

歷年來文大電機系成績優異提早畢業之學生

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歷年來的工數 TA 與專題生研究所上榜慶功宴





WebMail 請信 收件匣

我的收件匣

From: 廖正雄
 To: 'lyk@osa.gov.tw' <lyk@osa.gov.tw>
 Sent Date: Thu, 24 Mar 2022 14:28:57 Asia/Taipei
 Subject: 有關 111 年警政特考、一般特考、暨交通事業特考、命題委員、謝函。

李克治委員您好：
 本 (111) 年警政特考、一般特考、暨交通事業特考
 貴系無人勇毅 (國立臺北科技大學電子工程系教授) 想請您幫忙命題工作，不知能否幫忙？
 相關資訊如下：

科目	類別	類科	題數	題型	命題時限	命題時間
專業科目	警政人員	警政人員 (警政人員、警政人員、警政人員、警政人員)	20	測驗式	實地科目可自訂	111/3/18-6/19

Chapter 1 The First-Order Ordinary Differential Equations (ODE)

1-1 Separable Differential Equation $A(x)dx=B(y)dy$

Solution: $\int A(x)dx = \int B(y)dy + C$

Eg. Solve (a) $y'=3x^2+1$, $y(1)=4$, (b) $6x-2yy'=0$, and (c) $2 \cdot \frac{dy}{dx} - \frac{1}{y} = \frac{2x}{y}$, $y(0)=0$.

(Sol.) (a) $dy=(3x^2+1)dx$, $y=x^3+x+c$, $y(1)=4 \Rightarrow c=2$, $\therefore y=x^3+x+2$.

(b) $2ydy=6xdx \Rightarrow y^2=3x^2+c$.

(c) $2ydy = (1+2x)dx$, $y^2+c=x+x^2$, $y(0)=0 \Rightarrow c=0$,
 $\therefore y^2=x+x^2$.

Formulae:

$\int x^m dx = \begin{cases} \frac{x^{m+1}}{m+1} + c, & m \neq -1 \\ \ln x + c, & m = -1 \end{cases}$	$x^{-n} = \frac{1}{x^n}$	$\int e^{ax} dx = \frac{e^{ax}}{a} + c$
$\int \sin(ax) dx = -\frac{\cos(ax)}{a} + c$	$\int \cos(ax) dx = \frac{\sin(ax)}{a} + c$	

Eg. Solve $y'=xe^{x-y}$ with the boundary condition: $y=\ln 2$ at $x=0$. [2005 台大電研]

(Sol.) $e^y dy = x e^x dx \Rightarrow e^y = x e^x - e^x + c$, $y(x=0) = \ln 2 \Rightarrow c=3$, $\therefore e^y = x e^x - e^x + 3$.

Eg. Solve $y'=y^2 e^{2t}$ with $y(0)=2$. [2010 台大生醫電資所]

(Sol.) $\frac{1}{y^2} dy = e^{2t} dt \Rightarrow \frac{-1}{y} = \frac{e^{2t}}{2} + c$, $y(t=0) = 2 \Rightarrow c = -1$, $\therefore \frac{-1}{y} = \frac{e^{2t}}{2} - 1$.

1-2 The first-order Linear Differential Equation $y'+p(x)y=q(x)$

Solution: $y' \cdot e^{\int p(x)dx} + p(x)y \cdot e^{\int p(x)dx} = q(x) \cdot e^{\int p(x)dx}$

$$\Rightarrow \left[y \cdot e^{\int p(x)dx} \right]' = q(x) \cdot e^{\int p(x)dx} \Rightarrow y e^{\int p(x)dx} = \int \left[q(x) \cdot e^{\int p(x)dx} \right] dx + c$$

$$\Rightarrow y(x) = e^{-\int p(x)dx} \cdot \left\{ \int \left[q(x) \cdot e^{\int p(x)dx} \right] dx + c \right\}$$

Formulae:

$A^B = e^{B \ln(A)}$	$e^{\pm n \ln(x)} = x^{\pm n}$
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Eg. Solve $xy'+2y=3x^3$.

