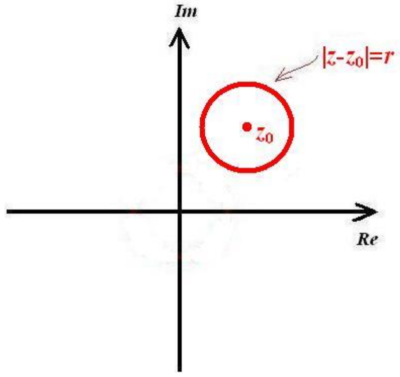
Chapter 2 Integration in the Complex Plane

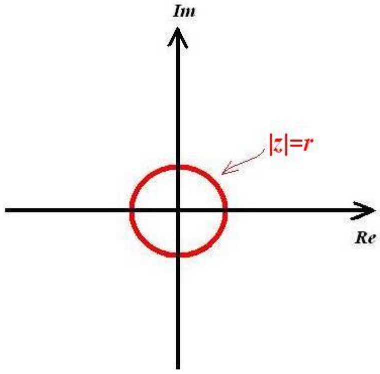
2-1 Complex Line Integrals and Some Integral Theorems

For smooth curve C: $z=z(t)$ for $a \leqq t \leqq b$, then $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$

Special case 1 C: $|z-z_0|=r \Leftrightarrow z(t)=z_0+re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$

Special case 2 C: $|z|=r \Leftrightarrow z(t)=re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$





Eg. Find $\oint_c \frac{1}{z} dz$**,** ***C*****: |*****z*****|=1.**

(Sol.) $|z|=1 \Leftrightarrow z=e^{it}$, $0 \leqq t \leqq 2\pi$, $z'(t)= ie^{it}$, $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$

Eg. Evaluate $\oint_C \frac{dz}{z-3i}$**,** ***C*****: |*****z*****-3*****i*****|=**$\frac{1}{3}$**.**

(Sol). $z(t)=3i+\frac{1}{3}e^{it}$, $z'(t)=\frac{1}{3}ie^{it}$, $\oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3}e^{it} dt = 2\pi i$

Eg. Evaluate $\oint_c \bar{z} dz$**,** ***C*****: |*****z*****|=1.**

(Sol.) $z(t)=e^{it}$, $\bar{z}=e^{-it}$, $z'(t)= ie^{it}$, $\oint_c \bar{z} dz = \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$

Eg. Evaluate $\oint_c [z - R_e(z)]dz$**,** ***C*****: |*****z*****|=2.**

(Sol.) $|z|=2 \Leftrightarrow z(t)=2e^{it}$, $z'(t)= i2e^{it}$

$$R_e(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(2e^{it}+2e^{-it}), \quad z - R_e(z) = \frac{1}{2}(z-\bar{z}) = \frac{1}{2}(2e^{it}-2e^{-it})$$

$$\oint_c [z - R_e(z)]dz = \int_0^{2\pi} [z(t) - R_e(z(t))] \cdot z'(t)dt$$

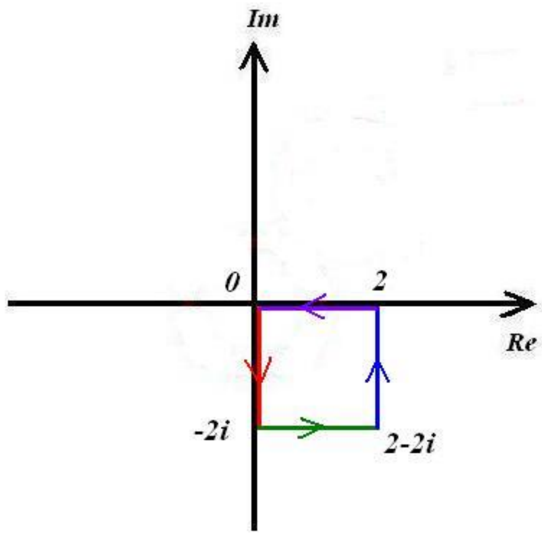
$$= \int_0^{2\pi} \frac{2}{2}(e^{it}-e^{-it}) \cdot 2ie^{it}dt = 2i\int_0^{2\pi} (e^{2it}-1)dt = -4\pi i$$

Eg. Evaluate $\oint_c [z^2 + I_m(z)]dz$ **, where** C **is the square with 0, -2*i*, 2-2*i*, and 2.**

(Sol.) $\oint_c [z^2 + I_m(z)]dz = \boxed{\int_0^{-2}(-t^2+t)idt} + \boxed{\int_0^{2}[(t-2i)^2-2]dt} + \boxed{\int_{-2}^{0}[(2+it)^2+t]idt}$

$+ \boxed{\int_2^{0}(t^2+0)dt} = \boxed{i\left(-\frac{t^3}{3}+\frac{t^2}{2}\right)\Big|_0^{-2}} + \boxed{\left(\frac{t^3}{3}-2it^2-6t\right)\Big|_0^{2}}$

$+ \boxed{i\left(4t+2it^2-\frac{t^3}{3}+\frac{t^2}{2}\right)\Big|_{-2}^{0}} + \boxed{\frac{t^3}{3}\Big|_2^{0}} = -4$



Cauchy's integral theorem Let *f*(*z*) be analytic in a simply-connected domain *D*, *C* is a simple closed curve in *D*, then $\oint_c f(z)dz = 0$.

Eg. Evaluate $\oint_c \frac{1}{z}dz$ **,** ***C*****: |*z*-2|=1.**

(Sol.) $f(z)=\frac{1}{z}$ is analytic except z=0. No poles are within C, ∴ $\oint_c \frac{dz}{z} = 0$

Eg. Evaluate $\oint_{|z|=1} \frac{dz}{z^2-4}$. **【1991 交大土木所】**

(Sol.) $f(z)=\frac{1}{z^2-4}$ is analytic except $z=\pm 2$. No poles are within C, ∴ $\oint_{|z|=1} \frac{dz}{z^2-4}=0$

Eg. Evaluate $\oint_C \frac{z}{\sin(z)(z-2i)^3}dz$ **,** ***C*****: |*z*-8*i*|=1.**

(Sol.) $f(z) = \frac{z}{\sin(z)(z-2i)^3}$ is analytic except $z=2i$, $n\pi$ (n=0, ±1, ±2, …)

No poles are within C, ∴ $\oint f(z)dz = 0$

Cauchy's integral formulae **Let $f(z)$ be analytic in a simply-connected region D, and let C be a simple curve enclosing z_0 in D, then $\oint_c \frac{f(z)}{z-z_0}dz = 2\pi i \cdot f(z_0)$ and**

$$\oint_c \frac{f(z)dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{(n-1)}(z_0).$$

Eg. Evaluate $\oint_{|z|=3} \frac{e^{2z}dz}{z-2}$. **[1991 交大土木所]**

(Sol.) $\oint_{|z|=3} \frac{e^{2z}}{z-2} = 2\pi i \cdot e^{2\cdot 2} = 2\pi i e^4$

Eg. Evaluate $\oint_c \frac{e^{iz}}{z^3}dz$, **$C$: $|z|=3$.**

(Sol.) $\oint_c \frac{e^{iz}}{z^3}dz = i2\pi \cdot \frac{(e^{iz})''}{(3-1)!}\Big|_{z_0=0} = -i\pi$

Eg. Given $\oint_C \frac{f(z)}{z}dz = 2\pi i$ **and** $\oint_C \frac{f(z)}{(z-1)^2}dz = 4\pi i$. **If let $f(z)=a+bz$, find a and b. [2005成大電研]**

(Sol.) $2\pi i = 2\pi i \cdot f(0) = 2\pi a i$ and $4\pi i = 2\pi i \cdot f'(0) = 2\pi b i$, $\therefore$ $a=1$ and $b=2$.

