

## Chapter 3 Complex Sequences and Series

### 3-1 Sequence and Series

**Sequence  $\{z_n\}$ :**  $z_0, z_1, z_2, z_3, \dots, z_n, \dots$

**Series:**  $\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \dots + z_n + \dots$



**Absolute convergence:**  $\sum_{n=0}^{\infty} |z_n|$  converges. (In this case,  $\sum_{n=0}^{\infty} z_n$  converges also.)

**Eg.**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$  is an absolutely convergent series because

$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n+1)^2} \right| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$  and  $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$  are both convergent.

**Conditional convergence:**  $\sum_{n=0}^{\infty} z_n$  converges but  $\sum_{n=0}^{\infty} |z_n|$  diverges.

**Eg.**  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$  is conditionally convergent because  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

is convergent but  $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$  is divergent.

**Radius of convergence for  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ :** There exists  $R$  (possibly  $R = \infty$ ) such that the series converges absolutely if  $|z - z_0| < R$  and diverges if  $|z - z_0| > R$ .

(Proof)  $\frac{|a_{n+1} (z - z_0)^{n+1}|}{|a_n (z - z_0)^n|} < 1 \Rightarrow |z - z_0| < \frac{|a_n|}{|a_{n+1}|}$

Define  $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$ , so the proof is complete.

**Eg. Find the radius of convergence for  $\sum_{n=0}^{\infty} \frac{n^n}{n!} (z - i)^n$ .**

(Sol.)  $\left| \frac{\frac{(n+1)^{n+1}}{(n+1)!} (z - i)^{n+1}}{\frac{n^n}{n!} (z - i)^n} \right| = \left| \left( \frac{n+1}{n} \right)^n (z - i) \right| < 1$

$|z - i| < \left( 1 + \frac{1}{n} \right)^{-n} \rightarrow e^{-1}$  as  $n \rightarrow \infty$ ,  $\therefore R = e^{-1}$

### 3-2 Complex Taylor's Series & Laurent's Series

**Taylor's series:**  $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n$ . In case  $z_0=0$ , it is called Maclaurin's series.

**Eg. Find the Maclaurin's expansion of  $f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-w^2} dw$ .**

$$\begin{aligned} \text{(Sol.) } e^{-w^2} &= 1 - w^2 + \frac{(-w^2)^2}{2!} + \frac{(-w^2)^3}{3!} + \frac{(-w^2)^4}{4!} + \frac{(-w^2)^5}{5!} + \dots \\ &= 1 - w^2 + \frac{w^4}{2!} - \frac{w^6}{3!} + \frac{w^8}{4!} - \frac{w^{10}}{5!} + \dots + (-1)^n \frac{w^{2n}}{n!} \\ \int_0^z e^{-w^2} dw &= z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)n!} + \dots \\ f(z) &= \frac{2}{\sqrt{\pi}} \left( z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)n!} + \dots \right) \end{aligned}$$

**Laurent's series:**

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots,$$

where  $a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw$ , and  $C: |z-z_0|=\rho, r_1 < \rho < r_2$

**Eg. Find the Laurent's series of  $f(z) = \frac{z - \sin z}{z^3}$  about  $z_0=0$ , indicate the type of singularity and the region of convergence of the series. [清大電研]**

$$\begin{aligned} \text{(Sol.) } f(z) &= \frac{z - \sin z}{z^3} = \frac{1}{z^3} \left[ z - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right] \\ &= \frac{1}{z^3} \left[ \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \frac{z^9}{9!} + \dots \right] = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \frac{z^6}{9!} + \dots \\ \therefore z_0 = 0 &\text{ is a removable singularity.} \end{aligned}$$

$$f(z) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \frac{z^6}{9!} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot z^{2n-2}}{(2n+1)!} = \sum_{n=1}^{\infty} a_n$$

