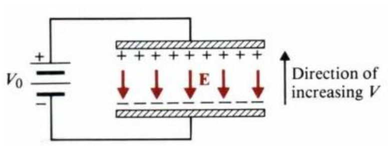
Chapter 4 Selected Topics for Circuits and Systems

4-1 Poisson's and Laplace's Equations

Poisson's equation: $\nabla \cdot \varepsilon \vec{E} = \rho \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon}$

Laplace's equation: If no charge exists, ρ=0, $\nabla^2 V = 0$

Eg. The two plates of a parallel-plate capacitor are separated by a distance d and maintained at potentials 0 and V_0. Assume negligible fringing effect at the edges, determine (a) the potential at any point between the plates, (b) the surface charge densities on the plates. [清大電研]



(Sol.) $\frac{d^2V}{dy^2} = 0 \Rightarrow V = c_1 y + c_2,\ V(y=0)=0,\ V(y=d)=V_0 \Rightarrow V = \frac{V_0}{d} y$

$\vec{E} = -\hat{y}\frac{dV}{dy} = -\hat{y}\frac{V_0}{d}, E_n = \hat{a}_n \cdot \vec{E} = \frac{\rho}{\varepsilon}$

At the lower plate: $\hat{a}_n = \hat{y},\ \rho_{sl} = -\frac{\varepsilon V_0}{d}$.

At the upper plate: $\hat{a}_n = -\hat{y},\ \rho_{ul} = \frac{\varepsilon V_0}{d}$

Eg. Show that uniqueness of electrostatic solutions.

(Proof) Let V_1 and V_2 satisfy $\nabla^2 V_1 = -\frac{\rho}{\varepsilon}$ and $\nabla^2 V_2 = -\frac{\rho}{\varepsilon}$. Define $V_d = V_1 - V_2$,

$\nabla^2 V_d = 0$

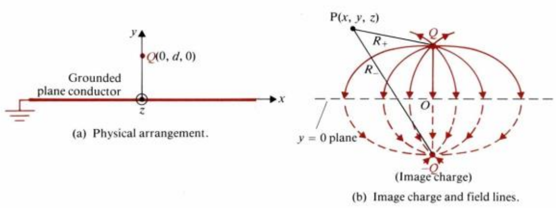
1. On the conducting boundaries, $V_d = 0 \Rightarrow V_1 = V_2$
2. Let $f = V_d$, $\vec{A} = \nabla V_d$

$$\nabla \cdot (V_d \nabla V_d) = \nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + A \cdot \nabla f = V_d \nabla^2 V_d + |\nabla V_d|^2 \Rightarrow \oiint_s (V_d \nabla V_d) \cdot \hat{a}_n ds = \iiint_{v'} |\nabla V_d|^2 dv$$

$$R \to \infty, V_d = V_1 - V_2 \propto \frac{1}{R}, \nabla V_d \propto \frac{1}{R^2}, ds \propto R^2 \Rightarrow \oiint_s (V_d \nabla V_d) \cdot \hat{a}_n ds \to 0,\ \therefore \iiint_v |\nabla V_d|^2 dv = 0 \Rightarrow$$

$V_d = 0 \Rightarrow V_1 = V_2$

Image Theorem $P(x,y,z)$ in the $y>0$ region is $V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$, where R_+ and R_- are the distances from Q and $-Q$ to the point P, respectively.

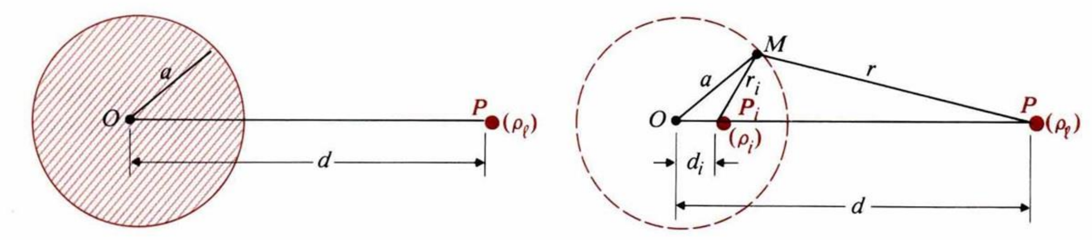


Eg. A line charge density ρ_l located at a distance d from the axis of a parallel conducting circular cylinder of radius a. Both are infinitely long. Find the image position of line charge.

(Sol.) Assume $\rho_i = -\rho_l, V = -\int_{r_0}^{r} E dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_l}{2\pi\varepsilon_0}\ell n\frac{r_o}{r}$

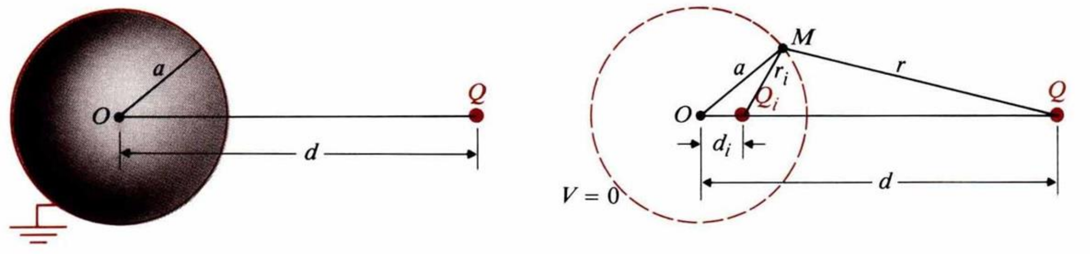
$$\Rightarrow V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ell n\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ell n\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ell n\frac{r_i}{r}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = C \Rightarrow d_i = \frac{a^2}{d}$$



(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.

Eg. A point charge Q is placed at a distance d to a conducting sphere. Find its image.



(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

(Sol.) $V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0 \Rightarrow \frac{r_i}{r} = -\frac{Q_i}{Q} = \frac{a}{d} \Rightarrow Q_i = -\frac{a}{d}Q$, $\frac{a-d_i}{d-a} = \frac{a}{d} \Rightarrow d_i = \frac{a^2}{d}$

4-2 Boundary-Value Problems in Rectangular Coordinates

$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$. Let $V(x,y,z)=X(x)Y(y)Z(z)$, $k_x^2+k_y^2+k_z^2=0$

$$\rightarrow \frac{d^2X(x)}{dx^2}+k_x^2X(x)=0,\ \frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0,\ \frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0$$

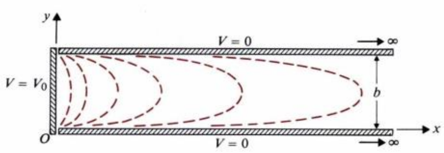
For $X(x)$, 1. $k_x^2=0$, $X(x)=A_0x+B_0$ is linear.

2. $k_x^2>0$, $X(x)=A_1\sin k_xx+B_1\cos k_xx$, $X(x=a)$ is finite, $X(x=b)$ is finite

3. $k_x^2<0$, $X(x)=A_2\sinh k_xx+B_2\cosh k_xx$, $X(\infty)$ is finite, $X(-\infty)$ is finite

Similar cases exist in $Y(y)$ and $Z(z)$.

Eg. Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance b. A third electrode perpendicular to and insulated from both is maintained at a constant potential V_0. Determine the potential distribution in the region enclosed by the electrodes. [高考]



(Sol.)