Eg. Evaluate $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz$ **and** $\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz$ **if** $C: \left|z-\frac{\pi}{6}\right| = \delta > 0$.

(Sol.) Let $f(z)=\sin^6(z)$ and $n=3$, $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz = 2\pi i \cdot \sin^6\left(\frac{\pi}{6}\right) = \frac{\pi i}{32}$,

$$\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz = \frac{2\pi i}{2!} \cdot [\sin^6(z)]''\Big|_{z_0=\frac{\pi}{6}} = \frac{21\pi i}{16}$$

Eg. Let z_0 be within C, find $\oint_c \frac{dz}{z-z_0}$ **and** $\oint_c \frac{dz}{(z-z_0)^n}$, **$n \geqq 2$.**

(Sol.) Let $f(z)=1$, $f^{(n-1)}(z_0)=0 \Rightarrow \oint_c \frac{dz}{z-z_0} = 2\pi i$ and $\oint_c \frac{dz}{(z-z_0)^n} = 0$.

Eg. Evaluate $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz$**, C is a closed curve not passing 1.**

(Sol.) If C does not enclose 1, $\frac{2\sin(z^2)}{(z-1)^4}$ is analytic within C, $\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz = 0$

If C encloses 1, let $f(z) = 2\sin(z^2) \Rightarrow \frac{2\sin(z^2)}{(z-1)^4} = \frac{f(z)}{(z-1)^4}$, n=4, n-1=3

$f^{(3)}(z) = -24z\sin(z^2) - 16z^3\cos(z^2)$

$\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz = \frac{2\pi i}{3!}[-24\sin(1) - 16\cos(1)] = \frac{\pi i}{3}[-24\sin(1) - 16\cos(1)]$

Gaussian mean-value theorem Let $f(z)$ be analytic within $|z-a|=r$, then $\int_0^{2\pi} f(a+re^{i\theta})d\theta$ **$=2\pi\cdot f(a)$.**

Eg. Evaluate $\int_0^{2\pi} \sin^2[\frac{\pi}{6} + 2e^{i\theta}]d\theta$**.**

(Sol.) $a=\frac{\pi}{6}$, r=2, and analytic within $|z-\frac{\pi}{6}|=2$,

$\therefore$ $\int_0^{2\pi} \sin^2[\frac{\pi}{6} + 2e^{i\theta}]d\theta = 2\pi \cdot \sin^2(\frac{\pi}{6}) = \pi/2$.

Argument principle Let $f(z)$ be a meromorphic function inside and on some closed contour C, and $f(z)$ has no zeros or poles on C, then $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz$ **$=N-P$, where N and P are the numbers of zeros and poles of $f(z)$ within C, respectively.**

Eg. Evaluate $\oint \frac{f'(z)}{f(z)}dz$ **for C: $|z|=\pi$ if (a) $f(z)=\sin(\pi z)$, (b) $f(z)=\cos(\pi z)$, and (c) $f(z)=\tan(\pi z)$..**

(Sol.) $\oint \frac{f'(z)}{f(z)}dz = 2\pi i(N-P)$

(a) There are 7 zeros: ±3, ±2, ±-1, 0 but no poles within C,

$\therefore$ $\oint \frac{f'(z)}{f(z)}dz = 2\pi i(7-0) = 14\pi i$

(b) There are 6 zeros: ±2.5, ±1.5, ±0.5 but no poles within C,

$\therefore$ $\oint \frac{f'(z)}{f(z)}dz = 2\pi i(6-0) = 12\pi i$

(b) There are 7 zeros: ±3, ±2, ±-1, 0 but 6 poles: ±2.5, ±1.5, ±0.5 within C,

$\therefore$ $\oint \frac{f'(z)}{f(z)}dz = 2\pi i(7-6) = 2\pi i$

Eg. Evaluate $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz$ **if** $f(z)=\frac{(z^2+1)^2}{(z^2+2z+2)^3}$ **and** C: $|z|=4$.

(Sol.) $N=4$ and $P=6$, $\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz=4-6=-2$.

Poisson's integral formulae for a circle **Let $f(z)$ be analytic within $|z|=R$ and $f(re^{i\theta})=u(r,\theta)+iv(r,\theta)$, then**

$f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)f(\mathrm{Re}^{i\phi})}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$, $u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)u(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$,

and $v(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)v(R,\phi)}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$ **for $z=re^{i\theta}$ within $|z|=R$.**

Eg. Show that $\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi=2\pi$.

(Proof) Let $f(re^{i\theta})=1=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi$,

$\therefore \int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi=2\pi$.

Eg. Evaluate $\int_0^{2\pi}\frac{e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi$ **and** $\int_0^{2\pi}\frac{e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi$.

(Sol.) Let $R=2$, $r=1$ and $f(re^{i\theta})=e^{e^{i\theta}}=e^{\cos\theta+i\sin\theta}=e^{\cos\theta}[\cos(\sin\theta)+i\sin(\sin\theta)]$,

$\therefore \frac{1}{2\pi}\int_0^{2\pi}\frac{3e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=e^{\cos\theta}\cos(\sin\theta)$

and $\frac{1}{2\pi}\int_0^{2\pi}\frac{3e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=e^{\cos\theta}\sin(\sin\theta)$.

$\Rightarrow \int_0^{2\pi}\frac{e^{\cos\phi}\cos(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=\frac{2\pi}{3}e^{\cos\theta}\cos(\sin\theta)$

and $\int_0^{2\pi}\frac{e^{\cos\phi}\sin(\sin\phi)}{5-4\cos(\theta-\phi)}d\phi=\frac{2\pi}{3}e^{\cos\theta}\sin(\sin\theta)$

Poisson's integral formulae on the upper half plane **Let $f(z)$ be analytic for $y\geqq 0$ on the z-plane and $f(z)=u(x,y)+iv(x,y)$, then**

$f(z)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)^2+y^2}dt$, $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(t,0)}{(t-x)^2+y^2}dt$, **and** $v(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{v(t,0)}{(t-x)^2+y^2}dt$.

2-2 Laurent's Theorem & the Residue Theorem

If $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \overbrace{\frac{a_{-3}}{(z-z_0)^3}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}}^{(principal\ part)}$

$$\underbrace{+a_0 + a_1(z-z_0)+a_2(z-z_0)^2+\cdots}_{(analytic\ part)}$$

Laurent's theorem $a_n = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$

Residue: $a_{-1} = \frac{1}{2\pi i}\oint_c f(z)dz$

Residue theorem Let $f(z)$ be analytic in D except $z_1, z_2, \ldots, z_n$ and C encloses $z_1, z_2, \ldots, z_n$ within D. Then we have $\oint_c f(z)dz = 2\pi i\cdot \sum_{j=1}^{n} \operatorname{Res}_{z_j}(f)$ **and**

$\operatorname{Res}_{z_j}(f) = \frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$, **where m is the order of a pole $z=z_j$.**

In case of $m=1$, $\operatorname{Res}_{z_j}(f) = \lim_{z\to z_j}[(z-z_j)\cdot f(z)]$.

Eg. Find the residues of $f(z)=\frac{z}{z-1}$.

(Sol.) At $z=1$, $m=1$, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{z-1}]=1$

Eg. Find the residues of $f(z)=\frac{z}{(z-1)(z+1)^2}$.

(Sol.) At $z=1$, $m=1$, $\operatorname{Res}_{1}(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z}{(z-1)(z+1)^2}]=\frac{1}{4}$

At $z=-1$, $m=2$, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{z}{(z+1)^2(z-1)}]=\left.\frac{(z-1)-z}{(z-1)^2}\right|_{z=-1}=-\frac{1}{4}$

Eg. Find the residues of $f(z)=\frac{1}{(z-1)^2(z+1)^2}$.

