e^{iz}}{z-1}] = \frac{e^{-i}}{-2} = \frac{-\cos(1)+i\sin(1)}{2}$$

$$\operatorname{Re}_{1} s[\frac{e^{iz}}{(z+1)(z-1)}] = \lim_{z\to 1}[\frac{e^{iz}}{(z+1)(z-1)}\cdot(z-1)] = \lim_{z\to 1}[\frac{e^{iz}}{z+1}] = \frac{e^{i}}{2} = \frac{\cos(1)+i\sin(1)}{2}$$

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{(z+1)(z-1)} dz = \pi i\cdot\{\operatorname{Re}_{-1} s[\frac{e^{iz}}{(z+1)(z-1)}] + \operatorname{Re}_{1} s[\frac{e^{iz}}{(z+1)(z-1)}]\} = \pi i\cdot[i\sin(1)]$$
$$= -\pi\sin(1)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)(x-1)} dx = -\pi\sin(1) \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(x)}{(x+1)(x-1)} dx = 0$$

Another method:

$$\frac{e^{iz}}{(z+1)(z-1)} = \frac{-\frac{1}{2}e^{iz}}{z+1} + \frac{\frac{1}{2}e^{iz}}{z-1}$$

$$\operatorname{Re}_{-1} s[\frac{e^{iz}}{z+1}] = \lim_{z\to -1}[\frac{e^{iz}}{z+1}\cdot(z+1)] = \lim_{z\to -1}[e^{iz}] = \cos(1) - i\sin(1)$$

$$\oint_c \frac{-\frac{1}{2}e^{iz}}{(z+1)} dz = -\frac{\pi i}{2}\cdot \operatorname{Re}_{-1} s[\frac{e^{iz}}{z+1}] = -\frac{i\pi}{2}\cos(1) - \frac{\pi}{2}\sin(1)$$

$$\operatorname{Re}_{1} s[\frac{e^{iz}}{z-1}] = \lim_{z\to 1}[\frac{e^{iz}}{z-1}\cdot(z-1)] = \lim_{z\to 1}[e^{iz}] = \cos(1) + i\sin(1)$$

$$\oint_c \frac{\frac{1}{2}e^{iz}}{z-1} dz = \frac{\pi i}{2}\cdot \operatorname{Re}_{1} s[\frac{e^{iz}}{z-1}] = \frac{i\pi}{2}\cos(1) - \frac{\pi}{2}\sin(1)$$

$$\oint_c \frac{e^{iz}}{(z+1)(z-1)} dz$$

$$= -\frac{i\pi}{2}\cos(1) - \frac{\pi}{2}\sin(1) + \frac{i\pi}{2}\cos(1) - \frac{\pi}{2}\sin(1)$$

$$= -\pi\sin(1)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{(x+1)(x-1)} dx = -\pi\sin(1) \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(x)}{(x+1)(x-1)} dx = 0$$

Eg. Evaluate $\int_0^{\infty}\frac{\sin(x)}{x}dx$ **. [1993 交大應數研、2003 中央光電所、2008 成大電研類似題]**

(Sol.)

Complex-plane integration method:

$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$ is an imaginary part and $\int_{-\infty}^{\infty}\frac{\cos(x)}{x}dx$ is a real part of an integral $\oint_c \frac{e^{iz}}{z}dz$. There exists only one pole **0** on the real axis.

By formula $\int_{-\infty}^{\infty} f(x)dx = 2\pi i\cdot\sum_{k=1}^{n}\operatorname{Re}s_{z_k}(f)+\pi i\cdot\sum_{j=1}^{m}\operatorname{Re}s_{a_j}(f)$, where z_k $(1\le k\le n)$ is the pole on the upper half-plane and a_j $(1\le j\le m)$ is the pole on the real axis.

$\Rightarrow \oint_c \frac{e^{iz}}{z}dz = \pi i\cdot \operatorname{Re}s_0[\frac{e^{iz}}{z}] = \pi i\cdot e^{i0} = \pi i$, $\therefore$ $\int_{-\infty}^{\infty}\frac{\cos(x)}{x}dx = 0$ and $\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx = \pi$

$\Rightarrow \int_0^{\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}$.