$V(x,y,z)=V(x,y)=X(x)Y(y)$

B.C.: $V(0,y)=V_0, V(x,0)=0$
$V(\infty,y)=0, V(x,b)=0$

$$\frac{d^2X(x)}{d^2x}=k_x^2X(x)\Rightarrow X(x)=D_1e^{k_xx}+D_2e^{-k_xx}=D_2e^{-k_xx}$$

$$\frac{d^2Y(y)}{dy^2}=-k_y^2Y(y)\Rightarrow Y(y)=A_1\sin k_yy$$

$$\Rightarrow V_n(x,y)=C_ne^{-k_xx}\sin k_yy\Rightarrow k_x=k_y=\frac{n\pi}{b}, n=1,2,3$$

$$\Rightarrow V_n(x,y)=C_ne^{-\frac{n\pi x}{b}}\sin\frac{n\pi y}{b}$$

$$V(0,y)=V_0=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi y}{b}$$

$$\int_0^b V_0\sin\frac{m\pi y}{b}dy=\sum_{n=1}^{\infty}C_n\int_0^b\sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}dy$$

$$=\begin{cases}\frac{2bV_0}{m\pi}, m: odd\\ 0, m: even\end{cases}=\begin{cases}\frac{C_n}{2}, m=n\\ 0, m\neq n\end{cases}$$

$$\Rightarrow C_n=\begin{cases}\frac{4V_0}{n\pi}, n: odd\\ 0, n: even\end{cases}$$

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n:odd}\frac{1}{n}e^{-\frac{n\pi x}{b}}\sin\frac{n\pi y}{b}, n=1,3,5,\cdots$$ for $x>0$, $0<y<b$.

4-3 Boundary-Value Problems in Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

1. Assume $\frac{\partial^2 V}{\partial z^2}=0$, then $V(r,\varphi)=R(r)\Phi(\varphi)$,

$\rightarrow \quad r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}-n^2R(r)=0,\quad \frac{d^2\Phi(\phi)}{d\phi^2}+n^2\Phi(\phi)=0$

$\rightarrow R(r)=A_r r^n+B_r r^{-n},\ \Phi(\varphi)=A_\varphi \sin n\varphi+B_\varphi \cos n\varphi$

$\rightarrow V_n(r,\varphi)=r^n(A\sin n\varphi+B\cos n\varphi)+r^{-n}(A'\sin n\varphi+B'\cos n\varphi),\ n\neq 0$

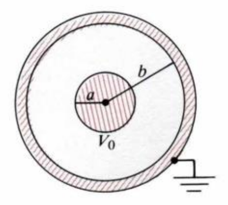
$\rightarrow V(r,\varphi)=\sum_{n=1}^{\infty}V_n(r,\phi)$

2. Assume $n=0$, $\frac{d^2\Phi(\phi)}{d\phi^2}=0\Rightarrow \Phi(\varphi)=A_0\varphi+B_0,\ R(r)=C_0\ln r+D_0$,

In the φ-independent case, $V(r)=C_1\ln r+D_1$

In the φ-dependent case, $\Phi(\varphi)=A\varphi+B,\ V(r,\varphi)=(C\ln r+D)(A\varphi+B)$

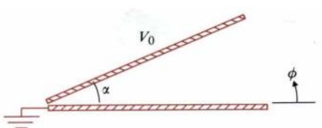
Eg. Consider a very long coaxial cable. The inner conductor has a radius *a* and is maintained at a potential V_0. The outer conductor has an inner radius *b* and is grounded. Determine the potential distribution in the space between the conductors. [電信特考]



(Sol.) $V(b)=0,\ V(a)=V_0\Rightarrow C_1\ell n(b)+C_2=0,\ C_1\ell n(a)+C_2=V_0$

$$C_1=\frac{V_0}{\ln(b/a)},C_2=-\frac{V_0\ln b}{\ln(b/a)},\ \therefore\ V(r)=\frac{V_0}{\ln(b/a)}\ln\left(\frac{r}{b}\right)$$

Eg. Two infinite insulated conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration. Determine the potential distributions for the regions: $\begin{matrix}(a)0<\phi<\alpha\\(b)\alpha<\phi<2\pi\end{matrix}$. **[中央光電所、成大電研]**



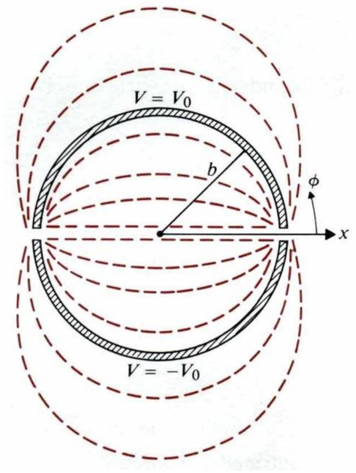
(Sol.)

(a) $V(\varphi)=A\varphi+B$, $\begin{cases}V(0)=0\Rightarrow B_0=0\\V(\alpha)=V_0=A_0\alpha\Rightarrow A_0=\frac{V_0}{\alpha}\end{cases}\Rightarrow V(\phi)=\frac{V_0}{\alpha}\phi, 0\le\phi\le\alpha$

(b) $\begin{cases}V(\alpha)=V_0=A_1\alpha+B_1\\V(2\pi)=0=2\pi A_1+B_1\end{cases}\Rightarrow A_1=-\frac{V_0}{2\pi-\alpha},B_1=\frac{2\pi V_0}{2\pi-\alpha}\Rightarrow V(\phi)=\frac{V_0}{2\pi-\alpha}(2\pi-\phi),$

$\alpha\le\phi\le 2\pi$

Eg. An infinitely long, thin, conducting circular tube of radius *b* is split in two halves. The upper half is kept at a potential *V=Vo* and the lower half at *V=-Vo*. Determine the potential distributions both inside and outside the tube.



(Sol.) $V(b,\phi)=\begin{cases}V_0, 0<\phi<\pi \\ -V_0, \pi<\varphi<2\pi\end{cases}$

(a) Inside the tube: $r<b \Rightarrow V_n(r,\phi)=A_n r^n \sin n\phi \Rightarrow V(r,\phi)=\sum_{n=1}^{\infty} A_n r^n \sin n\phi$

$r=b \Rightarrow \sum_{n=1}^{\infty} A_n b^n \sin n\phi = \begin{cases}V_0, 0<\phi<\pi \\ -V_0, \pi<\phi<2\pi\end{cases}$

$$\Rightarrow A_n=\begin{cases}\frac{4V_0}{n\pi b^n}, n: odd \\ 0, n: even\end{cases} \Rightarrow V(r,\varphi)=\frac{4V_0}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n \sin n\varphi, r<b$$

(b) Outside the tube: $r>b \Rightarrow V_n(r,\phi)=B_n r^{-n} \sin n\phi \Rightarrow V(r,\phi)=\sum_{n=1}^{\infty} B_n r^{-n} \sin n\phi$

$r=b \Rightarrow \sum_{n=1}^{\infty} B_n b^{-n} \sin n\phi = \begin{cases}V_0, 0<\phi<\pi \\ -V_0, \pi<\phi<2\pi\end{cases}$

$$\Rightarrow B_n=\begin{cases}\frac{4V_0 b^n}{n\pi}, n: odd \\ 0, n: even\end{cases} \Rightarrow V(r,\varphi)=\frac{4V_0}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\left(\frac{b}{r}\right)^n \sin n\varphi, r>b$$