(Sol.)

At $z=1$, $m=2$, $\operatorname{Res}_{1}(f)=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}[(z-1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z+1)}{(z+1)^4}\right|_{z=1}=-\frac{1}{4}$

At $z=-1$, $m=2$, $\operatorname{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}[(z+1)^2\cdot\frac{1}{(z+1)^2(z-1)^2}]=\left.\frac{-2(z-1)}{(z-1)^4}\right|_{z=-1}=\frac{1}{4}$

Eg. Find the residues of $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$.

(Sol.) $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}=\dfrac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}$

At z=-1, m=2,

$$\operatorname*{Res}_{-1}(f)=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}\right]=\left.\frac{(2z-2)(z^2+4)-(z^2-2z)(2z)}{(z^2+4)^2}\right|_{z=-1}$$

$=-\dfrac{14}{25}$

At z=2i, m=1, $\operatorname*{Res}_{2i}(f)=\lim_{z\to 2i}\left[(z-2i)\cdot\dfrac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}\right]=\dfrac{7+i}{25}$

At z=-2i, m=1, $\operatorname*{Res}_{-2i}(f)=\lim_{z\to -2i}\left[(z+2i)\cdot\dfrac{z^2-2z}{(z-2i)(z+2i)(z+1)^2}\right]=\dfrac{7-i}{25}$

Eg. Find the residues of $f(z)=\dfrac{\cot(z)\coth(z)}{z^3}$.

(Sol.) It is difficult to compute the residue at 0 by $\operatorname*{Res}_{z_j}(f)=\dfrac{1}{(m-1)!}\lim_{z\to z_j}\dfrac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$.

We utilize $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\dfrac{a_{-2}}{(z-z_0)^2}+\dfrac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+\cdots$

$f(z)=\dfrac{\cot(z)\coth(z)}{z^3}=\dfrac{\cos(z)\cosh(z)}{z^3\sin(z)\sinh(z)}$

$$=\frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\cdots\right)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+-\cdots\right)\left(z+\frac{z^3}{3!}+\cdots\right)}=\frac{\left(1-\frac{z^4}{6}+\cdots\right)}{\left(z^5-\frac{z^9}{90}+-\cdots\right)}==\frac{1}{z^5}-\frac{7}{45}\cdot\frac{1}{z}+\cdots$$

$\therefore$ $\operatorname*{Res}_{0}(f)=-\dfrac{7}{45}$

Eg. Evaluate $\oint_c \dfrac{z^2+1}{z^2-1}dz$, C: $|z-1|=1$. [1991 交大科管所]

(Sol.) There is only one pole 1 within C.

$\operatorname*{Res}_{1}(f)=\lim_{z\to 1}\left[(z-1)\cdot\dfrac{z^2+1}{z^2-1}\right]=\lim_{z\to 1}\dfrac{z^2+1}{z+1}=1$, $\therefore$ $\oint_c\dfrac{z^2+1}{z^2-1}dz=2\pi i\cdot 1=2\pi i$

Eg. Evaluate $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ **for (a)** ***C***: $|z|=\frac{1}{3}$, **(b)** ***C***: $|z-1|=\frac{1}{3}$, **(c)** ***C***: $|z|=2$. **[1991 台大機研]**

(Sol.)

(a) There is only one pole 0 within *C*.

$$\operatorname*{Res}_0(f)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}[z^2\cdot\frac{\cos(z)}{z^2(z-1)}]=\left.\frac{-(z-1)\sin(z)-\cos(z)}{(z-1)^2}\right|_{z=0}=-1,\quad \oint_c\frac{\cos(z)}{z^2(z-1)}dz=-2\pi i$$

(b) There is only one pole 1 within *C*.

$$\operatorname*{Res}_1(f)=\lim_{z\to 1}[(z-1)\cdot\frac{\cos(z)}{z^2(z-1)}]=\cos(1),\quad \oint_c\frac{\cos(z)}{z^2(z-1)}dz=2\pi i\cdot\cos(1)$$

(c) There are two poles 0 and 1 within *C*. $\oint_c\frac{\cos(z)}{z^2(z-1)}dz=2\pi i\cdot[-1+\cos(1)]$

Eg. Find the residue of $f(z)=\frac{\sin(z)}{(z-i)^3}$ **and evaluate** $\oint_c\frac{\sin(z)}{(z-i)^3}dz$, ***C***: $|z-i|=2$.

(Sol.) $m=3$, $\operatorname*{Res}_i(f)=\frac{1}{(3-1)!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\cdot\frac{\sin(z)}{(z-i)^3}]=\frac{-1}{2}\sin(i)=-\frac{1}{2}i\sinh(1)$

$$\therefore\ \oint\frac{\sin(z)}{(z-i)^2}dz=2\pi i\cdot\left(-\frac{1}{2}i\sinh(1)\right)=\pi\sinh(1)$$

Eg. Evaluate $\oint_c\tan z dz$, ***C***: $|z|=2$. **[1993 中山電研]**

(Sol.) There are two poles $\pm\pi/2$ within *C*.

$$\operatorname*{Res}_{\frac{\pi}{2}}(f)=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to\frac{\pi}{2}}\frac{z\sin(z)-\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$=\lim_{z\to\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)-\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$$\operatorname*{Res}_{-\frac{\pi}{2}}(f)=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to-\frac{\pi}{2}}\frac{z\sin(z)+\frac{\pi}{2}\sin(z)}{\cos(z)}$$

$$=\lim_{z\to-\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)+\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$$\oint_c\tan z dz=2\pi i\cdot[(-1)+(-1)]=-4\pi i$$

Eg. Evaluate $\oint_c \frac{\sin(z)}{z^2(z^2+4)}dz$**, *C* is any piecewise-smooth curve enclosing 0, 2*i*, and –2*i*.**

(Sol.) $f(z)=\frac{\sin(z)/z}{z(z+2i)(z-2i)}$, $\because \lim_{z\to 0}[\sin(z)/z]=1$, $\therefore$ $f(z)$ has a removable singularity at $0 \Rightarrow m$ of the pole z=0 in $f(z)$ is 1. $\operatorname{Res}_0(f)=\lim_{z\to 0} z\cdot\frac{\sin(z)}{z^2(z^2+4)}=\frac{1}{4}$

$$\operatorname{Res}_{2i}(f)=\lim_{z\to 2i}(z-2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\operatorname{Res}_{-2i}(f)=\lim_{z\to -2i}(z+2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\oint_c f(z)dz=2\pi i\left[\frac{1}{4}-\frac{1}{16}\sinh(2)-\frac{1}{16}\sinh(2)\right]=\frac{\pi i}{2}-\frac{\pi i}{4}\sinh(2)$$

Eg. Evaluate $\oint_c \frac{z^3e^{\frac{1}{z}}}{z^3+1}dz$**, *C*: |*z*|=3. [2013 中山電研]**

(Sol.) Let $t=1/z$, $z=1/t$, $dz=-dt/t^2$, C: $|t|=1/3$

$$\oint_c \frac{z^3e^{\frac{1}{z}}}{z^3+1}dz=\oint_c \frac{(\frac{1}{t})^3e^t}{(\frac{1}{t})^3+1}\cdot\left(\frac{-dt}{t^2}\right)=\oint_c \frac{-e^t}{t^2(t^3+1)}dt$$

There is only one pole 0 within *C*.

$$\operatorname{Res}_0(f)=\frac{1}{1!}\lim_{t\to 0}\frac{d}{dt}\left[t^2\cdot\frac{-e^t}{t^2(t^3+1)}\right]=-1,\ \therefore\ \oint_c \frac{z^3e^{\frac{1}{z}}}{z^3+1}dz=2\pi i\cdot(-1)=-2\pi i$$