Laplace transform method:

By formula $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty}F(u)du$,

$$L\left[\frac{\sin(t)}{t}\right] = \int_0^{\infty}e^{-st}\cdot\frac{\sin(t)}{t}dt = \int_s^{\infty}L[\sin(t)]ds = \int_s^{\infty}\frac{1}{s^2+1}ds = \tan^{-1}(s)\Big|_s^{\infty} = \frac{\pi}{2}-\tan^{-1}(s)$$

Set s=0, $\int_0^{\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}$

Fourier Transform method:

Consider a rectangular function $f(x)=\begin{cases}1, & |x|\le 1\\ 0, & |x|\rangle 1\end{cases}$,

$$\Im[f(x)] = \int_{-1}^{1}e^{-i\omega x}dx = \frac{2\sin(\omega)}{\omega} \Rightarrow f(x) = \Im^{-1}\left\{\frac{2\sin\omega}{\omega}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\omega}{\omega}e^{i\omega x}d\omega,$$

$f(0)=1 \Rightarrow \int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$, $\frac{\sin(x)}{x}$ is an even function $\Rightarrow \int_0^{\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}$.

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx$ **and** $\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}dx$.

(Sol.) Pole: 0 (on the real axis), $\oint_c \frac{e^{i2az}-e^{i2bz}}{z^2}dz=\pi i\cdot \operatorname{Re}\underset{0}{s}[\frac{e^{i2az}-e^{i2bz}}{z^2}]$

$\operatorname{Re}\underset{0}{s}[\frac{e^{i2az}-e^{i2bz}}{z^2}]=\lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}\left[z^2\cdot\frac{e^{i2az}-e^{i2bz}}{z^2}\right]=[i2ae^{i2az}-i2be^{i2bz}]\Big|_{z=0}=i2(a\text{-}b)$

$\therefore \oint_c \frac{e^{ii2az}-e^{i2bz}}{z^2}dz=\pi i\cdot\ i2(a\text{-}b)=2(b\text{-}a)\pi \Rightarrow \int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx=2(b\text{-}a)\pi$

Let $b=1$, $a=0\Rightarrow \int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx=\int_{-\infty}^{\infty}\frac{1-\cos(2x)}{x^2}dx=2(1\text{-}0)\pi=2\pi$

And $1\text{-}\cos(2x)=1\text{-}[1\text{-}2\sin^2(x)]=2\sin^2(x)$.

$\therefore\ \int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}dx=\pi$ and $\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x^2-2x+1)}dx$. **[2013 中央電研固態組、生醫電子組]**

(Sol.) Poles: 1 and -1 (on the real axis),

$\operatorname{Re}\underset{-1}{s}[\frac{e^{iz}}{(z+1)(z-1)^2}]=\lim_{z\to -1}[\frac{e^{iz}\cdot(z+1)}{(z+1)(z-1)^2}]=\lim_{z\to -1}[\frac{e^{iz}}{(z-1)^2}]=\frac{e^{-i}}{4}=\frac{\cos(1)-i\sin(1)}{4}$

$\operatorname{Re}\underset{1}{s}[\frac{e^{iz}}{(z+1)(z-1)^2}]=\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[\frac{e^{iz}\cdot(z-1)^2}{(z+1)(z-1)^2}]=\lim_{z\to 1}\frac{d}{dz}[\frac{e^{iz}}{z+1}]=\frac{i(z+1)e^{iz}-e^{iz}}{(z+1)^2}=$

$\frac{[-2\sin(1)-\cos(1)]+i[2\cos(1)-\sin(1)]}{4}$

$\int_{-\infty}^{\infty}\frac{e^{iz}}{(z+1)(z-1)^2}dz=\pi i\cdot\{\operatorname{Re}\underset{-1}{s}[\frac{e^{iz}}{(z+1)(z-1)^2}]+\operatorname{Re}\underset{1}{s}[\frac{e^{iz}}{(z+1)(z-1)^2}]\}$

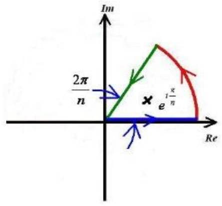
$=\pi i\cdot\{\frac{\cos(1)-i\sin(1)}{4}+\frac{[-2\sin(1)-\cos(1)]+i[2\cos(1)-\sin(1)]}{4}\}$

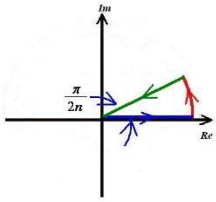
$=\pi i\cdot\frac{-2\sin(1)+i[2\cos(1)-2\sin(1)]}{4}=\pi\cdot\frac{-i\sin(1)-[\cos(1)-\sin(1)]}{2}$

$\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x^2-2x+1)}dx=\frac{-\pi\sin(1)}{2}$,

$\int_{-\infty}^{\infty}\frac{\cos(x)}{(x+1)(x^2-2x+1)}dx=\frac{-\pi[\cos(1)-\sin(1)]}{2}$

Case 4 $\int_0^\infty \begin{Bmatrix} \sin(x^n) \\ \cos(x^n) \end{Bmatrix} dx$ **or** $\int_0^\infty G(x^n)dx$





Choose C as a sector with angle $\dfrac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$ or a sector with angle $\dfrac{\pi}{2n}$ enclosing no poles.

