Chapter 3 Complex Sequences and Series

3-1 Sequence and Series

Sequence $\{z_n\}$: $z_0, z_1, z_2, z_3, \ldots, z_n, \ldots$

Series: $\sum_{n=0} z_n = z_0 + z_1 + z_2 + \cdots + z_n + \cdots$

Absolute convergence: $\sum_{n=0}^{\infty} |z_n|$ converges. (In this case, $\sum_{n=0}^{\infty} z_n$ converges also.)

Eg. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ **is an absolutely convergent series because** $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{(n+1)^2}\right| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ **and** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots$ **are both convergent.**

Conditional convergence: $\sum_{n=0}^{\infty} z_n$ converges but $\sum_{n=0}^{\infty} |z_n|$ diverges.

Eg. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ **is conditionally convergent because** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$ **is convergent but** $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2n+1}\right| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ **is divergent.**

Radius of convergence for $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ **:** There exists R (possibly $R=\infty$) such that the series converges absolutely if $|z-z_0|<R$ and diverges if $|z-z_0|>R$.

(Proof) $\frac{|a_{n+1}(z-z_0)^{n+1}|}{|a_n(z-z_0)^n|} < 1 \Rightarrow |z-z_0| < \frac{|a_n|}{|a_{n+1}|}$

Define $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, so the proof is complete.

Eg. Find the radius of convergence for $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z-i)^n$**.**

(Sol.) $\left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}(z-i)^{n+1}}{\frac{n^n}{n!}(z-i)^n} \right| = \left| \left(\frac{n+1}{n}\right)^n (z-i) \right| < 1$

$$|z-i| < \left(1+\frac{1}{n}\right)^{-n} \to e^{-1} \text{ as } n\to\infty, \quad \therefore\ R=e^{-1}$$

3-2 Complex Taylor's Series & Laurent's Series

Taylor's series: $f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)(z-z_0)^n$. In case z_0=0, it is called Maclaurin's series.

Eg. Find the Maclaurin's expansion of $f(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-w^2}dw$.

(Sol.) $e^{-w^2}=1-w^2+\frac{(-w^2)^2}{2!}+\frac{(-w^2)^3}{3!}+\frac{(-w^2)^4}{4!}+\frac{(-w^2)^5}{5!}+\cdots$

$$=1-w^2+\frac{w^4}{2!}-\frac{w^6}{3!}+\frac{w^8}{4!}-\frac{w^{10}}{5!}+\cdots+(-1)^n\frac{w^{2n}}{n!}+$$

$$\int_0^z e^{-w^2}dw=z-\frac{z^3}{3}+\frac{z^5}{5\cdot 2!}-+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)n!}+\cdots$$

$$f(z)=\frac{2}{\sqrt{\pi}}\left(z-\frac{z^3}{3}+\frac{z^5}{5\cdot 2!}-+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)n!}+\cdots\right)$$

Laurent's series:

$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$,

where $a_n=\frac{1}{2\pi i}\oint_c\frac{f(w)}{(w-z_0)^{n+1}}dw$, and C: $|z-z_0|=\rho$, $r_1<\rho<r_2$

Eg. Find the Laurent's series of $f(z)=\frac{z-\sin z}{z^3}$ about z_0=0, indicate the type of singularity and the region of convergence of the series. [清大電研]

(Sol.) $f(z)=\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-\cdots\right)\right]$

$$=\frac{1}{z^3}\left[\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-\frac{z^9}{9!}+-\cdots\right]=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\frac{z^6}{9!}+-\cdots$$

$\therefore$ $z_0=0$ is a removable singularity.

$$f(z)=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\frac{z^6}{9!}+-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot z^{2n-2}}{(2n+1)!}=\sum_{n=1}^{\infty}a_n$$

$$\frac{|a_{n+1}|}{|a_n|}=\left|\frac{(-1)^n\cdot z^{2n}}{(2n+3)!}\right|\Big/\left|\frac{(-1)^{n-1}\cdot z^{2n-2}}{(2n+1)!}\right|<1$$