Eg. Evaluate $\oint_c \frac{e^{-z}}{\cos(z)}dz$**, *C*: |*z*|=2. [2015 台聯大系統]**

(Sol.) $\cos(z)$ has two zeros at $z=\pm\pi/2$ within $|z|=2$.

$$\operatorname{Res}_{\frac{\pi}{2}}(f)=\lim_{z\to\frac{\pi}{2}}\left[(z-\frac{\pi}{2})\cdot\frac{e^{-z}}{\cos(z)}\right]=\lim_{z\to\frac{\pi}{2}}\frac{ze^{-z}-\frac{\pi}{2}e^{-z}}{\cos(z)}=\lim_{z\to\frac{\pi}{2}}\frac{e^{-z}-ze^{-z}+\frac{\pi}{2}e^{-z}}{-\sin(z)}=-e^{-\frac{\pi}{2}}$$

$$\operatorname{Res}_{-\frac{\pi}{2}}(f)=\lim_{z\to-\frac{\pi}{2}}\left[(z+\frac{\pi}{2})\cdot\frac{e^{-z}}{\cos(z)}\right]=\lim_{z\to-\frac{\pi}{2}}\frac{ze^{-z}+\frac{\pi}{2}e^{-z}}{\cos(z)}=\lim_{z\to-\frac{\pi}{2}}\frac{e^{-z}-ze^{-z}-\frac{\pi}{2}e^{-z}}{-\sin(z)}=e^{\frac{\pi}{2}}$$

$$\therefore\ \oint_c \frac{e^{-z}}{\cos(z)}dz=2\pi i\cdot(e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}})$$

Eg. Evaluate $\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz$**,** C**: $|z|$=0.5. [2014 中央電研固態組、生醫電子組]**

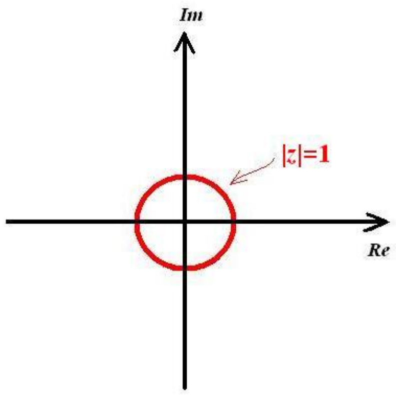
(Sol.) Let t=1/z, z=1/t, dz= -dt/t^2, C: $|t|$=2

$\oint_c \frac{e^{\frac{1}{z}}}{z+1}dz = \oint_c \frac{e^t}{(\frac{1}{t})+1}\cdot(\frac{-dt}{t^2}) = \oint_c \frac{-e^t}{t(t+1)}dt$. There are two poles 0 and -1 within C.

$$\operatorname{Res}_0 s(f) = \lim_{t\to 0}[t\cdot\frac{-e^t}{t(t+1)}] = -1,\quad \operatorname{Res}_{-1} s(f) = \lim_{t\to -1}[(t+1)\cdot\frac{-e^t}{t(t+1)}] = e^{-1},$$

$\therefore\ \oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz$ =$2\pi i\cdot(-1+e^{-1})$

2-3 Evaluation of Real Integrals



Case 1 $\int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta$

Choose C: $|z|=1$, $z=e^{i\theta}$

$$\Rightarrow \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right),$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right), \quad d\theta = \frac{dz}{iz}$$

$$\Rightarrow I = \int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta = \oint_c K\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\cdot\frac{dz}{iz}$$

Eg. Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$.

(Sol.) $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \oint_c \frac{dz/iz}{5-3\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_c \frac{2idz}{3z^2-10z+3} = \oint_c \frac{2idz}{3(z-\frac{1}{3})(z-3)}$

There is only one pole $\frac{1}{3}$ within $|z|=1$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = 2\pi i\cdot \operatorname{Res}_{\frac{1}{3}}(f) = 2\pi i\cdot\lim_{z\to\frac{1}{3}}(z-\frac{1}{3})\cdot\frac{2i}{3(z-\frac{1}{3})(z-3)} = 2\pi i\cdot\frac{2i}{-8} = \frac{\pi}{2}.$$

Eg. Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$.

(Sol.)

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{dz/iz}{3-2\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \oint_c \frac{2dz}{(1-2i)z^2+6iz-1-2i}$$

$$= \oint_c \frac{2dz}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}.$$

There is only one pole $\frac{2-i}{5}$ within $|z|=1$.

$$\therefore \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

$$= 2\pi i\cdot \operatorname{Res}_{\frac{2-i}{5}}(f) = 2\pi i\cdot\lim_{z\to\frac{2-i}{5}}\{(z-\frac{2-i}{5})\cdot\frac{2}{(1-2i)(z-\frac{2-i}{5})[z-(2-i)]}\} = 2\pi i\cdot(\frac{1}{2i}) = \pi.$$

Eg. Show that $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2\pi}{\sqrt{a^2-b^2}}$ **if $a>|b|$.**

(Proof) $\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=\oint_c\frac{dz/iz}{a+b\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)}=\oint_c\frac{2dz}{bz^2+2iaz-b}=\oint_c\frac{2dz}{b(z-z_1)(z-z_2)}$

Poles: $z_1=\frac{i}{b}\left(-a+\sqrt{a^2-b^2}\right)$ is within C: $|z|=1$, but $z_2=\frac{i}{b}\left(-a-\sqrt{a^2-b^2}\right)$ is not.

$$\operatorname*{Res}_{z_1}(f)=\lim_{z\to z_1}(z-z_1)\cdot\frac{2}{b(z-z_1)(z-z_2)}=\frac{2}{b}\cdot\frac{1}{z_1-z_2}=\frac{2}{b}\cdot\frac{b}{2i\sqrt{a^2-b^2}}=\frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore\ \int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}=2\pi i\cdot\frac{1}{i\sqrt{a^2-b^2}}=\frac{2\pi}{\sqrt{a^2-b^2}}$

Eg. Evaluate $\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}$**. [2011 成大電研]**

(Sol.) $\because\ \cos^2(t)=\frac{\cos(2t)+1}{2}$ and $\sin^2(t)=\frac{1-\cos(2t)}{2}$,

$\therefore\ a^2\cos^2(t)+b^2\sin^2(t)=\frac{(a^2+b^2)+(a^2-b^2)\cos(2t)}{2}$.

Let $\theta=2t$, $\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}=\int_0^{2\pi}\frac{2dt}{(a^2+b^2)+(a^2-b^2)\cos(2t)}$

$=\int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$\because$ The above integrand has a period of 2π,

$\therefore\ \int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}=2\int_0^{2\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$\because\ a^2+b^2>a^2-b^2$, using $\int_0^{2\pi}\frac{d\theta}{c+d\sin\theta}=\int_0^{2\pi}\frac{d\theta}{c+d\cos\theta}=\frac{2\pi}{\sqrt{c^2-d^2}}$ if $c>|d|$.

Set $c=a^2+b^2$ and $d=a^2-b^2$, we have

$\int_0^{2\pi}\frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}=\int_0^{4\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}$

$=2\int_0^{2\pi}\frac{d\theta}{(a^2+b^2)+(a^2-b^2)\cos(\theta)}=2\cdot\frac{2\pi}{\sqrt{(a^2+b^2)^2-(a^2-b^2)^2}}=\frac{2\pi}{ab}$

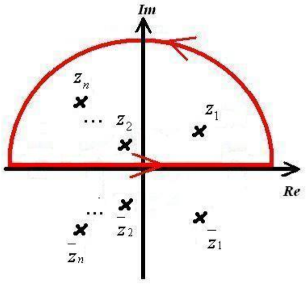
Eg. Show that $\int_0^{2\pi} e^{\cos\theta} \cdot \cos(\theta+\sin\theta)d\theta = \int_0^{2\pi} e^{\cos\theta} \cdot \sin(\theta+\sin\theta)d\theta = 0$.