$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{(-1)^n \cdot z^{2n}}{(2n+3)!} \right| \left/ \left| \frac{(-1)^{n-1} \cdot z^{2n-2}}{(2n+1)!} \right| \right| < 1$$

$$z^2 < (2n+3)(2n+2) \Rightarrow z < R \rightarrow \infty$$

**Basic formulae:**

$$\frac{1}{1 \pm r} = 1 \mp r + r^2 \mp r^3 + r^4 \mp r^5 + r^6 \mp r^7 + r^8 \mp \dots$$

$$\frac{1}{a-b} = \begin{cases} \frac{1}{1-\frac{b}{a}} = \frac{1}{a} \left[ 1 + \left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^3 + \left(\frac{b}{a}\right)^4 + \dots \right], & |a| > |b| \\ \frac{-\frac{1}{b}}{1-\frac{a}{b}} = \frac{-1}{b} \left[ 1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right)^4 + \dots \right], & |a| < |b| \end{cases}$$

$$\frac{1}{a+b} = \begin{cases} \frac{1}{1+\frac{b}{a}} = \frac{1}{a} \left[ 1 - \left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^3 + \left(\frac{b}{a}\right)^4 - \dots \right], & |a| > |b| \\ \frac{\frac{1}{b}}{1+\frac{a}{b}} = \frac{1}{b} \left[ 1 - \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right)^4 - \dots \right], & |a| < |b| \end{cases}$$

**Eg. Expand  $f(z) = \frac{z}{(z-1)(2-z)}$  by Laurent's series in case: (a)  $|z| < 1$ , (b)  $1 < |z| < 2$ , (c)  $|z| > 2$ , (d)  $|z-1| > 1$ , and (e)  $0 < |z-2| < 1$ .**

(Sol.)  $f(z) = \frac{z}{(z-1)(2-z)} = -\frac{z}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{2}{z-2}$

(a)  $|z| < 1 \Rightarrow 1 > \frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1} = \frac{-1}{1-z} = (-1)(1+z+z^2+z^3+\dots) & , \therefore f(z) = -\frac{1}{2}z - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \dots \\ = -1 - z - z^2 - z^3 - z^4 - \dots \\ \frac{-2}{z-2} = \frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \end{cases}$$

(b)  $1 < |z| < 2 \Rightarrow 1 > \frac{1}{|z|}, 1 > \frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) & , \therefore f(z) = \dots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \dots \\ = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \\ \frac{-2}{z-2} = \frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \end{cases}$$

$$(c) |z| > 2 \Rightarrow \frac{2}{|z|} < 1 \Rightarrow 1 > \frac{1}{|z|}$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) \\ = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots \\ \frac{-2}{z-2} = -\frac{2}{z} \cdot \frac{1}{1-\frac{2}{z}} = -\frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\dots\right) \\ = -\frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \dots \end{array} \right. , \therefore f(z) = -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \dots$$

$$(d) |z-1| > 1 \Rightarrow \frac{1}{|z-1|} < 1$$

$$\frac{-2}{z-2} = \frac{-2}{(z-1)-1} = -\frac{2}{z-1} \cdot \frac{1}{1-\frac{1}{z-1}} = -\frac{2}{z-1} \cdot [1+(z-1)^{-1}+(z-1)^{-2}+\dots] = -\frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \dots$$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-2} = \frac{-1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \frac{2}{(z-1)^4} - \dots$$

$$(e) 0 < |z-2| < 1 \Rightarrow \frac{1}{z-1} = \frac{1}{1+(z-2)} = 1 - (z-2) + (z-2)^2 - (z-2)^3 + \dots$$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-2} = -\frac{2}{z-2} + 1 - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4 - \dots$$

**Eg. Find the Laurent's series of  $f(z) = \frac{2}{z^2 - 1}$  for (a)  $|z-1| < 1$  and (b)  $|z-1| > 1$ .**

(Sol.)  $f(z) = \frac{2}{z^2 - 1} = \frac{1}{z-1} - \frac{1}{z+1} = \frac{1}{z-1} + \frac{-1}{2+(z-1)}$

(a)  $|z-1| < 1 \Rightarrow \frac{|z-1|}{2} < 1$

$$\frac{-1}{2+(z-1)} = \frac{\frac{-1}{2}}{1+\frac{z-1}{2}} = -\left(\frac{1}{2}\right) \cdot \left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \left(\frac{z-1}{2}\right)^4 - + \dots\right]$$

$$f(z) = \frac{1}{z-1} - \left(\frac{1}{2}\right) \cdot \left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \left(\frac{z-1}{2}\right)^4 - + \dots\right]$$

(b)  $|z-1| > 1 \Rightarrow \begin{cases} |z-1| > 2 \\ 2 > |z-1| > 1 \end{cases}$