$z^2<(2n+3)(2n+2)\Rightarrow z<R\to\infty$

Basic formulae:

$$\frac{1}{1\pm r}\underset{|r|<1}{\overset{\uparrow}{=}}1\mp r+r^2\mp r^3+r^4\mp r^5+r^6\mp r^7+r^8\mp+\cdots$$

$$\frac{1}{a-b}=\begin{cases}\dfrac{\frac{1}{a}}{1-\frac{b}{a}}=\dfrac{1}{a}\cdot\left[1+\left(\dfrac{b}{a}\right)+\left(\dfrac{b}{a}\right)^2+\left(\dfrac{b}{a}\right)^3+\left(\dfrac{b}{a}\right)^4+\ldots\right], & |a|>|b| \\ \dfrac{-\frac{1}{b}}{1-\frac{a}{b}}=\dfrac{-1}{b}\cdot\left[1+\left(\dfrac{a}{b}\right)+\left(\dfrac{a}{b}\right)^2+\left(\dfrac{a}{b}\right)^3+\left(\dfrac{a}{b}\right)^4+\ldots\right], & |a|<|b|\end{cases}$$

$$\frac{1}{a+b}=\begin{cases}\dfrac{\frac{1}{a}}{1+\frac{b}{a}}=\dfrac{1}{a}\cdot\left[1-\left(\dfrac{b}{a}\right)+\left(\dfrac{b}{a}\right)^2-\left(\dfrac{b}{a}\right)^3+\left(\dfrac{b}{a}\right)^4-+\ldots\right], & |a|>|b| \\ \dfrac{\frac{1}{b}}{1+\frac{a}{b}}=\dfrac{1}{b}\cdot\left[1-\left(\dfrac{a}{b}\right)+\left(\dfrac{a}{b}\right)^2-\left(\dfrac{a}{b}\right)^3+\left(\dfrac{a}{b}\right)^4-+\ldots\right], & |a|<|b|\end{cases}$$

Eg. Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ by Laurent's series in case: (a) $|z|<1$, (b) $1<|z|<2$, (c) $|z|>2$, (d) $|z-1|>1$, and (e) $0<|z-2|<1$.

(Sol.) $f(z)=\dfrac{z}{(z-1)(2-z)}=-\dfrac{z}{(z-1)(z-2)}=\dfrac{1}{z-1}-\dfrac{2}{z-2}$

(a) $|z|<1\Rightarrow\ 1>\dfrac{|z|}{2}$

$$\Rightarrow\begin{cases}\dfrac{1}{z-1}=\dfrac{-1}{1-z}=(-1)(1+z+z^2+z^3+\cdots) \\ \quad=-1-z-z^2-z^3-z^4-\cdots \\ \dfrac{-2}{z-2}=\dfrac{2}{2-z}=\dfrac{1}{1-\frac{z}{2}}=1+\dfrac{z}{2}+\left(\dfrac{z}{2}\right)^2+\left(\dfrac{z}{2}\right)^3+\cdots\end{cases},\ \therefore\ f(z)=-\frac{1}{2}z-\frac{3}{4}z^2-\frac{7}{8}z^3-\frac{15}{16}z^4-\cdots$$

(b) $1<|z|<2\Rightarrow 1>\dfrac{1}{|z|},\ \ 1>\dfrac{|z|}{2}$

$$\Rightarrow\begin{cases}\dfrac{1}{z-1}=\dfrac{1}{z\left(1-\frac{1}{z}\right)}=\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\cdots\cdots\right) \\ \quad=\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\cdots \\ \dfrac{-2}{z-2}=\dfrac{1}{1-\frac{z}{2}}=1+\dfrac{z}{2}+\left(\dfrac{z}{2}\right)^2+\left(\dfrac{z}{2}\right)^3+\cdots\end{cases},\ \therefore\ f(z)=\cdots+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+\frac{z}{2}+\frac{z^2}{4}+\cdots$$

(c) $|z|>2 \Rightarrow \dfrac{2}{|z|}<1 \Rightarrow 1>\dfrac{1}{|z|}$

$$\Rightarrow \begin{cases} \dfrac{1}{z-1}=\dfrac{1}{z\left(1-\frac{1}{z}\right)}=\dfrac{1}{z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right) \\ \quad =\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}+\dfrac{1}{z^4}+\cdots \\ \dfrac{-2}{z-2}=-\dfrac{2}{z}\cdot\dfrac{1}{1-\frac{2}{z}}=-\dfrac{2}{z}\left(1+\dfrac{2}{z}+\left(\dfrac{2}{z}\right)^2+\left(\dfrac{2}{z}\right)^3+\cdots\right) \\ \quad =-\dfrac{2}{z}-\dfrac{4}{z^2}-\dfrac{8}{z^3}-\cdots \end{cases} , \therefore \quad f(z)=-\frac{1}{z}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}-\cdots$$