Eg. Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$**, $n>1$.**

(Sol.) Choose C as a sector with angle $\dfrac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$.

$$\oint_c \frac{dz}{1+z^n} = 2\pi i \cdot \underset{e^{i\frac{\pi}{n}}}{\operatorname{Res}}(f) = 2\pi i \cdot \lim_{z\to e^{i\frac{\pi}{n}}}\left[\left(z-e^{i\frac{\pi}{n}}\right)\frac{1}{1+z^n}\right] = \frac{2\pi i}{nz^{n-1}}\bigg|_{z=e^{i\frac{\pi}{n}}} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$= \int_0^R \frac{dx}{1+x^n} + \int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} + \int_R^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n e^{i2\pi}}$$

As $R\to\infty$, $\displaystyle\int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} \to 0$ ($\because n>1$)

$$\therefore \int_0^\infty \frac{dx}{1+x^n} + \int_\infty^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n} = \left(1-e^{i\frac{2\pi}{n}}\right)\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^n} = \frac{e^{i\frac{\pi}{n}}}{1-e^{i\frac{2\pi}{n}}}\cdot\frac{-2\pi i}{n} = \frac{1}{\frac{e^{i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}}}{2i}}\cdot\frac{\pi}{n} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$

Eg. Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^{56}}$ **. (Ans.)** $\dfrac{\pi}{56\sin(\frac{\pi}{56})}$ **[2024 台聯大電機聯招]**

Eg. Show that $\displaystyle\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$.

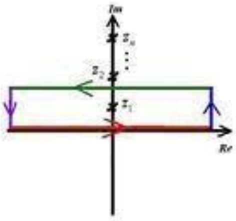
(Proof) Choose C as a sector with angle $\dfrac{\pi}{4}$ enclosing no poles.

$$\oint_c e^{iz^2}dz = 0 = \int_0^R e^{ix^2}dx + \int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta + \int_R^0 e^{iR^2e^{i\frac{\pi}{2}}}\cdot e^{i\frac{\pi}{4}}dR$$

As $R\to\infty$, $\displaystyle\left|\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta\right| = iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos 2\theta + i\sin 2\theta)}e^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}e^{i(\theta+R^2\cos 2\theta)}d\theta \to 0,$

$$\int_0^\infty[\cos(x^2)+i\sin(x^2)]dx = \int_0^\infty\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right)e^{-R^2}dR = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{\pi}}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2}i$$

$$\therefore \int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$



Case 5 $\int_{-\infty}^{\infty} G(e^x)dx$, **where** $G(x)=\dfrac{p(x)}{x^n+q_{n-1}(x)}$

Choose C as an infinitely-wide rectangle and there is one pole on the imaginary axis.

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx$, 0<$m$<1. 【1991台大機研】
【2014台聯大系統】

(Sol.) Pole: $i\pi$

$$\oint_c \frac{e^{mz}}{1+e^z}dz=\int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx+\int_0^{2\pi}\frac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy+\int_{\infty}^{-\infty}\frac{e^{mx}\cdot e^{i2m\pi}}{1+e^x\cdot e^{i2\pi}}dx+\int_{2\pi}^{0}\frac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy$$

$$=2\pi i\cdot \underset{\pi i}{\mathrm{Res}}(f)=2\pi i\cdot\lim_{z\to\pi i}\frac{(z-\pi i)e^{mz}}{1+e^z}$$

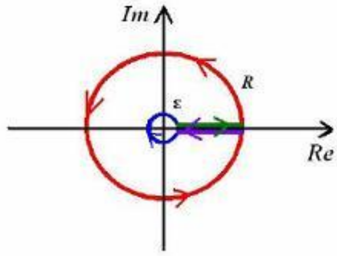
$$=2\pi i\cdot\lim_{z\to\pi i}\frac{e^{mz}+m(z-\pi i)e^{mz}}{e^z}=2\pi i(-1)^{m-1}=2\pi ie^{i(m-1)\pi}$$

$\because$ 0<m<1, $\therefore R\to\infty$, $\int_0^{2\pi}\frac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy\to 0$ and $\int_{2\pi}^{0}\frac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy\to 0$

$$\oint_c\frac{e^{mz}}{1+e^z}dz=2\pi ie^{i(m-1)\pi}=\int_{-\infty}^{\infty}(1-e^{i2m\pi})\cdot\frac{e^{mx}}{1+e^x}dx$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx=\frac{2\pi i}{1-e^{i2m\pi}}\cdot e^{im\pi}\cdot(-1)=\frac{\pi}{\dfrac{e^{im\pi}-e^{-im\pi}}{2i}}=\frac{\pi}{\sin(m\pi)}$$