Case 1: If  $|z-1| > 2 \Rightarrow 1 > \frac{2}{|z-1|}$ ,

$$\frac{-1}{2+(z-1)} = \frac{\frac{-1}{(z-1)}}{1+\frac{2}{(z-1)}} = -\left(\frac{1}{z-1}\right) \cdot \left[1 - \frac{2}{z-1} + \left(\frac{2}{z-1}\right)^2 - \left(\frac{2}{z-1}\right)^3 + \left(\frac{2}{z-1}\right)^4 - + \dots\right]$$

$$f(z) = \frac{1}{z-1} - \left(\frac{1}{z-1}\right) \cdot \left[1 - \frac{2}{z-1} + \left(\frac{2}{z-1}\right)^2 - \left(\frac{2}{z-1}\right)^3 + \left(\frac{2}{z-1}\right)^4 - + \dots\right]$$

$$= \frac{1}{z-1} \cdot \left[\frac{2}{z-1} - \left(\frac{2}{z-1}\right)^2 + \left(\frac{2}{z-1}\right)^3 - \left(\frac{2}{z-1}\right)^4 + \dots\right]$$

Case 2: If  $2 > |z-1| > 1 \Rightarrow 1 > \frac{|z-1|}{2}$

$$\frac{-1}{2+(z-1)} = \frac{\frac{-1}{2}}{1+\frac{z-1}{2}} = -\left(\frac{1}{2}\right) \cdot \left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \left(\frac{z-1}{2}\right)^4 - + \dots\right]$$

$$f(z) = \frac{1}{z-1} - \left(\frac{1}{2}\right) \cdot \left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \left(\frac{z-1}{2}\right)^4 - + \dots\right]$$

**Eg. Find the Laurent's series of  $f(z)=\frac{1}{6-z-z^2}$  for (a)  $|z|<2$ , (b)  $2<|z|<3$ , and (c)  $|z|>3$ . [2023 台聯大電機聯招]**

$$\text{(Sol.) } f(z) = \frac{1}{6-z-z^2} = \frac{-1}{z^2+z-6} = \frac{-1}{(z+3)(z-2)} = \frac{\frac{1}{5}}{z+3} - \frac{\frac{1}{5}}{z-2}$$

$$\text{(a) } |z|<2<3, \frac{1}{5} \cdot \frac{1}{z+3} = \frac{1}{5} \cdot \frac{\frac{1}{3}}{1+\frac{z}{3}} = \frac{1}{15} \cdot [1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots]$$

$$-\frac{1}{5} \cdot \frac{1}{z-2} = -\frac{1}{5} \cdot \frac{\frac{1}{2}}{1-\frac{z}{2}} = \frac{-1}{10} \cdot [1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots]$$

$$f(z) = \frac{1}{15} \cdot [1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots] - \frac{1}{10} \cdot [1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots]$$

$$\text{(b) } 2<|z|<3, \frac{1}{5} \cdot \frac{1}{z+3} = \frac{1}{5} \cdot \frac{\frac{1}{3}}{1+\frac{z}{3}} = \frac{1}{15} \cdot [1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots]$$

$$-\frac{1}{5} \cdot \frac{1}{z-2} = -\frac{1}{5} \cdot \frac{\frac{1}{z}}{1-\frac{2}{z}} = \frac{-1}{5z} \cdot [1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots]$$

$$f(z) = \frac{1}{15} \cdot [1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \dots] - \frac{1}{5z} \cdot [1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots]$$

$$\text{(c) } |z|>3>2, \frac{1}{5} \cdot \frac{1}{z+3} = \frac{1}{5} \cdot \frac{\frac{1}{z}}{1+\frac{3}{z}} = \frac{1}{5z} \cdot [1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots]$$

$$-\frac{1}{5} \cdot \frac{1}{z-2} = -\frac{1}{5} \cdot \frac{\frac{1}{z}}{1-\frac{2}{z}} = \frac{-1}{5z} \cdot [1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots]$$

$$f(z) = \frac{1}{5z} \cdot [1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \dots] - \frac{1}{5z} \cdot [1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots]$$