(d) $|z-1|>1 \Rightarrow \dfrac{1}{|z-1|}<1$

$$\frac{-2}{z-2}=\frac{-2}{(z-1)-1}=-\frac{2}{z-1}\cdot\frac{1}{1-\frac{1}{z-1}}=-\frac{2}{z-1}\cdot\left[1+(z-1)^{-1}+(z-1)^{-2}+\cdots\right]=-\frac{2}{z-1}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}-\cdots$$

$$\therefore \quad f(z)=\frac{1}{z-1}-\frac{2}{z-2}=\frac{-1}{z-1}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}-\frac{2}{(z-1)^4}-\cdots$$

(e) $0<|z-2|<1 \Rightarrow \dfrac{1}{z-1}=\dfrac{1}{1+(z-2)}=1-(z-2)+(z-2)^2-(z-2)^3+-\cdots$

$$\therefore \quad f(z)=\frac{1}{z-1}-\frac{2}{z-2}=-\frac{2}{z-2}+1-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4-+\cdots$$

Eg. Find the Laurent's series of $f(z)=\dfrac{2}{z^2-1}$ for (a) $|z-1|<1$ and (b) $|z-1|>1$.

(Sol.) $f(z)=\dfrac{2}{z^2-1}=\dfrac{1}{z-1}-\dfrac{1}{z+1}=\dfrac{1}{z-1}+\dfrac{-1}{2+(z-1)}$

(a) $|z-1|<1 \Rightarrow \dfrac{|z-1|}{2}<1$

$$\frac{-1}{2+(z-1)}=\frac{\frac{-1}{2}}{1+\frac{z-1}{2}}=-(\frac{1}{2})\cdot[1-\frac{z-1}{2}+(\frac{z-1}{2})^2-(\frac{z-1}{2})^3+(\frac{z-1}{2})^4-+\cdots]$$

$$f(z)=\frac{1}{z-1}-(\frac{1}{2})\cdot[1-\frac{z-1}{2}+(\frac{z-1}{2})^2-(\frac{z-1}{2})^3+(\frac{z-1}{2})^4-+\cdots]$$

(b) $|z-1|>1 \Rightarrow \begin{cases} |z-1|>2 \\ 2>|z-1|>1 \end{cases}$

Case 1: If $|z-1|>2 \Rightarrow 1>\dfrac{2}{|z-1|}$,

$$\frac{-1}{2+(z-1)}=\frac{\frac{-1}{(z-1)}}{1+\frac{2}{(z-1)}}=-(\frac{1}{z-1})\cdot[1-\frac{2}{z-1}+(\frac{2}{z-1})^2-(\frac{2}{z-1})^3+(\frac{2}{z-1})^4-+\cdots]$$

$$f(z)=\frac{1}{z-1}-(\frac{1}{z-1})\cdot[1-\frac{2}{z-1}+(\frac{2}{z-1})^2-(\frac{2}{z-1})^3+(\frac{2}{z-1})^4-+\cdots]$$

$$=\frac{1}{z-1}\cdot[\frac{2}{z-1}-(\frac{2}{z-1})^2+(\frac{2}{z-1})^3-(\frac{2}{z-1})^4+-\cdots]$$

Case 2: If $2>|z-1|>1 \Rightarrow 1>\dfrac{|z-1|}{2}$

$$\frac{-1}{2+(z-1)}=\frac{\frac{-1}{2}}{1+\frac{z-1}{2}}=-(\frac{1}{2})\cdot[1-\frac{z-1}{2}+(\frac{z-1}{2})^2-(\frac{z-1}{2})^3+(\frac{z-1}{2})^4+-\cdots]$$

$$f(z)=\frac{1}{z-1}-(\frac{1}{2})\cdot[1-\frac{z-1}{2}+(\frac{z-1}{2})^2-(\frac{z-1}{2})^3+(\frac{z-1}{2})^4+-\cdots]$$

Eg. Find the Laurent's series of $f(z)=\dfrac{1}{z^2-3z+2}$ for $1<|z|<2$. [1991 台大機研]