(Sol.) $y' + \frac{2}{x}y = 3x^2$, $p(x) = \frac{2}{x}$, $\int p(x)dx = 2 \ln(x)$, $e^{2 \ln(x)} = x^2 \Rightarrow x^2 y' + 2xy = 3x^4$

$$\Rightarrow (x^2 y)' = 3x^4, x^2 y = \frac{3x^5}{5} + c \Rightarrow y(x) = \frac{3x^3}{5} + \frac{c}{x^2}$$

Eg. Solve $y'+y=\sin(x)$.

(Sol.) $p(x)=1, \int p(x)dx=x \Rightarrow y' \cdot e^x + e^x \cdot y = e^x \cdot \sin(x) \Rightarrow (e^x \cdot y)' = e^x \cdot \sin(x),$
 $e^x \cdot y = \frac{e^x [\sin(x) - \cos(x)]}{2} + c \Rightarrow y(x) = \frac{\sin(x) - \cos(x)}{2} + c \cdot e^{-x}$

Eg. Solve (a) $y' = \frac{2}{x}y + x^2e^x$ and (b) $y' + 3x^2y = xe^{-x^3}$. [2013 成大電研]

(Sol.) (a) $y' - \frac{2}{x}y = x^2e^x, p(x) = -\frac{2}{x}, \int p(x)dx = -2\ln(x), e^{-2\ln(x)} = x^{-2}$

$\Rightarrow x^{-2}y' - 2x^{-3}y = e^x, [x^{-2}y]' = e^x, x^{-2}y = e^x + c \Rightarrow y = x^2e^x + cx^2$

(b) $y' + 3x^2y = xe^{-x^3}, p(x) = 3x^2, \int p(x)dx = x^3 \Rightarrow e^{x^3}y' + 3x^2e^{x^3}y = x, [e^{x^3}y]' = x,$

$e^{x^3}y = \frac{x^2}{2} + c \Rightarrow y = (\frac{x^2}{2} + c) \cdot e^{-x^3}$

Eg. Solve $y'+y\cot(x)=5e^{\cos(x)}$. [2013 師大應用電子所]

(Sol.) $p(x)=\cot(x), \int p(x)dx=\ln|\sin(x)|, e^{\ln|\sin(x)|}=\sin(x) \Rightarrow \sin(x)y'+y\cos(x)=5\sin(x)e^{\cos(x)}$
 $\Rightarrow [y\sin(x)]' = 5\sin(x)e^{\cos(x)} \Rightarrow y\sin(x) = -5e^{\cos(x)} + c, \therefore y(x) = \frac{-5e^{\cos(x)} + c}{\sin(x)}$

Eg. Solve $(x^2 + 1)y' + (x^2 + 2x + 1)y = x^2 + 1, y(0)=1$. [2017 台大生醫電資所]

(Sol.) $y' + \frac{x^2 + 2x + 1}{x^2 + 1}y = 1, y' + (1 + \frac{2x}{x^2 + 1})y = 1, \int (1 + \frac{2x}{x^2 + 1})dx = x + \ln(x^2 + 1)$

$\exp[x + \ln(x^2 + 1)] = (x^2 + 1)e^x, (x^2 + 1)e^x \cdot [y' + \frac{x^2 + 2x + 1}{x^2 + 1}y = 1],$

$(x^2 + 1)e^x \cdot y' + (x^2 + 2x + 1) \cdot e^x \cdot y = (x^2 + 1)e^x, [(x^2 + 1)e^x \cdot y]' = (x^2 + 1)e^x,$

$(x^2 + 1)e^x \cdot y = (x^2 - 2x + 3)e^x + C, y(0)=1 \Rightarrow C=-2, \therefore y = \frac{x^2 - 2x + 3 - 2e^{-x}}{x^2 + 1}$

1-3 Bernoulli Differential Equations $y'+p(x)y=r(x)y^\alpha$ (It is nonlinear if $\alpha \neq 1$)

Solution: Set $z = y^{1-\alpha}, \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{y^\alpha}{1-\alpha} \cdot \frac{dz}{dx} \Rightarrow \frac{y^\alpha}{1-\alpha} \cdot \frac{dz}{dx} + p(x) \cdot zy^\alpha = r(x)y^\alpha$