**Eg. Find the Laurent's series of  $f(z) = \frac{1}{z^2 - 3z + 2}$  for  $1 < |z| < 2$ . [1991 台大機研]**

$$\begin{aligned}
 \text{(Sol.) } \frac{1}{z^2 - 3z + 2} &= \frac{-1}{z-1} + \frac{1}{z-2} \\
 1 < |z| < 2 &\Rightarrow 1 > \frac{1}{|z|}, \frac{-1}{z-1} = \frac{-1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{-1}{z} \cdot \left[ 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] \\
 &\Rightarrow 1 > \frac{|z|}{2}, \frac{1}{z-2} = \frac{-1}{2} \cdot \frac{1}{1-\frac{z}{2}} = \frac{-1}{2} \left[ 1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right] \\
 &\Rightarrow f(z) = -\left( \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) - \frac{1}{2} \left( 1 + \frac{z}{2} + \frac{z^2}{4} + \dots \right)
 \end{aligned}$$

**Eg. (a)  $\frac{1}{(z-1)(z-3)}$  for  $1 < |z| < 3$  about  $z_0=0$ , (b)  $\frac{e^z}{(z-1)^2}$  for  $|z-1| > 0$  about  $z_0=1$ .**

**Find their respective Laurent's series. [1991 交大電信]**

$$\begin{aligned}
 \text{(Sol.) (a) } \frac{1}{(z-1)(z-3)} &= \frac{-\frac{1}{2}}{z-1} + \frac{\frac{1}{2}}{z-3} \\
 1 < |z| < 3 &\Rightarrow \frac{1}{|z|} < 1, \frac{-\frac{1}{2}}{z-1} = \frac{-\frac{1}{2}}{z} \cdot \frac{1}{(1-\frac{1}{z})} = \frac{-\frac{1}{2}}{z} \cdot \left[ 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right] \\
 1 < |z| < 3 &\Rightarrow 1 > \frac{|z|}{3}, \frac{\frac{1}{2}}{z-3} = \frac{-\frac{1}{2}}{3} \cdot \frac{1}{(1-\frac{z}{3})} = \frac{-1}{6} \left[ 1 + \frac{z}{3} + \frac{z^2}{9} + \dots \right] \\
 &\Rightarrow f(z) = -\frac{1}{2} \left( \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) - \frac{1}{6} \left( 1 + \frac{z}{3} + \frac{z^2}{9} + \dots \right) \\
 \text{(b) } \frac{e^z}{(z-1)^2} &= \frac{e \cdot e^{z-1}}{(z-1)^2} = \frac{e}{(z-1)^2} \cdot \left[ 1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \dots \right] \\
 &= e \cdot \left[ \frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \frac{(z-1)}{3!} + \dots \right]
 \end{aligned}$$

**Eg. Find the Laurent's series of  $f(z)=\frac{5z+2j}{z^2+jz}$  for  $1<|z-j|<2$ . [2013 中山電研]**

$$\text{(Sol.) } \frac{5z+2j}{z^2+jz} = \frac{3}{z+j} + \frac{2}{z}$$

$$|z-j|<2 \Rightarrow |z-j|<|2j| \Rightarrow \frac{|z-j|}{|2j|} < 1,$$

$$\frac{3}{z+j} = \frac{3}{2j+(z-j)} = \frac{3}{2j} \cdot \frac{1}{1+(\frac{z-j}{2j})} = \frac{3}{2j} \cdot [1 - \frac{z-j}{2j} + (\frac{z-j}{2j})^2 - (\frac{z-j}{2j})^3 + \dots]$$

$$1 < |z-j| \Rightarrow \frac{|j|}{|z-j|} < 1$$

$$\frac{2}{z} = \frac{2}{j+(z-j)} = \frac{2}{(z-j)} \cdot \frac{1}{1+(\frac{j}{z-j})} = \frac{2}{(z-j)} \cdot [1 - \frac{j}{z-j} + (\frac{j}{z-j})^2 - (\frac{j}{z-j})^3 + \dots]$$

$$\frac{5z+2j}{z^2+jz} = \frac{3}{2j} \cdot [1 - \frac{z-j}{2j} + (\frac{z-j}{2j})^2 - (\frac{z-j}{2j})^3 + \dots] + \frac{2}{(z-j)} \cdot [1 - \frac{j}{z-j} + (\frac{j}{z-j})^2 - (\frac{j}{z-j})^3 + \dots]$$