Eg. A long, grounded conducting cylinder of radius *b* is placed along the *z*-axis in an initially uniform electric field $\vec{E}=\hat{x}E_0$. Determine potential distribution *V(r,φ)* and electric field intensity $\vec{E}(r,\phi)$ outside the cylinder. Show that the electric field intensity at the surface of the cylinder may be twice as high as that in the distance, which may cause a local breakdown or corona (St. Elmo's fire.) [中央光電所、高考]

(Sol.) $V(r,\phi)=-E_0 r\cos\phi+\sum_{n=1}^{\infty} B_n r^{-n}\cos n\phi \quad \left(At\ r>>b,\ \vec{E}=\hat{x}E_0,\ V=-E_0 r\cos\phi\right)$

At $r=b: V(r,\phi)=-E_0 b\cos\phi+\sum_{n=1}^{\infty} B_n b^{-n}\cos n\phi=0 \Rightarrow B_1=E_0 b^2, B_n=0\ \ for\, n\neq 1.$

Outside the cylinder, $r\geq b: V(r,\phi)=-E_0 r\left(1-\frac{b^2}{r^2}\right)\cos\phi$

$$\vec{E}(r,\varphi)=-\nabla V=\hat{a}_r E_0\left(\frac{b^2}{r^2}+1\right)\cos\varphi+\hat{a}_\phi E_0\left(\frac{b^2}{r^2}-1\right)\sin\varphi$$

4-4 Boundary-Value Problems in Spherical Coordinates

$$\nabla^2 V=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Assume φ-independent: $\frac{\partial^2}{\partial^2\phi}=0$, $V(r,\theta)=R(r)\Theta(\theta)$

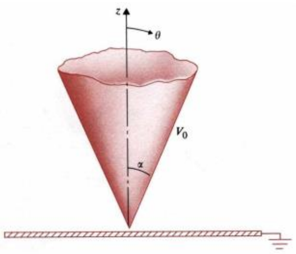
$$r^2\frac{d^2R(r)}{dr^2}+2r\frac{dR(R)}{dr}-k^2R(r)=0,\quad \frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]+n(n+1)\Theta(\theta)\sin(\theta)=0\text{, and}$$

$k^2=n(n+1)\rightarrow$ $R(\text{r})=A_nr^n+B_nr^{-n-1}$, $\Theta(\theta)=P_n(\cos\theta)\rightarrow V(r,\theta)=[A_nr^n+B_nr^{-n-1}]P_n(\cos\theta)$

Table of Legendre's Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta-1)$
3	$\frac{1}{2}(5\cos^3\theta-3\cos\theta)$

Eg. An infinite conducting cone of half-angle α is maintained at potential V_0 and insulated from a grounded conducting plane. Determine (a) the potential distribution $V(\theta)$ in the region $\alpha<\theta<\pi/2$, (b) the electric field intensity in the region $\alpha<\theta<\pi/2$, (c) the charge densities on the cone surface and on the grounded plane.



(Sol.)

$$\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right)=0,\ \frac{dV}{d\theta}=\frac{C_1}{\sin\theta}\Rightarrow V(\theta)=C_1\ell n\left(\tan\frac{\theta}{2}\right)+C_2$$

(a) $$\begin{aligned}V(\alpha)&=C_1\ell n\left(\tan\frac{\alpha}{2}\right)+C_2=V_0\\ V\left(\frac{\pi}{2}\right)&=C_1\ell n\left(\tan\frac{\pi}{4}\right)+C_2=0\end{aligned}\Rightarrow \begin{aligned}C_1&=\frac{V_0}{\ell n\left[\tan\left(\frac{\alpha}{2}\right)\right]}\\ C_2&=0\end{aligned}\Rightarrow V(\theta)=\frac{V_0\ell n\left[\tan\left(\frac{\theta}{2}\right)\right]}{\ell n\left[\tan\left(\frac{\alpha}{2}\right)\right]}$$

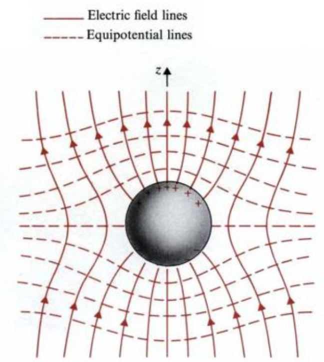
(b) $$\vec{E}=-\hat{a}_\theta\frac{dV}{Rd\theta}=-\hat{a}_\theta\frac{V_0}{R\ell n\left[\tan\left(\frac{\alpha}{2}\right)\right]\sin\theta},$$

(c) $$\theta=\alpha:\rho_s=\varepsilon_0E(\alpha)=\frac{\varepsilon_0V_0}{R\ell n\left[\tan\left(\frac{\alpha}{2}\right)\right]\sin\theta}$$

$$\theta=\frac{\pi}{2}:\rho_s=-\varepsilon_0E\left(\frac{\pi}{2}\right)=-\frac{\varepsilon_0V_0}{R\ell n\left[\tan\left(\frac{\alpha}{2}\right)\right]}$$

Eg. An uncharged conducting sphere of radius *b* is placed in an initially uniform electric field $\vec{E}=\hat{z}E_0$. Determine the potential distribution *V*(*R*,*θ*) and the electric field intensity $\vec{E}(R,\theta)$ after the introduction of the sphere. [中山電研]

(Sol.) $V(b,\theta)=0$



If $R>>b, V(R,\theta)=-E_0 z=-E_0 R\cos\theta$

$$V(R,\theta)=\sum_{n=0}^{\infty}\left[A_n R^n+B_n R^{-(n+1)}\right]P_n(\cos\theta),\quad R\ge b$$

($A_n=0, n\neq 1$; $A_1=-E_0$)

↓

$$=-E_0 R P_1(\cos\theta)+\sum_{n=0}^{\infty}B_n R^{-(n+1)}P_n(\cos\theta), R\ge b$$

(sphere is uncharged, $B_0=0$)

↓

$$=\left(\frac{B_1}{R^2}-E_0 R\right)\cos\theta+\sum_{n=2}^{\infty}B_n R^{-(n+1)}P_n(\cos\theta), R\ge b$$

$$R=b,\ 0=\left(\frac{B_1}{b^2}-E_0 b\right)\cos\theta+\sum_{n=2}^{\infty}B_n b^{-(n+1)}P_n(\cos\theta)\Rightarrow B_1=E_0 b^3,\ \ B_n=0,\ \ n\ge 2,$$

$$\therefore\ \ V(R,\theta)=-E_0\left[1-\left(\frac{b}{R}\right)^3\right]R\cos\theta, R\ge b$$

$$\vec{E}(R,\theta)=\hat{a}_R E_R+\hat{a}_\theta E_\theta=-\nabla V(R,\theta)=\hat{a}_R\left(-\frac{\partial V}{\partial R}\right)+\hat{a}_\theta\left(-\frac{\partial V}{R\partial\theta}\right)$$

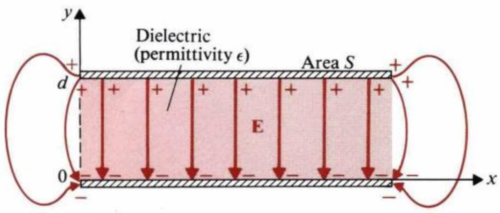
$$=\hat{a}_R E_0\left[1+2\left(\frac{b}{R}\right)^3\right]\cos\theta-\hat{a}_\theta E_0\left[1-\left(\frac{b}{R}\right)^3\right]\sin\theta, R\ge b$$

A dipole moment $\vec{P}=\hat{z}4\pi\varepsilon_0 b^2 E_0$ is at the center of the sphere. Surface charge density is $\rho_s(\theta)=\varepsilon_0 E_{R|R=b}=3\varepsilon_0 E_0\cos\theta$

4-5 Capacitors and Capacitances

$$Q=CV \Leftrightarrow C=Q/V$$

Eg. A parallel-plane capacitor consists of two parallel conducting plates of area S separate by uniform distance d, the space between the plates is filled with a dielectric of a constant permittivity. Determine the capacitance.