(Sol.) $\dfrac{1}{z^2-3z+2}=\dfrac{-1}{z-1}+\dfrac{1}{z-2}$

$$1<|z|<2 \Rightarrow 1>\frac{1}{|z|},\ \frac{-1}{z-1}=\frac{-1}{z}\cdot\frac{1}{1-\frac{1}{z}}=\frac{-1}{z}\cdot\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$\Rightarrow 1>\frac{|z|}{2},\quad \frac{1}{z-2}=\frac{-1}{2}\cdot\frac{1}{1-\frac{z}{2}}=\frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right]$$

$$\Rightarrow f(z)=-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

Eg. (a) $\dfrac{1}{(z-1)(z-3)}$ for $1<|z|<3$ about $z_0=0$, (b) $\dfrac{e^z}{(z-1)^2}$ for $|z-1|>0$ about $z_0=1$. Find their respective Laurent's series. [1991 交大電信]

(Sol.) (a) $\dfrac{1}{(z-1)(z-3)}=\dfrac{-\frac{1}{2}}{z-1}+\dfrac{\frac{1}{2}}{z-3}$

$$1<|z|<3 \Rightarrow \frac{1}{|z|}<1,\ \frac{-\frac{1}{2}}{z-1}=\frac{-\frac{1}{2}}{z}\cdot\frac{1}{(1-\frac{1}{z})}=\frac{-\frac{1}{2}}{z}\cdot\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$1<|z|<3 \Rightarrow 1>\frac{|z|}{3},\quad \frac{\frac{1}{2}}{z-3}=\frac{-\frac{1}{2}}{3}\cdot\frac{1}{(1-\frac{z}{3})}=\frac{-1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

$$\Rightarrow f(z)=-\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)$$

(b) $$\frac{e^z}{(z-1)^2}=\frac{e\cdot e^{z-1}}{(z-1)^2}=\frac{e}{(z-1)^2}\cdot[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\ldots]$$

$$=e\cdot[\frac{1}{(z-1)^2}+\frac{1}{(z-1)}+\frac{1}{2!}+\frac{(z-1)}{3!}+\ldots]$$

Eg. Find the Laurent's series of $f(z)=\dfrac{5z+2j}{z^2+jz}$ for $1<|z-j|<2$. [2013 中山電研]

(Sol.) $\dfrac{5z+2j}{z^2+jz}=\dfrac{3}{z+j}+\dfrac{2}{z}$

$|z-j|<2 \Rightarrow |z-j|<|2j| \Rightarrow \dfrac{|z-j|}{|2j|}<1$,

$$\frac{3}{z+j}=\frac{3}{2j+(z-j)}=\frac{3}{2j}\cdot\frac{1}{1+(\frac{z-j}{2j})}=\frac{3}{2j}\cdot[1-\frac{z-j}{2j}+(\frac{z-j}{2j})^2-(\frac{z-j}{2j})^3+-\cdots]$$

$1<|z-j| \Rightarrow \dfrac{|j|}{|z-j|}<1$

$$\frac{2}{z}=\frac{2}{j+(z-j)}=\frac{2}{(z-j)}\cdot\frac{1}{1+(\frac{j}{z-j})}=\frac{2}{(z-j)}\cdot[1-\frac{j}{z-j}+(\frac{j}{z-j})^2-(\frac{j}{z-j})^3+-\cdots]$$

$$\frac{5z+2j}{z^2+jz}=\frac{3}{2j}\cdot[1-\frac{z-j}{2j}+(\frac{z-j}{2j})^2-(\frac{z-j}{2j})^3+-\cdots]+\frac{2}{(z-j)}\cdot[1-\frac{j}{z-j}+(\frac{j}{z-j})^2-(\frac{j}{z-j})^3+-\cdots]$$

Eg. (a) Find the Laurent's series of $f(z)=\dfrac{\sin(z)\cdot\cos(2z)}{z^3}$ about $z_0=0$. (b) Evaluate $\oint_c f(z)dz$, where C is a circle $|z-1|=2$. [2012 中央電研固態組、生醫電子組]

(Sol.) (a) $f(z)=\dfrac{\sin(z)\cdot\cos(2z)}{z^3}=\dfrac{\sin(3z)+\sin(-z)}{2z^3}=\dfrac{\sin(3z)-\sin(z)}{2z^3}$

$$=\frac{1}{2z^3}\left[\left(3z-\frac{(3z)^3}{3!}+\frac{(3z)^5}{5!}-\frac{(3z)^7}{7!}+-\cdots\right)-(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-\cdots)\right]$$

$$=\frac{1}{2z^3}\left[2z-\frac{26z^3}{3!}+\frac{242z^5}{5!}-+-\cdots\right]=\frac{1}{z^2}-\frac{13}{3!}+\frac{121z^2}{5!}-+-\cdots$$

(b) There is only one pole 0 within $|z-1|=2$ and the residue at $z_0=0$ is 0.