$\Rightarrow \frac{dz}{dx} + (1-\alpha)p(x) \cdot z = r(x) \cdot (1-\alpha)$

$\Rightarrow z(x) = e^{-\int (1-\alpha)p(x)dx} \cdot \left[(1-\alpha) \cdot \int r(x) \cdot e^{\int (1-\alpha)p(x)dx} dx + c \right]$

$\Rightarrow [y(x)]^{1-\alpha} = e^{-(1-\alpha)\int p(x)dx} \cdot \left[(1-\alpha) \cdot \int r(x) \cdot e^{(1-\alpha)\int p(x)dx} dx + c \right]$

Eg. Solve $y'=y(xy^3-1)$. [2004 台大電研]

(Sol.) $y' + y = xy^4$. Set $z = y^{1-4} = y^{-3}$, $y = z^{-1/3}$, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -\frac{1}{3} z^{-4/3} \cdot z'$
 $-\frac{1}{3} z^{-4/3} \cdot z' + z^{-1/3} = xz^{-4/3}$, $(-3z^{4/3}) \times [-\frac{1}{3} z^{-4/3} \cdot z' + z^{-1/3} = xz^{-4/3}]$
 $\Rightarrow z' - 3z = -3x \Rightarrow [e^{-3x} \cdot z]' = -3x \cdot e^{-3x} \Rightarrow z = x + \frac{1}{3} + ce^{3x} \Rightarrow y^{-3} = x + \frac{1}{3} + ce^{3x}$

Eg. Solve $\frac{dy(x)}{dx} - y(x) + e^{2x}y^2(x) = 0$. [2015 台大電研]

(Sol.) $y' - y = -e^{2x}y^2$. Set $z = y^{1-2} = y^{-1}$, $y = z^{-1}$, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -\frac{1}{z^2} \cdot z'$
 $-z^{-2} \cdot z' - z^{-1} = -e^{2x}z^{-2}$, $(-z^2) \times [-z^{-2} \cdot z' - z^{-1} = -e^{2x}z^{-2}]$
 $\Rightarrow z' + z = e^{2x} \Rightarrow [e^x \cdot z]' = e^{3x} \Rightarrow z = \frac{e^{2x}}{3} + ce^{-x} \Rightarrow y^{-1} = \frac{e^{2x}}{3} + ce^{-x}$

Eg. Solve $xy' + y = 4x^2y^2$. [2015 師大電研]

(Sol.) $y' + \frac{1}{x}y = 4xy^2$. Set $z = y^{1-2} = y^{-1}$, $y = z^{-1}$, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -\frac{1}{z^2} \cdot z'$
 $-\frac{1}{z^2} \cdot z' + \frac{1}{xz} = \frac{4x}{z^2}$, $-z' + \frac{z}{x} = 4x$, $z' - \frac{z}{x} = -4x$, $p(x) = -\frac{1}{x}$, $\int p(x)dx = -\ln(x)$, $e^{-\ln(x)} = x^{-1}$
 $x^{-1}z' - \frac{z}{x^2} = -4$, $[x^{-1}z]' = -4$, $x^{-1}z = -4x + c$, $z = -4x^2 + cx \Rightarrow \frac{1}{y} = -4x^2 + cx$

Eg. Solve $y' + \frac{1}{3}y = \frac{(1-2x)}{3} \cdot y^4$. [2013 師大應用電子所]

(Sol.) Set $z = y^{1-4} = y^{-3}$, $y = z^{-1/3}$, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = -\frac{1}{3} z^{-4/3} \cdot z'$,
 $-\frac{1}{3} z^{-4/3} \cdot z' + \frac{1}{3} z^{-1/3} = \frac{(1-2x)}{3} \cdot z^{-4/3}$, $(-3z^{4/3}) \times [-\frac{1}{3} z^{-4/3} \cdot z' + \frac{1}{3} z^{-1/3} = \frac{(1-2x)}{3} \cdot z^{-4/3}]$
 $\Rightarrow z' - z = 2x - 1$, $[e^{-x} \cdot z]' = (2x-1)e^{-x} \Rightarrow e^{-x} \cdot z = (-2x-1)e^{-x} + c$,
 $z = -2x-1 + ce^x$, $\therefore y^{-3} = -2x-1 + ce^x$