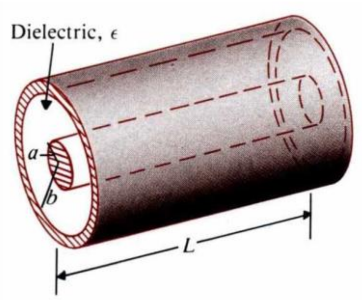


(Sol.) $\rho_s = \dfrac{Q}{S}$, $\vec{E} = -\hat{y}\dfrac{\rho_s}{\varepsilon} = -\hat{y}\dfrac{Q}{\varepsilon S}$

$$V = -\int_{y=0}^{y=d} \vec{E}\cdot d\vec{l} = \int_0^d \left(-\hat{y}\frac{Q}{\varepsilon S}\right)\cdot\left(\hat{y}\,dv\right) = \frac{Q}{\varepsilon S}d$$

$C = \dfrac{Q}{V} = \varepsilon\dfrac{S}{d}$. In this problem, $\vec{E} = -\hat{y}\dfrac{V}{d}$

Eg. A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor of radius b is filled with a dielectric of permittivity ε, and the length of the capacitor is L. Determine the capacitance of this capacitor.

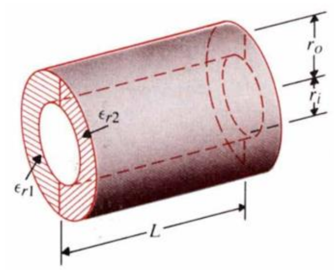


(Sol.) $\vec{E} = \hat{a}_r E_r = \dfrac{Q}{2\pi\varepsilon L r}$,

$$V_{ab} = -\int_{r=b}^{r=a} \vec{E}\cdot d\vec{l} = -\int_b^a \left(\hat{a}_r \frac{Q}{2\pi\varepsilon L r}\right)\cdot\left(\hat{a}_r\, dr\right)$$

$$= \frac{Q}{2\pi\varepsilon L}\ln\left(\frac{b}{a}\right),\quad C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}$$

Eg. A cylindrical capacitor of length L consists of coaxial conducting surface of radii r_i and r_o. Two dielectric media of different dielectric constants ε_{r1} and ε_{r2}, and fill the space between the conducting surface. Determine the capacitance. [台大物理所、高考電機技師]

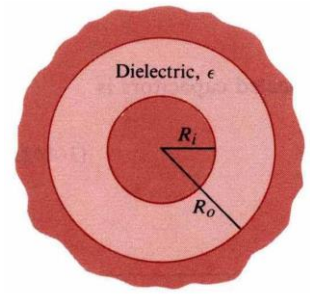


(Sol.) $\pi r L\left(\varepsilon_0\varepsilon_{r1} + \varepsilon_0\varepsilon_{r2}\right)E = \rho_l L \Rightarrow E = \dfrac{\rho_l}{\pi r\varepsilon_0\left(\varepsilon_{r1}+\varepsilon_{r2}\right)}$

$$V = -\int_{r_o}^{r_i} E dr = \frac{\rho_l}{\pi\varepsilon_0\left(\varepsilon_{r1}+\varepsilon_{r2}\right)}\ln\left(\frac{r_o}{r_i}\right)$$

$$C = \frac{\rho_l L}{V} = \frac{\pi\varepsilon_0\left(\varepsilon_{r1}+\varepsilon_{r2}\right)L}{\ln\left(r_o/r_i\right)}$$

Eg. A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a spherical wall of radius R_o. The space in between them is filled with dielectric of permittivity ε. Determine the capacitance. Assuming the earth to be a large conducting sphere (radius=6.37×10^3*km*) surrounded by air, find the capacitance of the earth and the maximum charge that can exist on the earth before the air breaks down.



(Sol.) $\vec{E} = \hat{a}_r E_r = \hat{a}_r \dfrac{Q}{4\pi\varepsilon R^2}$

$$V = -\int_{R_o}^{R_i} \vec{E}\cdot\left(\hat{a}_r\, dR\right) = -\int_{R_o}^{R_i} \frac{Q}{4\pi\varepsilon R^2}\, dR = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$$

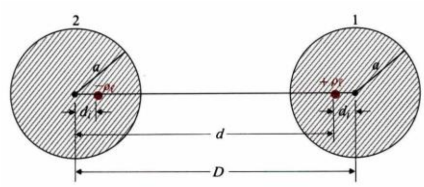
$$C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{R_i} - \dfrac{1}{R_o}}.$$

For an isolating conductor sphere with R_i , $R_o \to \infty$, $C = 4\pi\varepsilon R_i$

$$C = 4\pi\varepsilon_0 R = 4\pi \times \frac{1}{36\pi} \times 10^{-9} \times \left(6.37\times10^3 \times 10^3\right) = 7.08\times10^{-4} \quad (F)$$

$$E_b = 3\times10^6 = \frac{Q_{Max}}{4\pi\varepsilon_0 R^2} \quad \Rightarrow Q_{Max} = 1.35\times10^{10} \quad (C)$$

Eg. Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius *a*. The axes of the wires are separated by a distance *D*. [台大電研]



(Sol.) $V_2 = \dfrac{\rho_\ell}{2\pi\varepsilon}\ell n\dfrac{a}{d}$, $V_1 = -\dfrac{\rho_\ell}{2\pi\varepsilon}\ell n\dfrac{a}{d}$

$$C = \frac{\rho_\ell}{V_1 - V_2} = \frac{\pi\varepsilon}{\ell n(d/a)}, \quad d = D - d_i = D - \frac{a^2}{d}, \quad d = \frac{1}{2}\left(D + \sqrt{D^2 - 4a^2}\right)$$

$$C = \frac{\pi\varepsilon}{\ell n\left[(D/2a) + \sqrt{(D/2a)^2 - 1}\right]} = \frac{\pi\varepsilon}{\cosh^{-1}(D/2a)} \quad (F/m)$$

Eg. A straight conducting wire of radius *a* is parallel to and at height *h* from the surface of the earth. Assume that the earth is perfectly conducting; determine the capacitance and the force per unit length between the wire and the earth.