$\therefore \oint_c f(z)dz=2\pi i\cdot 0=0$.

3-3 Summation of Series by the Residue Theorem

Theorem Let $z_1, \ldots, z_m$ be the poles of $f(z)$, then $\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{j=1}^{m} \operatorname{Res}_{z_j}[\cot(\pi z)\cdot f(z)]$ **and** $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{j=1}^{m} \operatorname{Res}_{z_j}[\csc(\pi z)\cdot f(z)]$.

Eg. Evaluate $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots$. **[1991 清大電研]**

(Sol.) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = (-1)\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n-1} = \sum_{n=-\infty}^{0}\frac{(-1)^n}{2n-1} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = \left(\cdots+\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1\right) + \left(-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-+\cdots\right)$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n-1}$$

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n-1} = \frac{\pi}{2}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\csc(\pi z)}{2z-1}\right]$$

$$= \frac{\pi}{4}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\csc(\pi z)}{z-\frac{1}{2}}\right] = \frac{\pi}{4}\lim_{z\to\frac{1}{2}}\left[\frac{\csc \pi z}{z-\frac{1}{2}}\times\left(z-\frac{1}{2}\right)\right] = \frac{\pi}{4}$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots = \frac{\pi}{4}$$

Eg. Evaluate $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$.

(Sol.) $\sum_{n=-\infty}^{\infty}\frac{1}{(2n-1)^2} = \sum_{n=-\infty}^{0}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

$$= \left(\cdots+\frac{1}{5^2}+\frac{1}{3^2}+1\right) + \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) = 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\therefore\ 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{(2n-1)^2} = \frac{-\pi}{2}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\cot(\pi z)}{(2z-1)^2}\right]$$

$$= \frac{-\pi}{8}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\cot(\pi z)}{(z-\frac{1}{2})^2}\right] = \frac{-\pi}{8}\lim_{z\to\frac{1}{2}}\frac{1}{1!}\frac{d}{dz}\left[\frac{\cot(\pi z)}{(z-\frac{1}{2})^2}\times\left(z-\frac{1}{2}\right)^2\right]$$

$$= \frac{-\pi}{8}\cdot(-\pi)\cdot\csc^2\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8},\ \therefore\ \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \frac{\pi^2}{8}$$

Eg. Evaluate $1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+-\cdots$.

(Sol.) $1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+-\cdots=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2}=\sum_{n=-\infty}^{-1}\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\left(\cdots+\frac{1}{4^2}-\frac{1}{3^2}+\frac{1}{2^2}-1\right)$$

$$+\left(-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-+\cdots\right)=2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi}{2}\operatorname*{Res}_{0}\left[\frac{\csc(\pi z)}{z^2}\right]$$

$$\frac{\csc(\pi z)}{z^2}=\frac{1}{z^2\sin(\pi z)}=\frac{1}{z^2\left(\pi z-\dfrac{\pi^3z^3}{3!}+-\cdots\right)}=\frac{1}{\pi}\cdot\frac{1}{z^3}+\frac{\pi}{6}\cdot\frac{1}{z}+\cdots$$

$$\therefore\ \operatorname*{Res}_{0}\left(\frac{\csc(\pi z)}{z^2}\right)=\frac{\pi}{6}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\left(\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{6}\right)=\frac{\pi^2}{12}$$

Eg. Evaluate $1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots$.

(Sol.) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2}=-\frac{\pi}{2}\operatorname*{Res}_{0}\left(\frac{\cot(\pi z)}{z^2}\right)$

$$\frac{\cot(\pi z)}{z^2}=\frac{\cos(\pi z)}{z^2\sin(\pi z)}=\frac{\left(1-\dfrac{\pi^2z^2}{2!}+\dfrac{\pi^4z^4}{4!}-+\cdots\right)}{z^2\left(\pi z-\dfrac{\pi^3z^3}{3!}+-\cdots\right)}=\frac{1}{\pi}\cdot\frac{1}{z^3}-\frac{\pi}{3}\cdot\frac{1}{z}+\cdots$$

$$\therefore\ \operatorname*{Res}_{0}\left(\frac{\cot(\pi z)}{z^2}\right)=-\frac{\pi}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(-\frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{3}\right)=\frac{\pi^2}{6}$$

Eg. Evaluate $1+\dfrac{1}{2^4}+\dfrac{1}{3^4}+\cdots$.