Eg. Solve $\frac{dP(t)}{dt} = P(t) \cdot (c_1 - c_2P(t))$. [2003 台科大電研]

(Sol.) $\frac{dP}{dt} - c_1P = -c_2P^2$. Set $z = P^{1-2} = P^{-1}$, $P = z^{-1}$, $\frac{dP}{dt} = \frac{dP}{dz} \cdot \frac{dz}{dt} = -z^{-2} \cdot z'$,
 $-z^{-2} \cdot z' - c_1z^{-1} = -c_2z^{-2}$, $(-z^2) \times [-z^{-2} \cdot z' - c_1z^{-1} = -c_2z^{-2}]$
 $z' + c_1z = c_2 \Rightarrow [e^{c_1t} \cdot z]' = c_2e^{c_1t} \Rightarrow z = \frac{c_2}{c_1} + De^{-c_1t} \Rightarrow P(t)^{-1} = \frac{c_2}{c_1} + De^{-c_1t}$

1-4 Homogeneous & Quasi-homogeneous Differential Equations

The first-order homogeneous differential equation: $y'=f(y/x)$

Solution: Set $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x \frac{du}{dx} \Rightarrow x \frac{du}{dx} + u = f(u) \Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$ is a separable differential equation.

Eg. Solve $x \frac{dy}{dx} = \frac{y^2}{x} + y$.

(Sol.) Set $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} \Rightarrow u^2 + u = u + x \frac{du}{dx}$

$$\Rightarrow u^2 = x \frac{du}{dx}, \frac{dx}{x} = \frac{du}{u^2}, \ln|x| + c = -\frac{1}{u}$$

$$\Rightarrow u = \frac{-1}{\ln|x| + c}, y = xu = \frac{-x}{\ln|x| + c}$$

Eg. Solve $\frac{dy}{dx} = \frac{x+y}{x-y}$.

(Sol.) $\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right)} \Rightarrow \frac{1+u}{1-u} = u + x \frac{du}{dx} \Rightarrow x \frac{du}{dx} = \frac{1+u-u+u^2}{1-u} = \frac{1+u^2}{1-u}$

$$\Rightarrow \left(\frac{1-u}{1+u^2}\right) du = \frac{dx}{x}, \tan^{-1}(u) - \frac{1}{2} \ln|1+u^2| = \ln|x| + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln\left|1 + \left(\frac{y}{x}\right)^2\right| = \ln|x| + c \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \ln|x^2 + y^2| = c$$

Eg. Solve $(x - \sqrt{xy}) y' = y$. [1990 中山電研]

(Sol.) $(1 - \sqrt{\frac{y}{x}}) \frac{dy}{dx} = \frac{y}{x}$. Let $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x \frac{du}{dx}$

$$\Rightarrow \frac{dx}{x} = \left(\frac{1 - \sqrt{u}}{u\sqrt{u}}\right) du \Rightarrow \ln|x| + C = \int \frac{1}{u\sqrt{u}} du - \int \frac{1}{u} du = \int u^{-\frac{3}{2}} du - \ln|u| = -2u^{-\frac{1}{2}} - \ln|u|$$

$$\Rightarrow \ln|x| + 2\sqrt{\frac{x}{y}} + \ln\left(\frac{y}{x}\right) = C \Rightarrow \ln|y| + 2\sqrt{\frac{x}{y}} = C$$



Eg. Solve $y'=4y/(4x-y)$. [文化電機轉學考]

(Ans.) $\ln(|y|) = -\frac{4x}{y} + c$

Quasi-homogeneous differential equation: $\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+h}\right)$

Case 1 $ae-bd \neq 0$

Solution: Let A and B fulfill $\begin{cases} aA+bB+c=0 \\ dA+eB+h=0 \end{cases}$, and set $x = X + A$, $y = Y + B$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dY}{dX} = f\left(\frac{ax+by+c}{dx+ey+h}\right) = f\left(\frac{a(X+A)+b(Y+B)+c}{d(X+A)+e(Y+B)+h}\right) \\ &= f\left(\frac{aX+bY+aA+bB+c}{dX+eY+dA+eB+h}\right) \\ &\Rightarrow \frac{dY}{dX} = f\left(\frac{aX+bY}{dX+eY}\right) \text{ is a homogeneous equation.} \end{aligned}$$

Eg. Solve $\frac{dy}{dx} = \frac{2x+y-1}{x-2}$.