(Sol.) $D=2h$, $C = \dfrac{\pi\varepsilon_0}{\cosh^{-1}(D/2a)} = \dfrac{\pi\varepsilon_0}{\cosh^{-1}(h/a)}\left(F/m\right)$

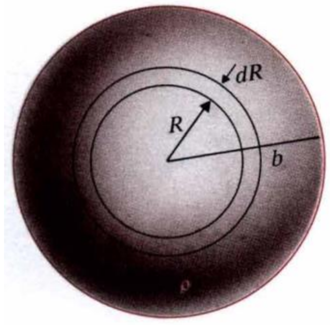
4-6 Electrostatic Energy

To remove Q_1 from infinite to a distance R_{12} from Q_2, the amount of work required is

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\varepsilon_0 R_{12}} = Q_1 \frac{Q_2}{4\pi\varepsilon_0 R_{12}} = Q_1 V_1 = \frac{1}{2}\left(Q_1 V_1 + Q_2 V_2\right)$$

$$\xrightarrow[\text{method}]{\text{induction}} W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k, \quad where \;\; V_k = \frac{1}{4\pi\varepsilon_0}\sum_{j=1(j\neq k)}^{N}\frac{Q_j}{R_{jk}}$$

Eg. Find the energy required to assemble a uniform charge of radius b and volume charge density ρ. [清大電研]



(Sol.) $V_R = \frac{Q_R}{4\pi\varepsilon_0 R}$ $\quad Q_R = \rho\frac{4}{3}\pi R^3$

$dQ_R = \rho 4\pi R^2 dR$, $\; dW = V_R\, dQ_R = \frac{4\pi}{3\varepsilon_0}\rho^2 R^4 dR$

$$W = \int dW = \frac{4\pi}{3\varepsilon_0}\rho^2\int_0^b R^4 dR = \frac{4\pi\rho^2 b^5}{15\varepsilon_0}$$

$$Q = \rho\frac{4\pi}{3}b^3, \quad W = \frac{3Q^2}{20\pi\varepsilon_0 b} \quad (J)$$

Eg. According to $W_e = \frac{1}{2}\iiint_{v'}\rho V dv = \frac{1}{2}\iiint_{v'}\left(\nabla\cdot\vec{D}\right)V dv$, show that the stored electric energy is $W_e = \frac{1}{2}\iiint_{v'}\vec{D}\cdot\vec{E}\,dv$

(Proof) $\because\;\; \nabla\cdot\left(V\vec{D}\right) = V\nabla\cdot\vec{D} + \vec{D}\cdot\nabla V$, $\;\therefore\;\; V\nabla\cdot\vec{D} = \nabla\cdot\left(V\vec{D}\right) - \vec{D}\cdot\nabla V$

$$\therefore\;\; W_e = \frac{1}{2}\iiint_{v'}\nabla\cdot\left(V\vec{D}\right)dv - \frac{1}{2}\iiint_{v'}\vec{D}\cdot\nabla V dv = \frac{1}{2}\oiint_{s'} V\vec{D}\cdot\hat{a}_n\,ds + \frac{1}{2}\iiint_{v'}\vec{D}\cdot\vec{E}\,dv$$

When $R\to\infty$, $S\propto R^2, V\propto\frac{1}{R}, \left|\vec{D}\right|\propto\frac{1}{R^2} \Rightarrow \frac{1}{2}\oiint_{s'} V\vec{D}\cdot\hat{a}_n\,ds \to 0 \Rightarrow W_e = \frac{1}{2}\iiint_{v'}\vec{D}\cdot\vec{E}\,dv$

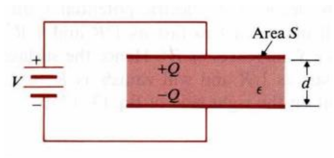
If $\vec{D} = \varepsilon\vec{E}$, then $W_e = \frac{1}{2}\iiint_{v'}\varepsilon\left|\vec{E}\right|^2 dv = \frac{1}{2}\iiint_{v'}\frac{\left|\vec{D}\right|^2}{\varepsilon}dv = \iiint_{v'} w_e\,dv$

Note: 1. SI unit for energy: Joule(J) and 1 eV =$1.6\times10^{-19}J$.

2. Work (or energy) is a scalar, not a vector.

Electrostatic energy density: $w_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon\left|\vec{E}\right|^2 = \frac{\left|\vec{D}\right|^2}{2\varepsilon}$

Eg. A parallel-plate capacitor of area S and separation d is charged by a d-c voltage source V. The permittivity of the dielectric is ε. Find the stored electrostatic energy.



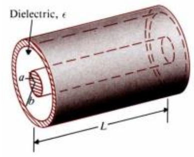
(Sol.) $E = \dfrac{V}{d}$,

$$W_e = \frac{1}{2}\iiint_{v'} \varepsilon \left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\varepsilon\left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2}\left(\varepsilon\frac{S}{d}\right)V^2$$

Eg. Use energy formulas to find the capacitance of a cylindrical capacitance having a length L, an inner conductor of radius a, an outer conductor of inner radius b, and dielectric of permittivity ε .

(Sol.) $\vec{E} = \hat{a}_r \dfrac{Q}{2\pi\varepsilon L r}$, $W_e = \dfrac{1}{2}\displaystyle\int_a^b \varepsilon\left(\frac{Q}{2\pi\varepsilon L r}\right)^2 (L 2\pi r dr) = \frac{Q^2}{4\pi\varepsilon L}\int_a^b \frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon L}\ln\left(\frac{b}{a}\right)$,

$$\frac{Q^2}{2C} = \frac{Q^2}{4\pi\varepsilon L}\ln\left(\frac{b}{a}\right) \Rightarrow C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}$$



4-7 Electrostatic Forces and Torques

Electrostatic force and torque due to the fixed charge:

$dW = \vec{F}_Q \cdot d\vec{l}$ is mechanic work done by the system, it costs the stored energy.

$$\because\ dW_e = -dW = -\vec{F}_Q \cdot d\vec{l} = (\nabla W_e)\cdot d\vec{l},\ \therefore\ \vec{F}_Q = -\nabla W_e \quad (N)$$

$$\left(\vec{F}_Q\right)_l = -\frac{\partial W_e}{\partial l} = -\frac{\partial}{\partial l}\left(\frac{Q^2}{2C}\right) = \frac{Q^2}{2C^2}\frac{\partial C}{\partial l},\quad dW = (T_Q)_z\, d\phi \Rightarrow (T_Q)_z = -\frac{\partial W_e}{\partial \phi}$$

Electrostatic force and torque due to the fixed potential:

$$dW_s = \sum_k V_k dQ_k,\quad dW = \vec{F}_v \cdot d\vec{l},\quad dW_e = \frac{1}{2}\sum_k V_k dQ_k = \frac{1}{2}dW_s$$

$$dW + dW_e = dW_s \quad \Rightarrow dW = \frac{1}{2}dW_s = dW_e = \vec{F}_v \cdot d\vec{l} = (\nabla W_e)\cdot d\vec{l}$$

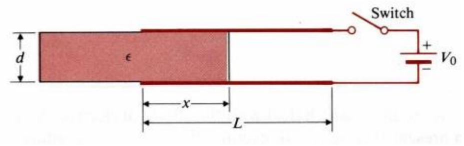
$$\therefore\ \vec{F}_v = \nabla W_e,\quad (T_v)_z = \frac{\partial W_e}{\partial \phi},\quad \left(\vec{F}_v\right)_l = \frac{\partial W_e}{\partial l} = \frac{\partial}{\partial l}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial C}{\partial l} = \frac{Q^2}{2C^2}\frac{\partial C}{\partial l}$$

Eg. A parallel-plate capacitor of width *w*, length *L*, and separation *d* has a solid dielectric slab of permittivity ε in the space between the plates. The capacitor is charged to a voltage V_0 by a battery. Assuming that the dielectric slab is withdrawn to the position shown, determine the force action on the slab. (a) with the switch closed, (b) after the switch is first opened. [台大電研、清大電研]

(Sol.) (a) $W_e = \frac{1}{2}CV_0^2,\ C = \frac{w}{d}[\varepsilon x + \varepsilon_0(L-x)] \Rightarrow \vec{F}_x = \nabla W_e = \hat{x}\frac{V_0^2}{2}\frac{\partial C}{\partial x} = \hat{x}\frac{V_0^2 w}{2d}(\varepsilon - \varepsilon_0)$

(b)