**Eg. (a) Find the Laurent's series of  $f(z)=\frac{\sin(z) \cdot \cos(2z)}{z^3}$  about  $z_0=0$ . (b) Evaluate**

**$\oint_c f(z)dz$ , where  $C$  is a circle  $|z-1|=2$ . [2012 中央電研固態組、生醫電子組]**

$$\text{(Sol.) (a) } f(z) = \frac{\sin(z) \cdot \cos(2z)}{z^3} = \frac{\sin(3z) + \sin(-z)}{2z^3} = \frac{\sin(3z) - \sin(z)}{2z^3}$$

$$= \frac{1}{2z^3} \left[ \left( 3z - \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!} - \frac{(3z)^7}{7!} + \dots \right) - \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right]$$

$$= \frac{1}{2z^3} \left[ 2z - \frac{26z^3}{3!} + \frac{242z^5}{5!} - \dots \right] = \frac{1}{z^2} - \frac{13}{3!} + \frac{121z^2}{5!} - \dots$$

(b) There is only one pole 0 within  $|z-1|=2$  and the residue at  $z_0=0$  is 0.

$$\therefore \oint_c f(z)dz = 2\pi i \cdot 0 = 0.$$



### 3-3 Summation of Series by the Residue Theorem

**Theorem** Let  $z_1, \dots, z_m$  be the poles of  $f(z)$ , then  $\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{j=1}^m \operatorname{Res}_{z_j} [\cot(\pi z) \cdot f(z)]$

and  $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{j=1}^m \operatorname{Res}_{z_j} [\csc(\pi z) \cdot f(z)]$ .

**Eg. Evaluate**  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ . [1991 清大電研]

$$\text{(Sol.) } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = (-1) \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} = \sum_{n=-\infty}^0 \frac{(-1)^n}{2n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \left( \dots + \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 \right) + \left( -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots \right)$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{2} \operatorname{Res}_{\frac{1}{2}} \left[ \frac{\csc(\pi z)}{2z-1} \right]$$

$$= \frac{\pi}{4} \operatorname{Res}_{\frac{1}{2}} \left[ \frac{\csc(\pi z)}{z - \frac{1}{2}} \right] = \frac{\pi}{4} \lim_{z \rightarrow \frac{1}{2}} \frac{\csc \pi z}{z - \frac{1}{2}} \times \left( z - \frac{1}{2} \right) = \frac{\pi}{4}$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$

**Eg. Evaluate**  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$ .

$$\text{(Sol.) } \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=-\infty}^0 \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$= \left( \dots + \frac{1}{5^2} + \frac{1}{3^2} + 1 \right) + \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\therefore 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{-\pi}{2} \operatorname{Res}_{\frac{1}{2}} \left[ \frac{\cot(\pi z)}{(2z-1)^2} \right]$$

$$= \frac{-\pi}{8} \operatorname{Res}_{\frac{1}{2}} \left[ \frac{\cot(\pi z)}{\left(z - \frac{1}{2}\right)^2} \right] = \frac{-\pi}{8} \lim_{z \rightarrow \frac{1}{2}} \frac{1}{1!} \frac{d}{dz} \left[ \frac{\cot(\pi z)}{\left(z - \frac{1}{2}\right)^2} \times \left(z - \frac{1}{2}\right)^2 \right]$$

$$= \frac{-\pi}{8} \cdot (-\pi) \cdot \csc^2\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8}, \therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

**Eg. Evaluate**  $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ .

(Sol.)  $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=-\infty}^{-1} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \left( \dots + \frac{1}{4^2} - \frac{1}{3^2} + \frac{1}{2^2} - 1 \right)$$

$$+ \left( -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi}{2} \operatorname{Re}_0 s \left[ \frac{\csc(\pi z)}{z^2} \right]$$

$$\frac{\csc(\pi z)}{z^2} = \frac{1}{z^2 \sin(\pi z)} = \frac{1}{z^2 \left( \pi z - \frac{\pi^3 z^3}{3!} + \dots \right)} = \frac{1}{\pi} \cdot \frac{1}{z^3} + \frac{\pi}{6} \cdot \frac{1}{z} + \dots$$

$$\therefore \operatorname{Re}_0 s \left( \frac{\csc(\pi z)}{z^2} \right) = \frac{\pi}{6} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \left( \frac{\pi}{2} \right) \cdot \left( \frac{\pi}{6} \right) = \frac{\pi^2}{12}$$