(Sol.) $a = 2, b = 1, c = -1, d = 1, e = 0, h = -2, ae - bd = 0 - 1 = -1 \neq 0$

$$\begin{cases} 2A+B-1=0 \\ A-2=0 \end{cases} \Rightarrow \begin{matrix} A=2 \\ B=-3 \end{matrix} \Rightarrow x = X+2, y = Y-3$$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{2(X+2)+(Y-3)-1}{X} = \frac{2X+Y}{X} = 2 + \left(\frac{Y}{X}\right) = u + 2$$

$$X \frac{du}{dX} + u = u + 2, \quad u = \ln(cX^2) \Rightarrow \frac{y+3}{x-2} = \ln[c(x-2)^2]$$

Case 2 $ae-bd=0$

Solution: Set $v = \frac{ax+by}{a} = \frac{dx+ey}{d}$, $y = \frac{a}{b}(v-x) \Rightarrow \frac{dy}{dx} = \frac{a}{b}\left(\frac{dv}{dx} - 1\right)$

$\therefore \frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+h}\right) \Rightarrow \frac{a}{b}\left(\frac{dv}{dx} - 1\right) = f\left(\frac{av+c}{dv+h}\right) \Rightarrow \frac{dv}{dx} = 1 + \frac{b}{a}f\left(\frac{av+c}{dv+h}\right)$ is a separable equation for v and x

Eg. Solve $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y-4}$.

(Sol.) $a = 2, b = 1, c = -1, d = 4, e = 2, h = -4$

$$\because ae - bd = 0, \therefore v = \frac{2x+y}{2} = \frac{4x+2y}{4}$$

$$\Rightarrow \frac{dv}{dx} = 1 + \frac{1}{2}\left(\frac{2v-1}{4v-4}\right) \Rightarrow \left(\frac{8v-8}{10v-9}\right)dv = dx \Rightarrow \frac{4v}{5} - \frac{2}{25} \ln|10v-9| + c = x$$

$$\frac{2}{5}(2x+y) - \frac{2}{25} \ln|10x+5y-9| + c = x$$

1-5 Exact Differential Equations and Integrating Factors

Exact differential equation: $M(x,y)dx+N(x,y)dy=0$ if $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$

Solution: $\exists F(x,y)$ fulfills $\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$

$$\Rightarrow dF = \frac{\partial F(x,y)}{\partial x} \cdot dx + \frac{\partial F(x,y)}{\partial y} dy = M(x,y)dx + N(x,y)dy = 0$$

Solve $\frac{\partial F(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial F(x,y)}{\partial y} = N(x,y)$

$\Rightarrow F(x,y)=C$ is its solution.

Eg. Solve $(6xy-y^3)dx+(4y+3x^2-3xy^2)dy=0$. [2011 中正電研]

$$(\text{Sol.}) \frac{\partial(6xy-y^3)}{\partial y} = 6x-3y^2 = \frac{\partial(4y+3x^2-3xy^2)}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x} = 6xy-y^3 \Rightarrow F(x,y) = 3x^2y - xy^3 + c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y} = 4y+3x^2-3xy^2 \Rightarrow F(x,y) = 2y^2 + 3x^2y - xy^3 + c_2(x)$$

$$\Rightarrow F(x,y) = 2y^2 + 3x^2y - xy^3 + c, \therefore 2y^2 + 3x^2y - xy^3 = C$$

Eg. Solve $y'[\sinh(3y)-2xy]=y^2$. [2013 中央電研固態組、生醫電子組]

$$(\text{Sol.}) y^2 dx + [2xy - \sinh(3y)] dy = 0$$

$$\frac{\partial(y^2)}{\partial y} = 2y = \frac{\partial(2xy - \sinh(3y))}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x} = y^2 \Rightarrow F(x,y) = xy^2 + c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y} = 2xy - \sinh(3y) \Rightarrow F(x,y) = xy^2 - \frac{\cosh(3y)}{3} + c_2(x)$$

$$\Rightarrow F(x,y) = xy^2 - \frac{\cosh(3y)}{3} + c, \therefore xy^2 - \frac{\cosh(3y)}{3} = C$$