(Proof) Let C: $|z|=1 \Rightarrow z(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$, $\therefore$ $f(z) = e^z$ is analytic within C.

$$\oint_c e^z dz = 0 = \int_0^{2\pi} e^{\cos\theta + i\sin\theta} de^{i\theta} = \int_0^{2\pi} ie^{\cos\theta} e^{i(\theta+\sin\theta)} d\theta$$

$$= i\int_0^{2\pi} e^{\cos\theta}\{[\cos(\theta+\sin\theta) + i[\sin(\theta+\sin\theta)]\}d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta} \cdot \{-\sin(\theta+\sin\theta) + i\cos(\theta+\sin\theta)\}d\theta, \ \therefore \ Re(\cdots)=Im(\cdots)=0$$



Case 2 $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$

Choose C as a semi-circle with infinite radius enclosing the upper half-plane. Poles: z_1, z_2, …, z_n are in the upper half-plane, $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n$ are in the lower half-plane. Assume *deg*(q) ≥ *deg*(p)+2, then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \oint_c f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z_j}(f)$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{xdx}{(x^2-2x+2)^2}$ **[1991中山電研]**

(Sol.) Poles: $1+i$ (upper half-plane), $1-i$ (lower half-plane)

$$\operatorname*{Res}_{1+i}\left[\frac{z}{(z^2-2z+2)^2}\right] = \frac{1}{(2-1)!}\lim_{z\to z_j}\frac{d^{2-1}}{dz^{2-1}}\left\{[z-(1+i)]^2 \cdot \frac{z}{[z-(1+i)]^2\cdot[z-(1-i)]^2}\right\}$$

$$= \lim_{z\to z_j}\frac{d}{dz}\left\{\frac{z}{[z-(1-i)]^2}\right\} = \left.\frac{[z-(1-i)]^2 - 2z[z-(1-i)]}{[z-(1-i)]^4}\right|_{1+i} = \frac{-i}{4},$$

$$\int_{-\infty}^{\infty}\frac{xdx}{(x^2-2x+2)^2} = 2\pi i \cdot \operatorname*{Res}_{1+i}\left[\frac{z}{(z^2-2z+1)^2}\right] = 2\pi i \cdot \left(\frac{-i}{4}\right) = \pi/2.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+64}dx$.

(Sol.) (a) Poles: $8i$ (upper half-plane), $-8i$ (lower half-plane)

$$\operatorname*{Res}_{8i}(f) = \lim_{z\to 8i}(z-8i)\cdot\frac{1}{(z+8i)(z-8i)} = \frac{1}{16i}, \quad \int_{-\infty}^{\infty}\frac{1}{x^2+64}dx = 2\pi i\cdot\frac{1}{16i} = \frac{\pi}{8}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$ **(2023 台聯大研究所工數 A).**

(Sol.) Poles: $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$ (upper half-plane), $e^{\frac{5\pi i}{4}}$, $e^{\frac{7\pi i}{4}}$ (lower half-plane)

$$\operatorname*{Res}_{e^{i\pi/4}}(f) = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{z-e^{\frac{i\pi}{4}}}{1+z^4}\right] = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{1}{4z^3}\right] = \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4}e^{-3\pi i/4} = \frac{1}{4}\left[-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right],$$

$$\operatorname*{Res}_{e^{i3\pi/4}}(f) = \frac{1}{4\left(e^{i3\pi/4}\right)^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}\left[\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right], \quad \int_{-\infty}^{\infty}\frac{dx}{x^4+1} = 2\pi i\left[\frac{1}{4}\cdot(-i\sqrt{2})\right] = \frac{\pi\sqrt{2}}{2}.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}dx$.

(Sol.) $\oint\frac{e^{iz}}{z^2+1}=2\pi i\cdot\operatorname{Res}_{i}(\frac{e^{iz}}{z^2+1})=2\pi i\cdot\lim_{z\to i}(z-i)\cdot\frac{e^{iz}}{(z-i)(z+i)}=2\pi i\cdot\frac{e^{-1}}{2i}=\pi e^{-1}$

$\therefore\ \int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}dx=\pi e^{-1}$ and $\int_{-\infty}^{\infty}\frac{\sin(x)}{x^2+1}dx=0$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx$ **and** $\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx$.

(Sol.) Poles: $2i$, $3i$ (upper half-plane), -$2i$, -$3i$ (lower half-plane)

$f(z)=\frac{e^{iz}}{(z^2+4)(z^2+9)}$, $\operatorname{Res}_{2i}(f)=\frac{e^{-2}}{20i}$, $\operatorname{Res}_{3i}(f)=\frac{-e^{-3}}{30i}$

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x^2+9)}dx=2\pi i\left(\frac{e^{-2}}{20i}-\frac{e^{-3}}{30i}\right)=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right)$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right),\quad \int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx=0.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx$.

(Sol.) $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx$. Poles: i (upper half-plane), -i (lower half-plane). $f(z)=\frac{z^2}{(z^2+1)^2}=\frac{z^2}{(z+i)^2(z-i)^2}$, m of $(z-i)^2$ is 2.

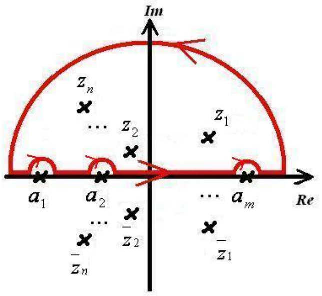
$$\lim_{z\to i}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-i)^2\cdot\frac{z^2}{(z+i)^2(z-i)^2}\right]=\frac{2z(z+i)^2-2z^2(z+i)}{(z+i)^4}\Bigg|_{z=i}=\frac{-8i+4i}{16}=-\frac{i}{4}$$

$$\therefore\ \int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{2\pi i}{2}\left(-\frac{i}{4}\right)=\frac{\pi}{4}.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x\sin(x)}{x^2+4}dx$. **[1991 交大電信所]** (Ans.) $\frac{\pi e^{-2}}{2}$

Eg. Evaluate $\int_{0}^{\infty}\frac{\cos(ax)}{x^2+b^2}dx$, $\begin{matrix}a\ge 0\\ b>0\end{matrix}$. **[1991 台大機研]**

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{x^3\sin(ax)}{x^4+4}dx$, a**>0. [2015 中央電研固態組、生醫電子組]**



Case 3 $\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cdot\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}dx$.

Some poles of $q(z)$ are located on the real axis.

Choose C as a semi-circle with infinite radius enclosing the upper half-plane, but excluding the poles on the real axis. Let z_k $(1\le k\le n)$ be the pole on the upper half-plane and a_j $(1\le j\le m)$ be the simple pole on the real axis. Then we have $\int_{-\infty}^{\infty} f(x)dx=2\pi i\cdot\sum_{k=1}^{n}\operatorname*{Res}_{z_k}(f)+\pi i\cdot\sum_{j=1}^{m}\operatorname*{Res}_{a_j}(f)$.