(Sol.) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^4}=-\frac{\pi}{2}\operatorname*{Res}_{0}\left(\frac{\cot(\pi z)}{z^4}\right)$

$$\frac{\cot(\pi z)}{z^4}=\frac{\cos(\pi z)}{z^4\sin(\pi z)}=\frac{\left(1-\dfrac{\pi^2z^2}{2!}+\dfrac{\pi^4z^4}{4!}-+\cdots\right)}{z^4\left(\pi z-\dfrac{\pi^3z^3}{3!}+\dfrac{\pi^5z^5}{5!}-+\cdots\right)}=\frac{1}{\pi}\cdot\frac{1}{z^5}-\frac{\pi}{3}\cdot\frac{1}{z^3}-\frac{\pi^3}{45}\cdot\frac{1}{z}\cdots$$

$$\therefore\ \operatorname*{Res}_{0}\left(\frac{\cot(\pi z)}{z^4}\right)=-\frac{\pi^3}{45}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(-\frac{\pi}{2}\right)\cdot\left(-\frac{\pi^3}{45}\right)=\frac{\pi^4}{90}$$

Eg. Evaluate (a) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=?$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=?$

(Sol.) (a) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=-\pi\cdot\left[\operatorname*{Re}_{ai}s\left(\frac{\cot(\pi z)}{z^2+a^2}\right)+\operatorname*{Re}_{-ai}s\left(\frac{\cot(\pi z)}{z^2+a^2}\right)\right]$

$$=-\pi\cdot\left[\frac{\cot(\pi ai)}{2ai}+\frac{\cot(-\pi ai)}{-2ai}\right]=-\frac{\pi\cot(\pi ai)}{ai}=\frac{\pi}{a}\coth(a\pi).$$

(b) $\displaystyle\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}+\frac{1}{a^2}+\sum_{n=-\infty}^{-1}\frac{1}{n^2+a^2}=2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}+\frac{1}{a^2}=\frac{\pi}{a}\coth(a\pi)$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{1}{2}\left[\frac{\pi}{a}\coth(a\pi)-\frac{1}{a^2}\right]=\frac{a\pi\coth(a\pi)-1}{2a^2}.$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{2}\lim_{a\to0}\frac{a\pi\coth(a\pi)-1}{a^2}=\frac{1}{2}\lim_{a\to0}\frac{a\pi\cdot\dfrac{e^{a\pi}+e^{-a\pi}}{e^{a\pi}-e^{-a\pi}}-1}{a^2}=\frac{1}{2}\lim_{a\to0}\frac{a\pi\cdot\dfrac{(1-\frac{a^2\pi^2}{2}+-\cdots)}{(a\pi-\frac{a^3\pi^3}{6}+-\cdots)}-1}{a^2}$$

$$=\frac{1}{2}\lim_{a\to0}\frac{(1+\frac{a^2\pi^2}{3}+\cdots)-1}{a^2}=\frac{\pi^2}{6}$$

Eg. Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+1}=?$ **[1999 台科大電研]**

(Sol.) a=1,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1^2}=\frac{1}{2}[\pi\coth(\pi)-1],\quad \sum_{n=0}^{\infty}\frac{1}{n^2+1}=1+\sum_{n=1}^{\infty}\frac{1}{n^2+1^2}=\frac{1}{2}[\pi\coth(\pi)+1]$$

3-4 Infinite products

Theorem If $\displaystyle\prod_{n=1}^{\infty}a_n\ (\neq 0)$ **converges, then** $\displaystyle\lim_{n\to\infty}a_n=1$.

