Chapter 4 Conformal Mapping

4-1 Introduction

Complex image transformation: *w=f(z)=f(x+iy)=u+iv*

$$|det(\text{Jacobian matrix})| = \left| \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \right| = |f'(z)|^2$$

Area magnification factor: $|f'(z)|^2$, **Rotation angle at z_0:** $\arg[f'(z_0)]$
Critical point z_c: z_c if $f'(z_c)=0$

Theorem **If $f(z)$ is analytic in R and $f'(z_0)\neq0$, then $w= f(z)$ is a conformal mapping.**

(Proof) $f(z)$ is analytic $\Leftrightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\left| \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \right| = \left| \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{bmatrix} \right| = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 = \left|\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right|^2 = |f'(z)|^2 \quad \text{is called}$$

the area magnification factor

$$C: z=z(t), \ C': w=w(t)=f(z), \ \frac{dw}{dt} = \frac{dw}{dz} \cdot \frac{dz}{dt} = f'(z)\frac{dz}{dt}$$

Let $\dfrac{dw}{dt} = \rho e^{i\phi}$, $\dfrac{dz}{dt} = re^{i\theta}$, $f'(z)=Re^{i\Psi}$, then

$$\rho e^{i\phi} = Rre^{i(\theta+\psi)} \Rightarrow \phi = \theta + \psi = \theta + \arg(f'(z))$$

$$c_1 \to c_1', \ c_2 \to c_2' \ \text{and} \ \phi_1 = \theta_1 + \arg(f'(z_0)), \ \phi_2 = \theta_2 + \arg(f'(z_0))$$

$$\Rightarrow \phi_1 - \phi_2 = \theta_1 - \theta_2$$

4-2 Conformal Mapping

Linear transformation: *w=Az+B, A≠0*
This transformation can magnify a closed region or rotate a straight line.

Reciprocal transformation: *w=k/z*
This conformal mapping is from the exterior/interior region of a circle to the interior/exterior region of a circle.

Eg. Map |z|<1 to |w|>4. (Sol.) $w = \dfrac{4}{z}$

Linear fractional transformation: $w=\dfrac{Az+B}{Cz+D}$

This conformal mapping is from a circle/line to another circle/line. For $z_1 \rightarrow w_1$, $z_2 \rightarrow w_2$, $z_3 \rightarrow w_3$, then $\dfrac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \Rightarrow w=\dfrac{Az+B}{Cz+D}$

Eg. Find a linear fractional transformation $w=f(z)$ with $3 \rightarrow i$, $1\text{-}i \rightarrow 4$, and $2\text{-}i \rightarrow 6+2i$.

(Sol.) $\dfrac{w-i}{w-(6+2i)} \cdot \dfrac{-2-2i}{4-i} = \dfrac{z-3}{z-2+i} \cdot \dfrac{-1}{-2-i} \Rightarrow w = \dfrac{(20+4i)z-(16i+68)}{(6+5i)z-(22+7i)}$
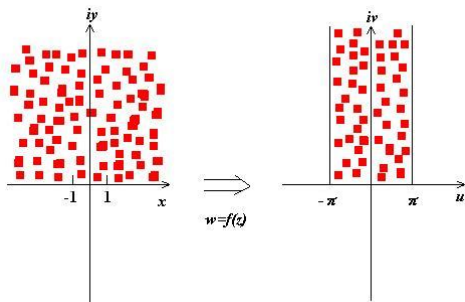
Eg. Find a linear fractional transformation $w=f(z)$ with $i \rightarrow 4i$, $1 \rightarrow 3\text{-}i$, and $2+i \rightarrow \infty$.

(Sol.) $\dfrac{w-4i}{w-\infty} \cdot \dfrac{3-i-\infty}{3-5i} = \dfrac{z-i}{z-2-i} \cdot \dfrac{-1-i}{1-i} \Rightarrow w = \dfrac{-1+3i+(5-i)z}{2+i-z}$

Eg. Find a fixed point (invariant) for $w=\dfrac{2z-5}{z+4}$.

(Sol.) $w = \dfrac{2z-5}{z+4} = z \Rightarrow z^2+2z+5=0 \Rightarrow z=-1\pm 2i$

Schwartz-Christoffel transformation: This conformal mapping is from the upper half plane to a polygon. For $x_1 \rightarrow w_1$ (inner angle θ_1), $x_2 \rightarrow w_2$ (inner angle θ_2), ..., $x_n \rightarrow w_n$ (inner angle θ_n), then $w=A\displaystyle\int_0^z (z-x_1)^{\frac{\theta_1}{\pi}-1}(z-x_2)^{\frac{\theta_2}{\pi}-1} \cdots (z-x_n)^{\frac{\theta_n}{\pi}-1}dz+B$.

Eg. Find a conformal mapping $w=f(z)$ from the upper half plane to an infinitely deep rectangular well and $-\infty \rightarrow -\pi+i\infty$, $-1 \rightarrow -\pi$, $1 \rightarrow \pi$, and $\infty \rightarrow \pi+i\infty$.