Eg. Solve $y=(y^2-x)y'$.

$$(\text{Sol.}) y dx + (x - y^2) dy = 0$$

$$\Rightarrow \frac{\partial(y)}{\partial y} = \frac{\partial(x - y^2)}{\partial x} = 1, F(x,y) = \int y dx + c_1 = \int (x - y^2) dy + c_2 = xy - \frac{y^3}{3} + c_2,$$

$$\therefore xy - \frac{y^3}{3} = C$$

Eg. Solve $\frac{dy}{dx} = \frac{-2xy^3 - 2}{3x^2y^2 + e^y}$.

(Sol.) $(2xy^3 + 2)dx + (3x^2y^2 + e^y)dy = 0$, $\frac{\partial(2xy^3 + 2)}{\partial y} = 6xy^2 = \frac{\partial(3x^2y^2 + e^y)}{\partial x}$

$$\frac{\partial F(x, y)}{\partial x} = 2xy^3 + 2 \Rightarrow F(x, y) = x^2y^3 + 2x + c_1(y)$$

$$\frac{\partial F(x, y)}{\partial y} = 3x^2y^2 + e^y \Rightarrow F(x, y) = x^2y^3 + e^y + c_2(x)$$

$$\Rightarrow F(x, y) = x^2y^3 + 2x + e^y + c, \therefore x^2y^3 + 2x + e^y = C$$

Integrating factor $u(x,y)$: For $M(x,y)dx + N(x,y)dy = 0$, in case

$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$ but $\frac{\partial [u(x, y)M(x, y)]}{\partial y} = \frac{\partial [u(x, y)N(x, y)]}{\partial x}$, and then $u(x, y)$ is called the integrating factor.

Eg. Solve $(y^2 - 6xy)dx + (3xy - 6x^2)dy = 0$.

(Sol.) $\frac{\partial(y^2 - 6xy)}{\partial y} = 2y - 6x \neq 3y - 12x = \frac{\partial(3xy - 6x^2)}{\partial x}$

Choose $u(x, y) = y \Rightarrow (y^3 - 6xy^2)dx + (3xy^2 - 6x^2y)dy = 0$

$$\frac{\partial(y^3 - 6xy^2)}{\partial y} = 3y^2 - 12xy = \frac{\partial(3xy^2 - 6x^2y)}{\partial x}$$

$$\frac{\partial F(x, y)}{\partial x} = y^3 - 6xy^2 \Rightarrow F(x, y) = xy^3 - 3x^2y^2 + c_1(y)$$

$$\frac{\partial F(x, y)}{\partial y} = 3xy^2 - 6x^2y \Rightarrow F(x, y) = xy^3 - 3x^2y^2 + c_2(x). \therefore xy^3 - 3x^2y^2 = C$$

Another Method: $\frac{dy}{dx} = -\frac{y^2 - 6xy}{3xy - 6x^2}$ is the first-order homogeneous equation.

$$\frac{dy}{dx} = -\frac{y^2 - 6xy}{3xy - 6x^2} = -\frac{\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right)}{3\left(\frac{y}{x}\right) - 6}. \text{ Let } u = \frac{y}{x} \Rightarrow y' = u + x \frac{du}{dx}$$

$$\frac{3u - 6}{-4u^2 + 12u} du = \frac{dx}{x} \Rightarrow -\frac{3}{4} \int \frac{u - 2}{u(u - 3)} du = \ln|x| + c$$

$$\Rightarrow -\frac{1}{4} \left[\int \frac{2du}{u} + \int \frac{du}{u - 3} \right] = \ln|x| + c \Rightarrow \ln|u^2(u - 3)| = -4\ln|x| - 4c$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 \left(\frac{y}{x} - 3\right) = \frac{A}{x^4}, xy^3 - 3x^2y^2 = A, xy^3 - 3x^2y^2 = A$$

Eg. Solve $(x+e^y)dy-dx=0$.