***Case 6* Other types**



Eg. For 0<p<1, $\int_0^\infty \frac{x^p\,dx}{x(1+x)} = ?$

(Sol.) ∵ 0<p<1, ∴ Poles are 0 and -1.

$$\oint \frac{z^p\,dz}{z(1+z)} = 2\pi i \cdot \operatorname*{Res}_{-1}(f) = 2\pi i \cdot \lim_{z\to -1}(z+1)\cdot\frac{z^p}{z(z+1)} = 2\pi i \cdot e^{i\pi(p-1)}$$, and

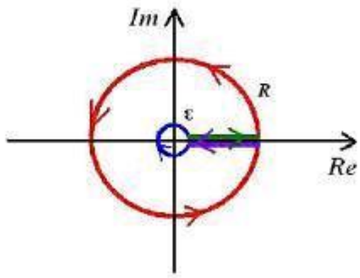
$$\oint \frac{z^p\,dz}{z(1+z)} = \int_\varepsilon^R \frac{x^p\,dx}{x(1+x)} + \int_0^{2\pi} \frac{i(Re^{i\theta})^p\,d\theta}{1+Re^{i\theta}} + \int_R^\varepsilon \frac{(xe^{i2\pi})^{p-1}\,d(xe^{i2\pi})}{1+xe^{i2\pi}} + \int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p\,d\theta}{1+\varepsilon e^{i\theta}}$$

∵ 0<p<1, $R\to\infty$, ∴ $\int_0^{2\pi} \frac{i(Re^{i\theta})^p\,d\theta}{1+Re^{i\theta}} \to 0$

∵ $\varepsilon \to 0$, ∴ $\int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p\,d\theta}{1+\varepsilon e^{i\theta}} \to 0$

$$\Rightarrow \oint \frac{z^p\,dz}{z(1+z)} = \int_0^\infty \frac{x^p\,dx}{x(1+x)} + \int_\infty^0 \frac{(xe^{i2\pi})^{p-1}\,d(xe^{i2\pi})}{1+xe^{i2\pi}} = \int_0^\infty \frac{[1-e^{i2\pi(p-1)}]\cdot x^p\,dx}{x(1+x)}$$

$$\therefore \int_0^\infty \frac{x^p\,dx}{x(1+x)} = \frac{2\pi i\cdot e^{i\pi(p-1)}}{1-e^{i2\pi(p-1)}} = \frac{\pi}{(e^{ip\pi}-e^{-ip\pi})/2i} = \frac{\pi}{\sin(p\pi)}$$



Eg. For -1<a<1, $\int_0^\infty \frac{x^a\,dx}{(1+x)^2} = ?$ **【交大電信研究所、2003 中央光電所】**

(Sol.) –1 is a multiple-order pole.

$$\oint \frac{z^a\,dz}{(1+z)^2} = 2\pi i \cdot \operatorname*{Res}_{-1}(f) = 2\pi i \cdot \lim_{z\to -1}\frac{1}{1!}d[(z+1)^2\cdot\frac{z^a}{(z+1)^2}]/dz = 2\pi i \cdot a e^{i\pi(a-1)}$$, and

$$\oint \frac{z^a\,dz}{(1+z)^2} = \int_\varepsilon^R \frac{x^a\,dx}{(1+x)^2} + \int_0^{2\pi} \frac{(Re^{i\theta})^a\,iRe^{i\theta}\,d\theta}{(1+Re^{i\theta})^2} + \int_R^\varepsilon \frac{(xe^{i2\pi})^a\,d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a\,i\varepsilon e^{i\theta}\,d\theta}{(1+\varepsilon e^{i\theta})^2}$$

∵ -1<a<1, $R\to\infty$, ∴ $\int_0^{2\pi} \frac{(Re^{i\theta})^a\,iRe^{i\theta}\,d\theta}{(1+Re^{i\theta})^2} \to 0$

∵ $\varepsilon \to 0$, ∴ $\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a\,i\varepsilon e^{i\theta}\,d\theta}{(1+\varepsilon e^{i\theta})^2} \to 0$

$$\Rightarrow \oint \frac{z^a\,dz}{(1+z)^2} = \int_0^\infty \frac{x^a\,dx}{(1+x)^2} + \int_\infty^0 \frac{(xe^{i2\pi})^a\,d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} = \int_0^\infty \frac{[1-e^{i2\pi(a+1)}]\cdot x^a\,dx}{(1+x)^2}$$

$$\therefore \int_0^\infty \frac{x^a\,dx}{(1+x)^2} = \frac{2\pi i\cdot a e^{i\pi(a-1)}}{1-e^{i2\pi(a+1)}} = \frac{\pi a}{(e^{i\pi a}-e^{-i\pi a})/2i} = \frac{a\pi}{\sin(a\pi)}$$