(Proof) Let $\displaystyle p_N=\prod_{n=1}^{N}a_n$, $\displaystyle 1=\lim_{N\to\infty}p_N\Big/\lim_{N\to\infty}p_{N-1}=\lim_{N\to\infty}\left(\prod_{n=1}^{N}a_n\Big/\prod_{n=1}^{N-1}a_n\right)=\lim_{N\to\infty}a_N$

Theorem $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ **converges if and only if** $\displaystyle\sum_{n=1}^{\infty}\log(1+a_n)$ **converges.**

Theorem $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$ **is absolutely convergent if and only if** $\displaystyle\sum_{n=1}^{\infty}a_n$ **is absolutely convergent.**

Eg. Determine the domain D of convergence for $\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2}\cdot z^n\right]$.

(Sol.) $\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2} z^n\right]$ diverges if $\lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{n^2} z^n\right|>1$

$$\left(1+\frac{1}{n}\right)^{n^2}=e^{n^2\ln\left(1+\frac{1}{n}\right)}=e^{n\ln\left(1+\frac{1}{n}\right)^n}\to e^{n\ln e}=e^n$$

$$\lim_{n\to\infty}\left|e^n\cdot z^n\right|>1\Rightarrow |z|>e^{-1},\ \prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2} z^n\right] \text{ diverges}$$

$\therefore\ D=\{z:|z|<e^{-1}\}$ for convergence.

Weierstrass' theorem Let $f(z)$ be analytic. All zeros a_1, a_2, a_3, …, a_n, … are single and $0<|a_1|<|a_2|<|a_3|<$ …, $\lim_{n\to\infty}|a_n|=\infty$, then $f(z)=$ $f(0)e^{zf'(0)/f(0)}\cdot\prod_{k=1}^{\infty}\left[(1-z/a_k)e^{z/a_k}\right]$.

Eg. Show that $\cos(z)=\left[1-\frac{z^2}{(\pi/2)^2}\right]\left[1-\frac{z^2}{(3\pi/2)^2}\right]\left[1-\frac{z^2}{(5\pi/2)^2}\right]\cdots$.

(Proof) Zeros of $\cos(z)=\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\cdots$

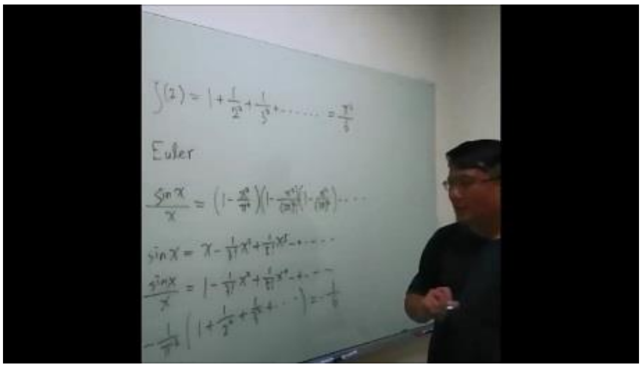
$$\cos(0)=1,\ \ z[\cos(z)]'/\cos(z)_{|z=0}=0,\ \ e^{z/a_k}\cdot e^{z/(-a_k)}=1$$

$$\therefore\ \cos(z)=\left[1-\frac{z^2}{(\pi/2)^2}\right]\cdot\left[1-\frac{z^2}{(3\pi/2)^2}\right]\cdots$$

Note: $\sin(z)=z\left[1-\frac{z^2}{\pi^2}\right]\cdot\left[1-\frac{z^2}{(2\pi)^2}\right]\cdot\left[1-\frac{z^2}{(3\pi)^2}\right]\cdots$

Appendices

1. Apply Weierstrass' theorem to prove $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$.



2. A weird proof of 1+2+3+4+5+6+……=$\frac{-1}{12}$

S_1=1-1+1-1+1-1+1-1+-……

S_1+S_1=(1-1+1-1+1-1+1-1+1-……)

+ (1-1+1-1+1-1+1-1+-……)=1, ∴ $S_1=\frac{1}{2}$

S_2=1-2+3-4+5-6+7-8+-……

S_2+S_2=(1-2+3-4+5-6+7-8+9-……)

+ (1- 2+3-4+5-6+7-8+-……)= 1-1+1-1+1-1+1-1+-……=$\frac{1}{2}$, ∴ $S_2=\frac{1}{4}$

S=1+2+3+4+5+6+……

$S-S_2$=(1+2+3+4+5+6+7+8……)

-(1- 2+3- 4+5- 6+7-8+-……)=4+8+12+16+……=4S, ∴ $S=-S_2/3=\frac{-1}{12}$