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{1}{x(x^2-4x+5)}dx$. **[2003交大電信所]**

(Sol.) $f(z)=\frac{1}{z(z^2-4z+5)}=\frac{1}{z[z-(2+i)][z-(2-i)]}$ has 3 poles:

0 (on the real axis), $2+i$ (upper half-plane), $2-i$ (lower half-plane)

$$\int_{-\infty}^{\infty}\frac{1}{x(x^2-4x+5)}dx=2\pi i\cdot\operatorname*{Res}_{2+i}(f)+\pi i\cdot\operatorname*{Res}_{0}(f)$$

$$=2\pi i\cdot\lim_{z\to 2+i}[[z-(2+i)]\cdot\frac{1}{z[z-(2+i)][z-(2-i)]}]+\pi i\cdot\lim_{z\to 0}[z\cdot\frac{1}{z(z^2-4z+5)}]$$

$$=\frac{2\pi i}{(2+i)\cdot 2i}+\frac{\pi i}{5}=\frac{\pi(2-i)}{5}+\frac{\pi i}{5}=\frac{2\pi}{5}.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{dx}{x^3+1}$. **[2012台聯大系統類似題]**

(Sol.) Poles: $e^{\frac{\pi i}{3}}$ (upper half-plane), -1 (on the real axis), $e^{\frac{5\pi i}{3}}$ (lower half-plane)

$$\operatorname*{Res}_{e^{\frac{i\pi}{3}}}(f)=\lim_{z\to e^{\frac{i\pi}{3}}}[\frac{z-e^{\frac{i\pi}{3}}}{z^3+1}]=\lim_{z\to e^{\frac{i\pi}{3}}}[\frac{1}{3z^2}]=\frac{1}{3\left(e^{\pi i/3}\right)^2}=\frac{1}{3}e^{-2\pi i/3}=\frac{1}{3}\left[-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right],$$

$$\operatorname*{Res}_{-1}(f)=\frac{1}{3(-1)^2}=\frac{1}{3},\quad \int_{-\infty}^{\infty}\frac{dx}{x^3+1}=2\pi i\cdot\left[\frac{1}{6}(-1-i\sqrt{3})\right]+\pi i\cdot\left[\frac{1}{3}\right]=\frac{\pi\sqrt{3}}{3}.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos(\frac{\pi x}{2})}{x-2}dx$ **and** $\int_{-\infty}^{\infty}\frac{\sin(\frac{\pi x}{2})}{x-2}dx$.

(Sol.) Pole: 2 (on the real axis), $\oint_c\frac{e^{i\frac{\pi z}{2}}}{z-2}dz=\pi i\cdot\operatorname*{Res}_{2}[\frac{e^{i\frac{\pi z}{2}}}{z-2}]=\pi i\cdot e^{i\frac{\pi}{2}\cdot 2}=\pi i\cdot e^{i\pi}=-\pi i$,

$\therefore\ \int_{-\infty}^{\infty}\frac{\cos(\frac{\pi x}{2})}{x-2}dx=0$ and $\int_{-\infty}^{\infty}\frac{\sin(\frac{\pi}{2}x)}{x-2}dx=-\pi$.

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x-1)}dx$

(Sol.) Poles: 1 and -1 (on the real axis),

$$\operatorname*{Res}_{-1}[\frac{e^{iz}}{(z+1)(z-1)}]=\lim_{z\to -1}[\frac{e^{iz}}{(z+1)(z-1)}\cdot(z+1)]=\lim_{z\to -1}[\frac{e^{iz}}{z-1}]=\frac{e^{-i}}{-2}=\frac{-\cos(1)+i\sin(1)}{2}$$

$$\operatorname*{Res}_{1}[\frac{e^{iz}}{(z+1)(z-1)}]=\lim_{z\to 1}[\frac{e^{iz}}{(z+1)(z-1)}\cdot(z-1)]=\lim_{z\to 1}[\frac{e^{iz}}{z+1}]=\frac{e^{i}}{2}=\frac{\cos(1)+i\sin(1)}{2}$$

$$\int_{-\infty}^{\infty}\frac{e^{iz}}{(z+1)(z-1)}dz=\pi i\cdot\{\operatorname*{Res}_{-1}[\frac{e^{iz}}{(z+1)(z-1)}]+\operatorname*{Res}_{1}[\frac{e^{iz}}{(z+1)(z-1)}]\}=\pi i\cdot[i\sin(1)]$$

$$=-\pi\sin(1)$$

$$\therefore\ \int_{-\infty}^{\infty}\frac{\cos(x)}{(x+1)(x-1)}dx=-\pi\sin(1)\quad\text{and}\quad\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x-1)}dx=0$$

Another method:

$$\frac{e^{iz}}{(z+1)(z-1)}=\frac{-\frac{1}{2}e^{iz}}{z+1}+\frac{\frac{1}{2}e^{iz}}{z-1}$$

$$\operatorname*{Res}_{-1}[\frac{e^{iz}}{z+1}]=\lim_{z\to -1}[\frac{e^{iz}}{z+1}\cdot(z+1)]=\lim_{z\to -1}[e^{iz}]=\cos(1)-i\sin(1)$$

$$\oint_c\frac{-\frac{1}{2}e^{iz}}{(z+1)}dz=-\frac{\pi i}{2}\cdot\operatorname*{Res}_{-1}[\frac{e^{iz}}{z+1}]=-\frac{i\pi}{2}\cos(1)-\frac{\pi}{2}\sin(1)$$

$$\operatorname*{Res}_{1}[\frac{e^{iz}}{z-1}]=\lim_{z\to 1}[\frac{e^{iz}}{z-1}\cdot(z-1)]=\lim_{z\to 1}[e^{iz}]=\cos(1)+i\sin(1)$$

$$\oint_c\frac{\frac{1}{2}e^{iz}}{z-1}dz=\frac{\pi i}{2}\cdot\operatorname*{Res}_{1}[\frac{e^{iz}}{z-1}]=\frac{i\pi}{2}\cos(1)-\frac{\pi}{2}\sin(1)$$

$$\oint_c\frac{e^{iz}}{(z+1)(z-1)}dz$$

$$=-\frac{i\pi}{2}\cos(1)-\frac{\pi}{2}\sin(1)+\frac{i\pi}{2}\cos(1)-\frac{\pi}{2}\sin(1)$$

$$=-\pi\sin(1)$$

$$\therefore\ \int_{-\infty}^{\infty}\frac{\cos(x)}{(x+1)(x-1)}dx=-\pi\sin(1)\quad\text{and}\quad\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x-1)}dx=0$$

Eg. Evaluate $\int_0^{\infty}\frac{\sin(x)}{x}dx$. [1993 交大應數研、2003 中央光電所、2008 成大電研類似題]

(Sol.)

Complex-plane integration method:

$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$ is an imaginary part and $\int_{-\infty}^{\infty}\frac{\cos(x)}{x}dx$ is a real part of an integral $\oint_c \frac{e^{iz}}{z}dz$. There exists only one pole **0** on the real axis.

By formula $\int_{-\infty}^{\infty} f(x)dx = 2\pi i\cdot\sum_{k=1}^{n}\operatorname{Re}s_{z_k}(f)+\pi i\cdot\sum_{j=1}^{m}\operatorname{Re}s_{a_j}(f)$, where z_k $(1\le k\le n)$ is the pole on the upper half-plane and a_j $(1\le j\le m)$ is the pole on the real axis.

$\Rightarrow \oint_c \frac{e^{iz}}{z}dz=\pi i\cdot\operatorname{Re}s_0[\frac{e^{iz}}{z}]=\pi i\cdot e^{i0}=\pi i$, $\therefore$ $\int_{-\infty}^{\infty}\frac{\cos(x)}{x}dx=0$ and $\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx=\pi$

$\Rightarrow \int_0^{\infty}\frac{\sin(x)}{x}dx=\frac{\pi}{2}$.