$W_e = \frac{Q^2}{2C},\ \vec{F}_Q = -\nabla W_e = -\hat{x}\frac{Q^2}{2}\frac{\partial}{\partial x}\left(\frac{1}{C}\right) = \hat{x}\frac{V_0^2 w}{2d}(\varepsilon - \varepsilon_0)$



4-8 Resistors and Resistances

Ohm's law: $V=RI$

$$V=E\ell\Rightarrow E=\frac{V}{\ell},I=\iint_s \vec{J}\cdot d\vec{S}=JS\Rightarrow J=\frac{I}{S}=\sigma\frac{V}{\ell}\Rightarrow V=\left(\frac{\ell}{\sigma S}\right)I=RI$$

$$\therefore\ R=\frac{\ell}{\sigma S},\ G=\frac{1}{R}=\sigma\frac{S}{\ell}$$

Power dissipation: $P=\iiint_{v'}\vec{E}\cdot\vec{J}dv=\int\vec{E}\cdot d\vec{\ell}\iint_{ss}\vec{J}\cdot d\vec{S}=-VI=-I^2R$

Eg. A *d-c* voltage of 6*V* applied to the ends of 1*km* of a conducting wire of 0.5*mm* radius results in a current of 1/6*A*. Find (a) the conductivity of the wire, (b) the electric field intensity of the wire, (c) the power dissipation in the wire, (d) the electron drift velocity, assuming electron mobility in the wire to be $1.4\times10^{-3}\left(m^2/\ V\cdot s\right)$.

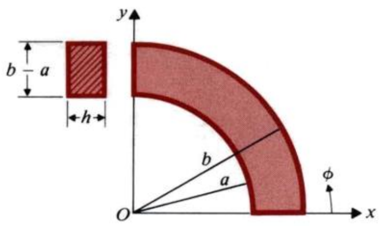
(Sol.) (a) $R=\frac{\ell}{\sigma S}=\frac{V}{I}\Rightarrow\sigma=\frac{\ell I}{SV}=3.54\times10^7\left(S/m\right)$, (b) $E=\frac{V}{\ell}=6\times10^{-3}(V/m)$, (c)

$P=VI=1Watt$, (d) $v_e=\mu E=8.4\times10^{-6}(m/\sec)$

Calculation of resistance:

$$\nabla^2V=0\Rightarrow V\Rightarrow\vec{E}=-\nabla V\Rightarrow\vec{J}=\sigma\vec{E}\Rightarrow\vec{I}=\oiint\vec{J}ds\Rightarrow R=V/I$$

Eg. A conducting material of uniform thickness *h* and conductivity *σ*, has the shape of a quarter of a flat circular washer, with inner radius *a* and outer radius *b*. Determine the resistance between the end faces. [清大電研]



(Sol.) $\nabla^2V=0$, $V=0$ at $\phi=0$, $V=V_0$ at $\phi=\frac{\pi}{2}$

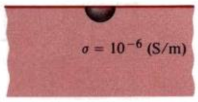
$\frac{d^2V}{d\phi^2}=0$, $V=c_1\varphi+c_2$, $V=\frac{2V_0}{\pi}\phi$,

$\vec{J}=\sigma\vec{E}=-\sigma\nabla V=-\hat{a}_\phi\sigma\frac{\partial V}{r\partial\phi}=-\hat{a}_\phi\frac{2\sigma V_0}{\pi r}$

$$I=\int_S\vec{J}\cdot d\vec{s}=\frac{2\sigma V_0}{\pi}h\int_a^b\frac{dr}{r}=\frac{2\sigma hV_0}{\pi}\ln\frac{b}{a},\quad R=\frac{V_0}{I}=\frac{\pi}{2\sigma h\ln\left(\frac{b}{a}\right)}$$

Eg. A ground connection is made by burying a hemispherical conductor of radius 25*mm* in the earth with its base up. Assuming the earth conductivity to $\sigma=10^{-6}$ *S/m*, find the resistance of the conductor to far-away points in the ground. [交大電信所]

(Sol.) $\vec{J}=\hat{a}_R\dfrac{I}{2\pi R^2}$, $\vec{E}=\hat{a}_R\dfrac{I}{2\pi\sigma R^2}$ $\Rightarrow V_0=-\int_{\infty}^{b}E\,dR=\dfrac{I}{2\pi\sigma b}$



$$R=\frac{V_0}{I}=\frac{1}{2\pi\sigma b}=\frac{1}{2\pi(10^{-6})(25\times10^{-3})}=6.36\times10^{6}.$$

Relation between *R* and *C*: $C=\dfrac{Q}{V}=\dfrac{\oint_S \vec{D}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}}=\dfrac{\oint_S \varepsilon\vec{E}\cdot d\vec{s}}{-\int_L \vec{E}\cdot d\vec{l}}$,

$$R=\frac{V}{I}=\frac{-\int_L \vec{E}\cdot d\vec{l}}{\oint_S \vec{J}\cdot d\vec{s}}=\frac{-\int_L \vec{E}\cdot d\vec{l}}{\oint_S \sigma\vec{E}\cdot d\vec{s}},\quad \therefore\ \boxed{RC=\frac{C}{G}=\frac{\varepsilon}{\sigma}}$$

Eg. Find the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1<R_2$) if the space between the surfaces is filled with a homogeneous and isotropic material having a conductivity σ.

(Sol.) $C=\dfrac{4\pi\varepsilon}{\frac{1}{R_1}-\frac{1}{R_2}}$, $RC=\dfrac{\varepsilon}{\sigma}$ $\Rightarrow R=\dfrac{1}{C}\cdot\dfrac{\varepsilon}{\sigma}$ $\Rightarrow R=\dfrac{1}{4\pi\sigma}\left(\dfrac{1}{R_1}-\dfrac{1}{R_2}\right)$

Eg. Find the leakage resistance per unit length (a) between the inner and outer conductors of a coaxial cable that has an inner conductor of radius *a*, an outer conductor of inner radius *b*, and a medium with conductivity *σ*, and (b) of a parallel-wire transmission line consisting of wires of radius *a* separated by a distance *D* in a medium with conductivity *σ*. [台科大電研]

(Sol.)

(a) $C=\dfrac{2\pi\varepsilon}{\ln(\frac{b}{a})}$, $R=\dfrac{\varepsilon}{\sigma C}=\dfrac{1}{2\pi\sigma}\ln\left(\dfrac{b}{a}\right)$, (b) $C=\dfrac{\pi\varepsilon}{\cosh^{-1}(\frac{D}{2a})}$, $R=\dfrac{\varepsilon}{\sigma C}=\dfrac{1}{\pi\sigma}\cosh^{-1}(\dfrac{D}{2a})$

4-9 Inductors and Inductances

Mutual flux: $\Phi_{12} = \iint_{S_2} \vec{B}_1 \cdot \mathrm{d}\vec{S}_2 = L_{12} I_1 \quad (Wb)$

General mutual inductance: $L_{12} = \dfrac{N_2 \Phi_{12}}{I_1} = \dfrac{\Lambda_{12}}{I_1} \quad (H)$

Self-Inductance: $L_{11} = \dfrac{\Lambda_{11}}{I_1}$

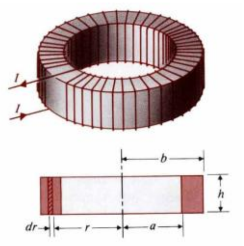
Neumaun formula: $L_{12} = \dfrac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \dfrac{d\vec{\ell}_1 d\vec{\ell}_2}{R}$

$$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \iint_S \vec{B}_1 \cdot d\vec{S}_2 = \frac{N_2}{I_1} \iint_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{S}_2 = \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{\ell}_2$$

$$\because \quad \vec{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi} \oint_{C_1} \frac{d\vec{\ell}_1}{R_1}, \quad \therefore \quad L_{12} = \frac{\mu_0 N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 d\vec{\ell}_2}{R}$$

Eg. Assume that *N* turns of wire are tightly wound on a toroidal frame of a rectangular cross section. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil. [台大電研]

(Sol.)