(Sol.) $M(x, y) = -1$, $N(x, y) = x + e^y$, $\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$

Choose integrating factor: $e^{-y} \Rightarrow -e^{-y}dx + (xe^{-y} + 1)dy = 0$

$$\frac{\partial(-e^{-y})}{\partial y} = e^{-y} = \frac{\partial(xe^{-y} + 1)}{\partial x}$$

$$F(x, y) = -xe^{-y} + c_1(x, y) = -xe^{-y} + y + c_2(x, y), \therefore -xe^{-y} + y = C$$

Another method: $\frac{dx}{dy} - x = e^y$ is the first-order linear differential equation for $x(y)$.

$$\frac{dx}{dy} - x = e^y, \quad p(y) = -1, \quad e^{\int p(y)dy} = e^{-y}, \quad e^{-y} \frac{dx}{dy} - e^{-y}x = e^{-y}e^y = 1$$

$$(e^{-y}x)' = 1, \quad e^{-y}x = y + c, \quad -xe^{-y} + y = C$$

1-6 Riccati's Equation $y' = P(x)y^2 + Q(x)y + R(x)$

Suppose that there exists one specific solution $y=S(x)$, then a general solution can be obtained as follows

$$y = S(x) + \frac{1}{z}, \quad y' = S'(x) - \frac{1}{z^2} \cdot z'$$

$$S'(x) - \frac{1}{z^2} \cdot z' = P(x) \cdot \left[S^2(x) + \frac{2S(x)}{z} + \frac{1}{z^2} \right] + Q(x) \cdot \left[S(x) + \frac{1}{z} \right] + R(x)$$

$$-\frac{1}{z^2} \cdot z' = P(x) \cdot \frac{1}{z^2} + 2P(x)S(x) \frac{1}{z} + Q(x) \cdot \frac{1}{z}$$

$z' + [2P(x)S(x) + Q(x)]z = -P(x)$ is the 1st-order linear differential equation.

Eg. Solve $y' = e^{-3x}y^2 - y + 3e^{3x}$.

(Sol.) $y = e^{3x}$ is a solution $\Rightarrow y = e^{3x} + \frac{1}{z}, y' = 3e^{3x} - \frac{z'}{z^2}$

$$3e^{3x} - \frac{z'}{z^2} = e^{-3x} \cdot \left(e^{6x} + \frac{2e^{3x}}{z} + \frac{1}{z^2} \right) - \left(e^{3x} + \frac{1}{z} \right) + 3e^{3x}$$

$$z' = -2z - e^{-3x} + z, z' + z = -e^{-3x}, z'e^x + e^x \cdot z = -e^{-2x}$$

$$(z \cdot e^x)' = -e^{-2x}, \quad ze^x = \frac{1}{2}e^{-2x} + c, \quad \therefore y = e^{3x} + \frac{2}{e^{-3x} + 2ce^{-x}}$$

1-7 Some Theorems on the First-order Ordinary Differential Equations

A family of curves $F(x,y,k)=0$ is a solution of $y'=f(x,y)$.

Eg. A family of circles $x^2+y^2-k^2=0$ is a solution of $y'=-x/y$.

Theorem An oblique trajectory intersecting $y'=f(x,y)$ at an angle α is $y' = \frac{f(x,y) + \tan(\alpha)}{1 - f(x,y)\tan(\alpha)}$; particularly, if $\alpha=\pi/2$, then the orthogonal trajectory is $y'=-1/f(x,y)$.

Eg. Find the families of oblique trajectories intersecting the circle $x^2+y^2=k^2$ at angles of 45° and 90° .

(Sol.) $x^2 + y^2 = k^2 \leftrightarrow y' = -\frac{x}{y} = f(x, y)$

$$1. \tan(45^\circ)=1, y' = \frac{-\frac{x}{y} + 1}{1 + \frac{x}{y}} = \frac{y-x}{y+x} = \frac{\left(\frac{y}{x}\right) - 1}{\left(\frac{y}{x}\right) + 1} = \frac{v-1}{v+1}$$

$$v + xv' = \frac{v-1}{v+1}, v' = \frac{v-1-v^2-v}{(v+1)x} = -\frac{1+v^2}{v+1} \cdot \frac{1}{x}$$