**Eg. Evaluate**  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ .

(Sol.)  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^2} = -\frac{\pi}{2} \operatorname{Re}_0 s \left( \frac{\cot(\pi z)}{z^2} \right)$

$$\frac{\cot(\pi z)}{z^2} = \frac{\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{\left( 1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \dots \right)}{z^2 \left( \pi z - \frac{\pi^3 z^3}{3!} + \dots \right)} = \frac{1}{\pi} \cdot \frac{1}{z^3} - \frac{\pi}{3} \cdot \frac{1}{z} + \dots$$

$$\therefore \operatorname{Re}_0 s \left( \frac{\cot(\pi z)}{z^2} \right) = -\frac{\pi}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left( -\frac{\pi}{2} \right) \cdot \left( -\frac{\pi}{3} \right) = \frac{\pi^2}{6}$$

**Eg. Evaluate**  $1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$ .

(Sol.)  $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n^4} = -\frac{\pi}{2} \operatorname{Re}_0 s \left( \frac{\cot(\pi z)}{z^4} \right)$

$$\frac{\cot(\pi z)}{z^4} = \frac{\cos(\pi z)}{z^4 \sin(\pi z)} = \frac{\left( 1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \dots \right)}{z^4 \left( \pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \dots \right)} = \frac{1}{\pi} \cdot \frac{1}{z^5} - \frac{\pi}{3} \cdot \frac{1}{z^3} + \frac{\pi^3}{45} \cdot \frac{1}{z} - \dots$$

$$\therefore \operatorname{Re}_0 s \left( \frac{\cot(\pi z)}{z^4} \right) = -\frac{\pi^3}{45} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \left( -\frac{\pi}{2} \right) \cdot \left( -\frac{\pi^3}{45} \right) = \frac{\pi^4}{90}$$

**Eg. Evaluate (a)**  $\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = ?$  **(b)**  $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = ?$

$$\begin{aligned} \text{(Sol.) (a)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} &= -\pi \cdot \left[ \operatorname{Res}_{ai} \left( \frac{\cot(\pi z)}{z^2 + a^2} \right) + \operatorname{Res}_{-ai} \left( \frac{\cot(\pi z)}{z^2 + a^2} \right) \right] \\ &= -\pi \cdot \left[ \frac{\cot(\pi ai)}{2ai} + \frac{\cot(-\pi ai)}{-2ai} \right] = -\frac{\pi \cot(\pi ai)}{ai} = \frac{\pi}{a} \coth(a\pi). \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} + \sum_{n=-\infty}^{-1} \frac{1}{n^2 + a^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} = \frac{\pi}{a} \coth(a\pi) \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} &= \frac{1}{2} \left[ \frac{\pi}{a} \coth(a\pi) - \frac{1}{a^2} \right] = \frac{a\pi \coth(a\pi) - 1}{2a^2}. \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{2} \lim_{a \rightarrow 0} \frac{a\pi \coth(a\pi) - 1}{a^2} = \frac{1}{2} \lim_{a \rightarrow 0} \frac{a\pi \cdot \frac{e^{a\pi} + e^{-a\pi}}{e^{a\pi} - e^{-a\pi}} - 1}{a^2} = \frac{1}{2} \lim_{a \rightarrow 0} \frac{a\pi \cdot \frac{(1 - \frac{a^2\pi^2}{2} + \dots)}{(a\pi - \frac{a^3\pi^3}{6} + \dots)} - 1}{a^2} \\ &= \frac{1}{2} \lim_{a \rightarrow 0} \frac{(1 + \frac{a^2\pi^2}{3} + \dots) - 1}{a^2} = \frac{\pi^2}{6} \end{aligned}$$

**Eg. Evaluate**  $\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = ?$  [1999 台科大電研]

(Sol.)  $a=1$ ,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1^2} = \frac{1}{2} [\pi \coth(\pi) - 1], \quad \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2 + 1^2} = \frac{1}{2} [\pi \coth(\pi) + 1]$$

### 3-4 Infinite products

**Theorem** If  $\prod_{n=1}^{\infty} a_n$  ( $\neq 0$ ) converges, then  $\lim_{n \rightarrow \infty} a_n = 1$ .

(Proof) Let  $p_N = \prod_{n=1}^N a_n$ ,  $1 = \lim_{N \rightarrow \infty} p_N / \lim_{N \rightarrow \infty} p_{N-1} = \lim_{N \rightarrow \infty} \left( \prod_{n=1}^N a_n / \prod_{n=1}^{N-1} a_n \right) = \lim_{N \rightarrow \infty} a_N$