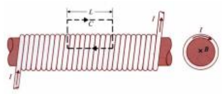


$$d\vec{l} = \hat{a}_\phi r d\phi, \quad \oint_C \vec{B} \cdot \mathrm{d}\vec{l} = \int B r \,\mathrm{d}\phi = 2\pi r B = \mu_o N I \Rightarrow B = \frac{\mu_o N I}{2\pi r}$$

$$\Rightarrow \Phi = \iint_S \vec{B} \cdot \mathrm{d}\vec{s} = \frac{\mu_o N I h}{2\pi} \int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_o N I h}{2\pi} \ln(\frac{b}{a}),$$

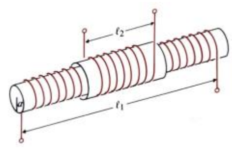
$$L = \frac{N\Phi}{I} = \frac{\mu_o N^2 h}{2\pi} \ln(\frac{b}{a})$$

Eg. Find the inductance per unit length of a very long solenoid with air core having *n* turns per unit length. And *S* is the cross-sectional area.



(Sol.) $B = \mu_o n I$, $\Phi = BS = \mu_o n S I$ $\quad \wedge' = n\Phi = \mu_o n^2 S I$, $L' = \mu_o n^2 S$

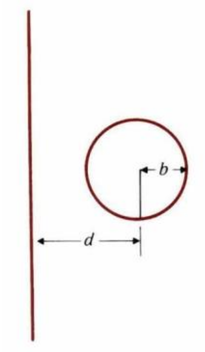
Eg. Two coils of N_1 and N_2 turns are wound concentrically on a straight cylindrical core of radius *a* and permeability μ. The windings have lengths l_1 and l_2, respectively. Find the mutual inductance between the coils.



(Sol.) $\Phi_{12} = \mu(\dfrac{N_1}{\ell_1})(\pi a^2) I_1$, $\Lambda_{12} = N_2 \Phi_{12} = \dfrac{\mu}{\ell_1} N_1 N_2 \pi a^2 I_1$

$$=> L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu}{\ell_1} N_1 N_2 \pi a^2$$

Eg. Determine the mutual inductance between a very long, straight wire and a conducting circular loop. [台大電研、清大物理所]



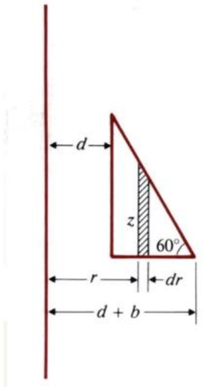
(Sol.)

B at p is $\dfrac{\mu_0 I}{2\pi(d+r\cos\theta)}$

$$\Lambda=\frac{\mu_0 I}{2\pi}\int_0^b\int_0^{2\pi}\frac{rd\theta dr}{d+r\cos\theta}=\frac{\mu_0 I}{2\pi}\int_0^b\frac{2\pi rdr}{\sqrt{d^2-r^2}}=\mu_0 I(d-\sqrt{d^2-b^2})$$

$$L=\mu_0(d-\sqrt{d^2-b^2})$$

Eg. Determine the mutual inductance between a conducting triangular loop and a very long straight wire.



(Sol.)

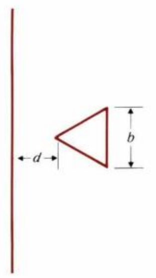
$$\vec{B}=\hat{a}_\varphi\frac{\mu_0 I}{2\pi r},\quad \Lambda=\Phi=\int_S\vec{B}\cdot d\vec{s},\quad \text{where}\quad d\vec{S}=\hat{a}_\phi zdr$$

$$z=\sqrt{3}(d+b-r)$$

$$\Lambda=\frac{\sqrt{3}\mu_0 I}{2\pi}\int_d^{d+b}\frac{1}{r}(d+b-r)dr=\frac{\sqrt{3}\mu_0 I}{2\pi}[(d+b)\ln(1+b/d)-b]$$

$$L=\frac{\Lambda}{I}=\frac{\sqrt{3}\mu_0}{2\pi}[(d+b)\ln(1+b/d)-b]$$

Eg. Determine the mutual inductance between a very long, straight wire and a conducting equilateral triangular loop. [高考]



(Sol.)

$$\vec{B}=\hat{a}_\Phi\frac{\mu_0 I}{2\pi r}=\hat{a}_\Phi B_\Phi$$

$$\Lambda=\int_d^{d+\frac{\sqrt{3}}{2}b}B_\Phi\cdot\frac{2}{\sqrt{3}}(r-d)dr=\frac{\mu_0 I}{\sqrt{3}\pi}[\frac{\sqrt{3}}{2}b-d\ln(1+\frac{\sqrt{3}b}{2d})]$$

$$L=\frac{\Lambda}{I}=\frac{\mu_0}{\sqrt{3}\pi}[\frac{\sqrt{3}}{2}b-d\ln(1+\frac{\sqrt{3}b}{2d})]$$

Eg. A rectangular loop of width *w* and height *h* is situated near a very long wire carrying a current i_1. Assume i_1 to be a rectangular pulse. Find the induced current i_2 in the rectangular loop whose self-inductance is *L*.

(Sol.)

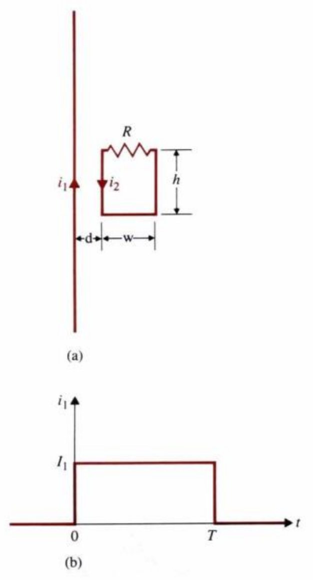
$$L_{12}\frac{di_1}{dt} = L\frac{di_2}{dt} + Ri_2,$$

where $L_{12} = \frac{\Phi_{12}}{i_1} = \frac{h}{i_1}\int_d^{d+w}\frac{\mu_0 i_1}{2\pi r}dr = \frac{\mu_0 h}{2\pi}\ln(1+\frac{w}{d})$

t=0, $L\frac{di_2}{dt} + Ri_2 = L_{12}I_1\delta(t) \Rightarrow i_2 = \frac{L_{12}}{L}I_1 e^{-(\frac{R}{L})t}$

t=T, $i_2 = \frac{L_{12}}{L}I_1 e^{-\frac{RT}{L}}$, when $-I_1$ is applied

t>T, $i_2 = -\frac{L_{12}}{L}I_1 e^{-(\frac{R}{L})(t-T)}$



4-10 Magnetic Energy

$W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\iiint_{V'} \vec{A}\cdot\vec{J}dv'$

Let $V_1 = L_1 \frac{di_1}{dt} \Rightarrow W_1 = \int V_1 i_1 dt = L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L_1 I_1^2 = \frac{1}{2} I_1 \Phi_1$: Magnetic energy

Similarly, $V_{21} = L_{21} \frac{di_2}{dt} \Rightarrow W_{21} = \int V_{21} I_1 dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2$