(Sol.) $w = A\displaystyle\int (z+1)^{\frac{(\pi/2)}{\pi}-1}(z-1)^{\frac{(\pi/2)}{\pi}-1}dz+B$

$= A\displaystyle\int \dfrac{dz}{\sqrt{z^2-1}}+B = \dfrac{A}{i}\int \dfrac{dz}{\sqrt{1-z^2}}+B = A'\sin^{-1}(z)+B$

$-1 \rightarrow -\pi,\ 1 \rightarrow \pi \Rightarrow \begin{cases} \pi = A'\dfrac{\pi}{2}+B \\ -\pi = -A'\dfrac{\pi}{2}+B \end{cases} \Rightarrow \begin{cases} A'=2 \\ B=0 \end{cases}, \therefore w=2\sin^{-1}(z)$

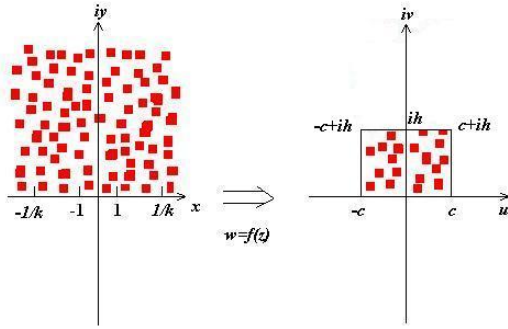**Eg. Find a conformal mapping $w=f(z)$ from the upper half plane to a rectangular region and $-\infty \to ih$, $-1/k \to -c+ih$, $-1 \to -c$, $0 \to 0$, $1 \to c$, $1/k \to c+ih$, and $\infty \to ih$. ($k<1$)**

(Sol.) $w = A\int \left(z+\dfrac{1}{k}\right)^{\frac{(\pi/2)}{\pi}-1} \cdot (z+1)^{\frac{(\pi/2)}{\pi}-1} \cdot (z-0)^{\frac{\pi}{\pi}-1} \cdot (z-1)^{\frac{(\pi/2)}{\pi}-1} \cdot \left(z-\dfrac{1}{k}\right)^{\frac{(\pi/2)}{\pi}-1} dz + B$

$$= A\int \frac{dz}{(z^2-1)^{1/2}\left(z^2-\dfrac{1}{k^2}\right)^{1/2}} + B = kA\int \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}} + B$$

$\because \ z=0 \to w=0, \ \therefore \ B=0 \Rightarrow w = C\displaystyle\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}}$

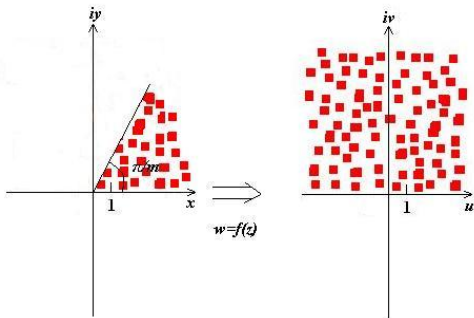**Eg. Find a conformal mapping $w=f(z)$ from a sector to the upper half plane and**

$$(\lim_{R\to\infty} \mathrm{Re}^{i\frac{\pi}{m}}) \to -\infty, \ 0 \to 0, \text{ and } 1 \to 1.$$

(Sol.) $z = f^{-1}(w) = A\int (w-0)^{\frac{(\pi/m)}{\pi}-1} \cdot (w-1)^{\frac{\pi}{\pi}-1} dw + B$

$$= A\int w^{\frac{1}{m}-1} dw + B = Amw^{\frac{1}{m}} + B = A'w^{\frac{1}{m}} + B$$

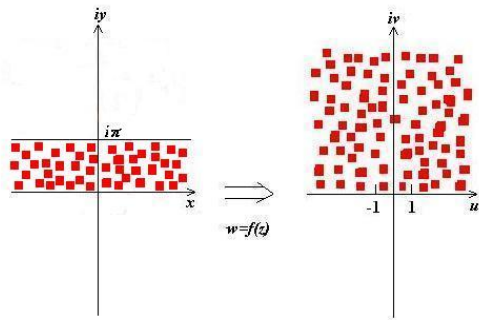$\because \ z=0 \to w=0$ and $z=1 \to w=1, \ \therefore \ A'=1, \ B=0 \Rightarrow z = w^{\frac{1}{m}} \Rightarrow w = z^m$

Eg. Find a conformal mapping $w=f(z)$ from an infinitely long belt to the upper half plane and $\infty+\pi i \to -\infty$, $\pi i \to -1$, $-\infty+\pi i \to 0^-$, $-\infty \to 0^+$, $0 \to 1$, and $\infty \to \infty$.

(Sol.) $z = f^{-1}(w) = A\int (w+1)^{\frac{\pi}{\pi}-1}(w-0^-)^{\frac{(\pi/2)}{\pi}-1}(w-0^+)^{\frac{(\pi/2)}{\pi}-1}(w-1)^{\frac{\pi}{\pi}-1}\,dw + B$

$$= A\int \frac{dw}{w} + B = A\ln(w) + B$$

$z = \pi i \to w = -1$
$z = 0 \to w = 1$ $\Rightarrow$ $\pi i = A\ln(-1) + B = A\ln(e^{i\pi}) + B = i\pi A + B$
$0 = A\ln(1) + B = B$

$\Rightarrow A=1,\ B=0 \Rightarrow z = \ln(w) \Rightarrow w = f(z) = e^z$