**Theorem**  $\prod_{n=1}^{\infty} (1 + a_n)$  converges if and only if  $\sum_{n=1}^{\infty} \log(1 + a_n)$  converges.

**Theorem**  $\prod_{n=1}^{\infty} (1 + a_n)$  is absolutely convergent if and only if  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

**Eg. Determine the domain  $D$  of convergence for  $\prod_{n=1}^{\infty} \left[ \left(1 + \frac{1}{n}\right)^{n^2} \cdot z^n \right]$ .**

(Sol.)  $\prod_{n=1}^{\infty} \left[ \left(1 + \frac{1}{n}\right)^{n^2} z^n \right]$  diverges if  $\lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^{n^2} z^n \right| > 1$

$$\left(1 + \frac{1}{n}\right)^{n^2} = e^{n^2 \ln\left(1 + \frac{1}{n}\right)} = e^{n \ln\left(1 + \frac{1}{n}\right)^n} \rightarrow e^{n \ln e} = e^n$$

$$\lim_{n \rightarrow \infty} |e^n \cdot z^n| > 1 \Rightarrow |z| > e^{-1}, \quad \prod_{n=1}^{\infty} \left[ \left(1 + \frac{1}{n}\right)^{n^2} z^n \right] \text{ diverges}$$

$\therefore D = \{z : |z| < e^{-1}\}$  for convergence.

**Weierstrass' theorem** Let  $f(z)$  be analytic. All zeros  $a_1, a_2, a_3, \dots, a_n, \dots$  are single and  $0 < |a_1| < |a_2| < |a_3| < \dots, \lim_{n \rightarrow \infty} |a_n| = \infty$ , then  $f(z) = f(0)e^{zf'(0)/f(0)} \cdot \prod_{k=1}^{\infty} \left[ \left(1 - z/a_k\right) e^{z/a_k} \right]$ .

**Eg. Show that  $\cos(z) = \left[1 - \frac{z^2}{(\pi/2)^2}\right] \left[1 - \frac{z^2}{(3\pi/2)^2}\right] \left[1 - \frac{z^2}{(5\pi/2)^2}\right] \dots$ .**

(Proof) Zeros of  $\cos(z) = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

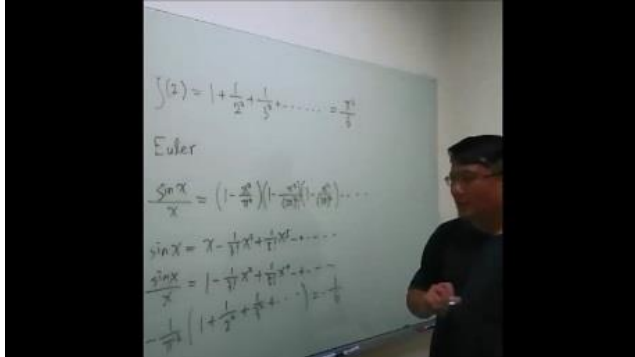
$$\cos(0) = 1, \quad z[\cos(z)]'/\cos(z)|_{z=0} = 0, \quad e^{z/a_k} \cdot e^{z/(-a_k)} = 1$$

$$\therefore \cos(z) = \left[1 - \frac{z^2}{(\pi/2)^2}\right] \cdot \left[1 - \frac{z^2}{(3\pi/2)^2}\right] \dots$$

**Note:**  $\sin(z) = z \left[1 - \frac{z^2}{\pi^2}\right] \cdot \left[1 - \frac{z^2}{(2\pi)^2}\right] \cdot \left[1 - \frac{z^2}{(3\pi)^2}\right] \dots$

## Appendices

**1. Apply Weierstrass' theorem to prove**  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$ .



**2. A weird proof of**  $1+2+3+4+5+6+\dots = \frac{-1}{12}$

$$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

$$S_1 + S_1 = (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots)$$

$$+ (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots) = 1, \therefore S_1 = \frac{1}{2}$$

$$S_2 = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \dots$$

$$S_2 + S_2 = (1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - \dots)$$

$$+ (1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \dots) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots = \frac{1}{2}, \therefore S_2 = \frac{1}{4}$$

$$S = 1 + 2 + 3 + 4 + 5 + 6 + \dots$$

$$S - S_2 = (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \dots)$$

$$- (1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \dots) = 4 + 8 + 12 + 16 + \dots = 4S, \therefore S = -S_2/3 = \frac{-1}{12}$$