And $W_2 = \frac{1}{2} L_2 I_2^2 \quad \Rightarrow W_m = \frac{1}{2} L_1 I_1^2 + L_{21} I_1 I_2 + \frac{1}{2} L_2 I_2^2 = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} L_{jk} I_j I_k$

Generally, $W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k$ when $\Phi_k = \sum_{j=1}^{N} L_{jk} I_j$

$\because \quad \Phi_k = \iint_{S_k} \vec{B}\cdot d\vec{S}_n' = \oint_{C_k} \vec{A}\cdot d\vec{\ell}_k$

$\therefore \quad W_m = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \oint_{C_k} \vec{A}\cdot d\vec{l}_k = \frac{1}{2}\iiint_{V'} \vec{A}\cdot\vec{J}dv' \quad (\Delta I_k dl_k' = J(\Delta \hat{a}_k')dl_k' = J\cdot v_k')$

$\because \quad \nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H}) \Rightarrow \vec{A}\cdot(\nabla\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \nabla\cdot(\vec{A}\times\vec{H})$

And $\vec{J} = \nabla\times\vec{H} \Rightarrow \vec{A}\cdot\vec{J} = \vec{H}\cdot\vec{B} - \nabla\cdot(A\times\vec{H})$

$\Rightarrow W_m = \frac{1}{2}\iiint_{V'} \vec{H}\cdot\vec{B}dv' - \frac{1}{2}\oiint_{S'} (\vec{A}\times\vec{H})\cdot\vec{a}_n dS'$ as $R\to\infty \Rightarrow |\vec{A}| \propto \frac{1}{R}$, $\quad |\vec{H}| \propto \frac{1}{R^2}$,

$d\vec{S} \propto R^2 \Rightarrow -\frac{1}{2}\oiint_{S'} (\vec{A}\times\vec{H})\cdot\hat{a}_n dS' \to 0$

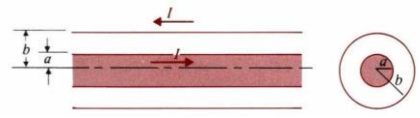
$\therefore \quad W_m = \frac{1}{2}\iiint_{V'} \vec{H}\cdot\vec{B}dv' = \iiint_{V'} w_m dv'$

$$w_m = \frac{1}{2}\vec{H}\cdot\vec{B}$$

Magnetic energy density: $w_m = \frac{|B|^2}{2\mu}$ and $L = \frac{2W_m}{I^2}$.

$$w_m = \frac{1}{2}\mu |H|^2$$

Eg. Determine the inductance per unit length of an air coaxial transmission line that has a solid inner conductor of radius *a* and a very thin outer conductor of radius *b*. [台科大電機所]



(Sol.)

$W_{m1} = \frac{1}{2\mu_0}\int_0^a B_1^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3 dr = \frac{\mu_0 I^2}{16\pi}$

$W_{m2} = \frac{1}{2\mu_0}\int_a^b B_2^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r} dr = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}$, $L' = \frac{2}{I^2}(W_{m1}+W_{m2}) = \frac{\mu_0}{8\pi}+\frac{\mu_0}{2\pi}\ln\frac{b}{a}$

Eg. Consider two coupled circuits having self-inductance L_1 and L_2, which carry currents I_1 and I_2, respectively. The mutual inductance between the circuits is *M*.

a) Find the ratio I_1/I_2 that makes the stored magnetic energy W_m a minimum.

b) Show that $M \le \sqrt{L_1 L_2}$. [清大核工所]

(Sol.) $W_m = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2$

(a) $W_m = \frac{I_2^2}{2}[L_1(\frac{I_1}{I_2})^2 + 2M(\frac{I_1}{I_2}) + L_2] = \frac{I_2^2}{2}[L_1 x^2 + 2Mx + L_2]$, $x \equiv \frac{I_1}{I_2}$

$\frac{dW_m}{dx} = 0 = \frac{I_2^2}{2}(2L_1 x + 2M) \Rightarrow x = \frac{I_1}{I_2} = -\frac{M}{L_1}$ for minimum W_m

(b) $(W_m)_{min} = \frac{I_2^2}{2}(-\frac{M^2}{L_1} + L_2) \ge 0 \Rightarrow M \le \sqrt{L_1 L_2}$

4-11 Magnetic Forces and Torques

Force due to constant flux linkage:

$$\vec{F}_\phi \cdot d\vec{\ell} = -dW_m = -(\nabla W_m)\cdot d\vec{\ell} \Rightarrow \vec{F}_\phi = -\nabla W_m \quad \text{and} \quad (T_\phi)_z = -\frac{\partial W_m}{\partial \phi}$$

Force due to constant current:

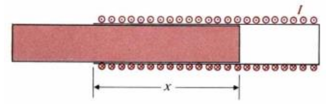
$$dW_s = \sum_k I_k' d\Phi_k = dW + dW_m$$

$$dW_m = \frac{1}{2}\sum_k I_k \Phi_k = \frac{1}{2}dW_s \Rightarrow dW = \vec{F}_I \cdot d\vec{l} = dW_m = (\nabla W_m)\cdot d\vec{l} \Rightarrow \vec{F}_I = \nabla W_m$$

Torque in terms of mutual inductance:

$$W_m = \frac{1}{2}L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2}L_2 I_2^2 \Rightarrow F_I = I_1 I_2 (\nabla L_{12}),\ T_I = I_1 I_2 \frac{\partial L_{12}}{\partial \phi}$$

Eg. A current I flows in a long solenoid with n closely wound coil-turns per unit length. The cross-sectional area of its iron core, which has permeability μ, is S. Determine the force acting on the core if it is withdrawn to the position. [高考電機技師]



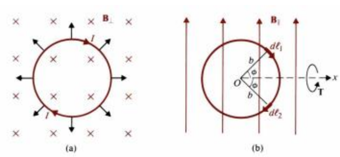
(Sol.)

$$W_m = \frac{1}{2}\iiint \mu H^2 dv,\ W_m(x+\Delta x) - W_m(x) = (\frac{1}{2}\mu_0\mu_r n^2 I^2 - \frac{1}{2}\mu_0 n^2 I^2)S\Delta x = \frac{1}{2}\mu_0(\mu_r - 1)n^2 I^2 S\Delta x$$

$$\Rightarrow (F_I)_x = \frac{\partial W_m}{\partial x} = \frac{\mu_0}{2}(\mu_r - 1)n^2 I^2 S$$

Magnetic torque: $\vec{T} = \vec{m} \times \vec{B}$ **($B=B_\perp+B_\parallel$, $m \parallel B_\perp \Rightarrow m\times B_\perp=0$)**

$$d\vec{T} = \hat{x}dF\,2b\sin\phi = \hat{x}(IdlB_\parallel \sin\phi)2b\sin\phi = \hat{x}2Ib^2 B_\parallel \sin^2\phi d\phi$$



$$\vec{T} = \int d\vec{T} = \hat{x}2Ib^2 B_{//}\int_0^\pi \sin^2\phi d\phi = \hat{x}I(\pi b^2)B_{//} = \hat{x}mB_{//}$$

$$\Rightarrow \vec{T} = \vec{m} \times \vec{B}$$

Eg. A rectangular loop in the *xy*-plane with sides b_1 and b_2 carrying a current I has in a uniform magnetic field $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$. Determine the force and torque on the loop.

(Sol.) $\vec{T} = \vec{m} \times \vec{B} = Ib_1b_2(\hat{x}B_y - \hat{y}B_x)$