Laplace transform method:

By formula $L\left[\frac{f(t)}{t}\right]=\int_s^{\infty}F(u)du$,

$$L\left[\frac{\sin(t)}{t}\right]=\int_0^{\infty}e^{-st}\cdot\frac{\sin(t)}{t}dt=\int_s^{\infty}L[\sin(t)]ds=\int_s^{\infty}\frac{1}{s^2+1}ds=\tan^{-1}(s)\Big|_s^{\infty}=\frac{\pi}{2}-\tan^{-1}(s)$$

Set s=0, $\int_0^{\infty}\frac{\sin(x)}{x}dx=\frac{\pi}{2}$

Fourier Transform method:

Consider a rectangular function $f(x)=\begin{cases}1, |x|\le 1\\ 0, |x|\rangle 1\end{cases}$,

$$\Im[f(x)]=\int_{-1}^{1}e^{-i\omega x}dx=\frac{2\sin(\omega)}{\omega}\Rightarrow f(x)=\Im^{-1}\left\{\frac{2\sin\omega}{\omega}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\omega}{\omega}e^{i\omega x}d\omega,$$

$f(0)=1\Rightarrow\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi$, $\frac{\sin(x)}{x}$ is an even function $\Rightarrow \int_0^{\infty}\frac{\sin(x)}{x}dx=\frac{\pi}{2}$.

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx$ **and** $\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}dx$.

(Sol.) Pole: 0 (on the real axis), $\oint_c \frac{e^{i2az}-e^{i2bz}}{z^2}dz = \pi i\cdot \operatorname*{Re s}_0[\frac{e^{i2az}-e^{i2bz}}{z^2}]$

$$\operatorname*{Re s}_0[\frac{e^{i2az}-e^{i2bz}}{z^2}]=\lim_{z\to 0}\frac{1}{(2-1)!}\frac{d}{dz}\left[z^2\cdot\frac{e^{i2az}-e^{i2bz}}{z^2}\right]=[i2ae^{i2az}-i2be^{i2bz}]\,\Big|_{z=0}=i2(a\text{-}b)$$

$$\therefore \oint_c \frac{e^{ii2az}-e^{i2bz}}{z^2}dz=\pi i\cdot\ i2(a\text{-}b)=2(b\text{-}a)\pi \Rightarrow \int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx=2(b\text{-}a)\pi$$

Let b=1, a=0$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos(2ax)-\cos(2bx)}{x^2}dx=\int_{-\infty}^{\infty}\frac{1-\cos(2x)}{x^2}dx=2(1\text{-}0)\pi=2\pi$

And 1-cos(2x)=1-[1-2sin^2(x)]=2sin^2(x).

$\therefore \int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}dx=\pi$ and $\int_{0}^{\infty}\frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x^2-2x+1)}dx$. **[2013 中央電研固態組、生醫電子組]**

(Sol.) Poles: 1 and -1 (on the real axis),

$$\operatorname*{Re s}_{-1}[\frac{e^{iz}}{(z+1)(z-1)^2}]=\lim_{z\to -1}[\frac{e^{iz}\cdot(z+1)}{(z+1)(z-1)^2}]=\lim_{z\to -1}[\frac{e^{iz}}{(z-1)^2}]=\frac{e^{-i}}{4}=\frac{\cos(1)-i\sin(1)}{4}$$

$$\operatorname*{Re s}_{1}[\frac{e^{iz}}{(z+1)(z-1)^2}]=\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}[\frac{e^{iz}\cdot(z-1)^2}{(z+1)(z-1)^2}]=\lim_{z\to 1}\frac{d}{dz}[\frac{e^{iz}}{z+1}]=\frac{i(z+1)e^{iz}-e^{iz}}{(z+1)^2}=$$

$$\frac{[-2\sin(1)-\cos(1)]+i[2\cos(1)-\sin(1)]}{4}$$

$$\int_{-\infty}^{\infty}\frac{e^{iz}}{(z+1)(z-1)^2}dz=\pi i\cdot\{\operatorname*{Re s}_{-1}[\frac{e^{iz}}{(z+1)(z-1)^2}]+\operatorname*{Re s}_{1}[\frac{e^{iz}}{(z+1)(z-1)^2}]\}$$

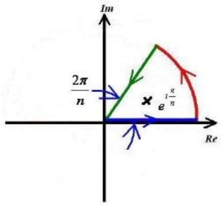
$$=\pi i\cdot\{\frac{\cos(1)-i\sin(1)}{4}+\frac{[-2\sin(1)-\cos(1)]+i[2\cos(1)-\sin(1)]}{4}\}$$

$$=\pi i\cdot\frac{-2\sin(1)+i[2\cos(1)-2\sin(1)]}{4}=\pi\cdot\frac{-i\sin(1)-[\cos(1)-\sin(1)]}{2}$$

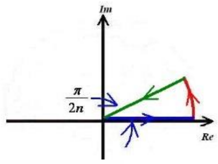
$$\int_{-\infty}^{\infty}\frac{\sin(x)}{(x+1)(x^2-2x+1)}dx=\frac{-\pi\sin(1)}{2},$$

$$\int_{-\infty}^{\infty}\frac{\cos(x)}{(x+1)(x^2-2x+1)}dx=\frac{-\pi[\cos(1)-\sin(1)]}{2}$$

Case 4 $\int_0^\infty \begin{Bmatrix}\sin(x^n)\\ \cos(x^n)\end{Bmatrix}dx$ **or** $\int_0^\infty G(x^n)dx$



Choose C as a sector with angle $\dfrac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$ or a sector with angle $\dfrac{\pi}{2n}$ enclosing no poles.



Eg. Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$, **$n>1$.**

(Sol.) Choose C as a sector with angle $\dfrac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$.

$$\oint_c \frac{dz}{1+z^n} = 2\pi i\cdot \underset{e^{i\frac{\pi}{n}}}{\mathrm{Res}}(f) = 2\pi i\cdot \lim_{z\to e^{i\frac{\pi}{n}}}\left[\left(z-e^{i\frac{\pi}{n}}\right)\frac{1}{1+z^n}\right] = \frac{2\pi i}{nz^{n-1}}\bigg|_{z=e^{i\frac{\pi}{n}}} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$= \boxed{\int_0^R \frac{dx}{1+x^n}} + \boxed{\int_0^{\frac{2\pi}{n}} \frac{i\mathrm{Re}^{i\theta}\,d\theta}{1+R^n e^{in\theta}}} + \boxed{\int_R^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n e^{i2\pi}}}$$

As $R\to\infty$, $\displaystyle\int_0^{\frac{2\pi}{n}} \frac{i\mathrm{Re}^{i\theta}\,d\theta}{1+R^n e^{in\theta}} \to 0$ ($\because$ $n>1$)

$$\therefore\ \boxed{\int_0^\infty \frac{dx}{1+x^n}} + \boxed{\int_\infty^0 \frac{e^{i\frac{2\pi}{n}}\,dR}{1+R^n}} = \left(1-e^{i\frac{2\pi}{n}}\right)\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^n} = \frac{e^{i\frac{\pi}{n}}}{1-e^{i\frac{2\pi}{n}}}\cdot\frac{-2\pi i}{n} = \frac{1}{\dfrac{e^{i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}}}{2i}}\cdot\frac{\pi}{n} = \frac{\dfrac{\pi}{n}}{\sin\left(\dfrac{\pi}{n}\right)}$$

Eg. Show that $\displaystyle\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$.

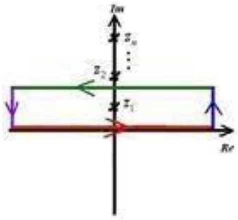
(Proof) Choose C as a sector with angle $\dfrac{\pi}{4}$ enclosing no poles.

$$\oint_c e^{iz^2}dz = 0 = \boxed{\int_0^R e^{ix^2}dx} + \boxed{\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot i\mathrm{Re}^{i\theta}\,d\theta} + \boxed{\int_R^0 e^{iR^2e^{i\frac{\pi}{2}}}\cdot e^{i\frac{\pi}{4}}dR}$$

As $R\to\infty$, $\boxed{\displaystyle\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot i\mathrm{Re}^{i\theta}\,d\theta = iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos2\theta+i\sin2\theta)}e^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin2\theta}e^{i(\theta+R^2\cos2\theta)}d\theta \to 0,}$

$$\boxed{\int_0^\infty [\cos(x^2)+i\sin(x^2)]dx} = \underline{\int_0^\infty\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right)}e^{-R^2}dR = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{\pi}}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2}i$$

$$\therefore\ \int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$



Case 5 $\int_{-\infty}^{\infty} G(e^x)dx$ **, where** $G(x)=\dfrac{p(x)}{x^n+q_{n-1}(x)}$

Choose C as an infinitely-wide rectangle and there is one pole on the imaginary axis.

Eg. Evaluate $\int_{-\infty}^{\infty} \dfrac{e^{mx}}{1+e^x}dx$ **, 0<*m*<1.** **【1991台大機研】**

【2014台聯大系統】

(Sol.) Pole: $i\pi$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = \boxed{\int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx} + \boxed{\int_0^{2\pi}\frac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy} + \boxed{\int_{\infty}^{-\infty}\frac{e^{mx}\cdot e^{i2m\pi}}{1+e^x\cdot e^{i2\pi}}dx} + \boxed{\int_{2\pi}^{0}\frac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy}$$

$$= 2\pi i\cdot \operatorname*{Res}_{\pi i}(f) = 2\pi i\cdot \lim_{z\to\pi i}\frac{(z-\pi i)e^{mz}}{1+e^z}$$

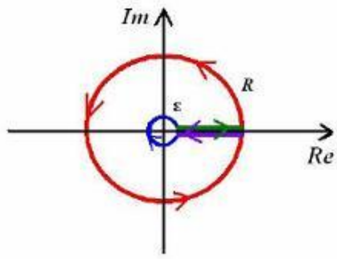
$$= 2\pi i\cdot \lim_{z\to\pi i}\frac{e^{mz}+m(z-\pi i)e^{mz}}{e^z} = 2\pi i(-1)^{m-1} = 2\pi i e^{i(m-1)\pi}$$

$\because\ 0<m<1,\ \therefore R\to\infty,$ $\boxed{\int_0^{2\pi}\dfrac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy \to 0}$ and $\boxed{\int_{2\pi}^{0}\dfrac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy \to 0}$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = 2\pi i e^{i(m-1)\pi} = \int_{-\infty}^{\infty}(1-e^{i2m\pi})\cdot\frac{e^{mx}}{1+e^x}dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx = \frac{2\pi i}{1-e^{i2m\pi}}\cdot e^{im\pi}\cdot(-1) = \frac{\pi}{\dfrac{e^{im\pi}-e^{-im\pi}}{2i}} = \frac{\pi}{\sin(m\pi)}$$

Case 6 **Other types**



Eg. For 0<*p*<1, $\int_0^\infty \frac{x^p dx}{x(1+x)} = ?$

(Sol.) $\because$ $0<p<1$, $\therefore$ Poles are 0 and -1.

$$\oint \frac{z^p dz}{z(1+z)} = 2\pi i \cdot \underset{-1}{\operatorname{Res}}(f) = 2\pi i \cdot \lim_{z\to -1}(z+1)\cdot\frac{z^p}{z(z+1)} = 2\pi i \cdot e^{i\pi(p-1)}\text{, and}$$

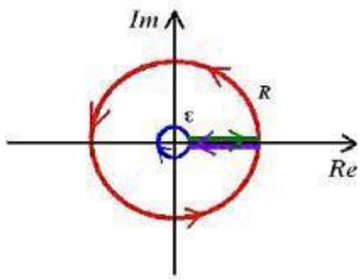
$$\oint \frac{z^p dz}{z(1+z)} = \boxed{\int_\varepsilon^R \frac{x^p dx}{x(1+x)}} + \boxed{\int_0^{2\pi} \frac{i(\mathrm{Re}^{i\theta})^p d\theta}{1+\mathrm{Re}^{i\theta}}} + \boxed{\int_R^\varepsilon \frac{(xe^{i2\pi})^{p-1} d(xe^{i2\pi})}{1+xe^{i2\pi}}} + \boxed{\int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p d\theta}{1+\varepsilon e^{i\theta}}}$$

$\because$ $0<p<1$, $R\to\infty$, $\therefore$ $\boxed{\int_0^{2\pi} \frac{i(\mathrm{Re}^{i\theta})^p d\theta}{1+\mathrm{Re}^{i\theta}} \to 0}$

$\because$ $\varepsilon \to 0$, $\therefore$ $\boxed{\int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p d\theta}{1+\varepsilon e^{i\theta}} \to 0}$

$$\Rightarrow \oint \frac{z^p dz}{z(1+z)} = \boxed{\int_0^\infty \frac{x^p dx}{x(1+x)}} + \boxed{\int_\infty^0 \frac{(xe^{i2\pi})^{p-1} d(xe^{i2\pi})}{1+xe^{i2\pi}}} = \int_0^\infty \frac{[1-e^{i2\pi(p-1)}]\cdot x^p dx}{x(1+x)}$$

$$\therefore \int_0^\infty \frac{x^p dx}{x(1+x)} = \frac{2\pi i \cdot e^{i\pi(p-1)}}{1-e^{i2\pi(p-1)}} = \frac{\pi}{(e^{ip\pi}-e^{-ip\pi})/2i} = \frac{\pi}{\sin(p\pi)}$$



Eg. For -1<*a*<1, $\int_0^\infty \frac{x^a dx}{(1+x)^2} = ?$ **【交大電信研究所、2003 中央光電所】**

(Sol.) -1 is a multiple-order pole.

$$\oint \frac{z^a dz}{(1+z)^2} = 2\pi i \cdot \underset{-1}{\operatorname{Res}}(f) = 2\pi i \cdot \lim_{z\to -1}\frac{1}{1!}d[(z+1)^2\cdot\frac{z^a}{(z+1)^2}]/dz = 2\pi i \cdot ae^{i\pi(a-1)}\text{, and}$$

$$\oint \frac{z^a dz}{(1+z)^2} = \boxed{\int_\varepsilon^R \frac{x^a dx}{(1+x)^2}} + \boxed{\int_0^{2\pi} \frac{(\mathrm{Re}^{i\theta})^a i\mathrm{Re}^{i\theta} d\theta}{(1+\mathrm{Re}^{i\theta})^2}} + \boxed{\int_R^\varepsilon \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2}} + \boxed{\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a i\varepsilon e^{i\theta} d\theta}{(1+\varepsilon e^{i\theta})^2}}$$

$\because$ $-1<a<1$, $R\to\infty$, $\therefore$ $\boxed{\int_0^{2\pi} \frac{(\mathrm{Re}^{i\theta})^a i\mathrm{Re}^{i\theta} d\theta}{(1+\mathrm{Re}^{i\theta})^2} \to 0}$

$\because$ $\varepsilon \to 0$, $\therefore$ $\boxed{\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a i\varepsilon e^{i\theta} d\theta}{(1+\varepsilon e^{i\theta})^2} \to 0}$

$$\Rightarrow \oint \frac{z^a dz}{(1+z)^2} = \boxed{\int_0^\infty \frac{x^a dx}{(1+x)^2}} + \boxed{\int_\infty^0 \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2}} = \int_0^\infty \frac{[1-e^{i2\pi(a+1)}]\cdot x^a dx}{(1+x)^2}$$

$$\therefore \int_0^\infty \frac{x^a dx}{(1+x)^2} = \frac{2\pi i \cdot ae^{i\pi(a-1)}}{1-e^{i2\pi(a+1)}} = \frac{\pi a}{(e^{i\pi a}-e^{-i\pi a})/2i} = \frac{a\pi}{\sin(a\pi)}$$