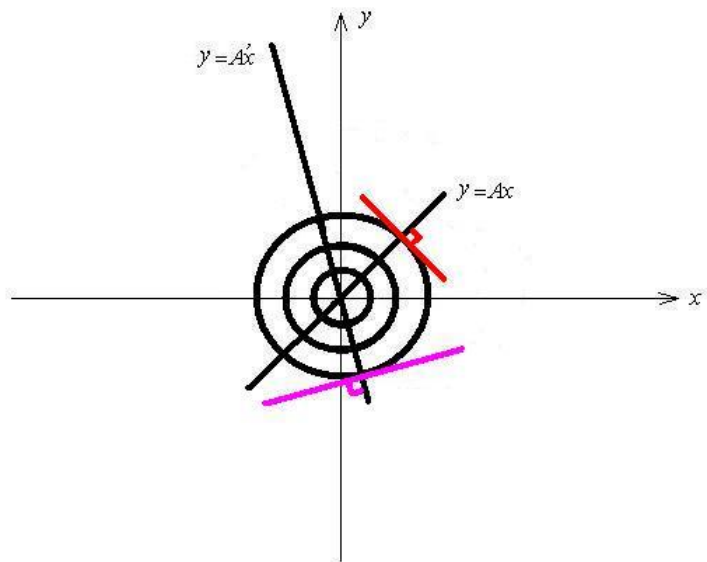
$$\left(\frac{1+v}{1+v^2}\right)dv = -\frac{dx}{x} \Rightarrow \frac{1}{2} \ln|1+v^2| + \tan^{-1}(v) = -\ln|x| + c$$

$$\therefore \frac{1}{2} \ln\left|1 + \left(\frac{y}{x}\right)^2\right| + \tan^{-1}\left(\frac{y}{x}\right) = -\ln|x| + c$$

2.

$$y' = -\frac{1}{\left(-\frac{x}{y}\right)} = \frac{y}{x}, \frac{dy}{y} = \frac{dx}{x},$$

$$\therefore y = Ax$$

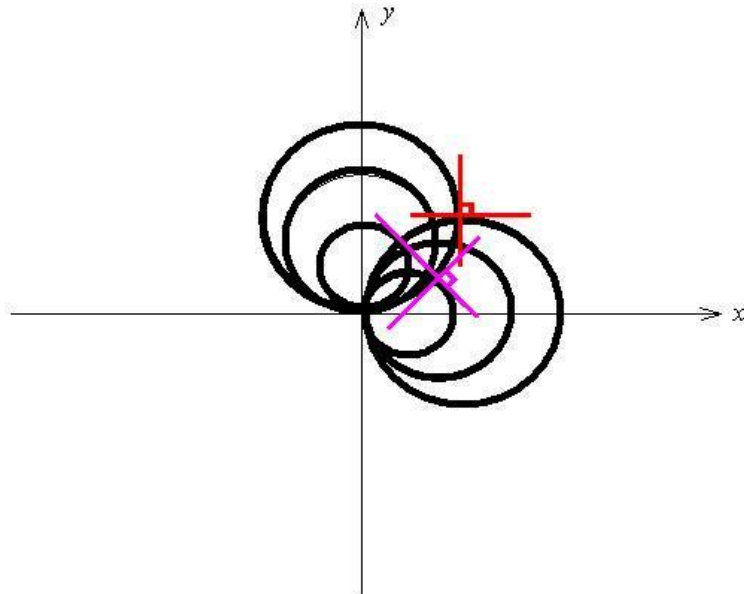


Theorem A family of curves $F(\theta, r, k) = 0$, of which differential equation is $f(\theta, r, r') = 0$. Then the family of orthogonal trajectories has differential equation $f(\theta, r, -r^2/r') = 0$, where $r' = dr/d\theta$.

Eg. Find the family of trajectories orthogonal to $r = k \cos(\theta)$.

(Sol.) $r = k \cos(\theta)$, $r' = -k \sin(\theta) \Rightarrow r = -\cot(\theta) r'$

Family of orthogonal trajectories: $r = -\cot(\theta) \cdot \left(-\frac{r^2}{r'}\right) \Rightarrow \frac{r}{r'} = \tan(\theta) \Rightarrow r = k' \sin(\theta)$



1-8 Solutions of the First-order Ordinary Differential Equations by Matlab language and Wolfram Alpha

In **Matlab** language, we can use the following instructions to obtain the solution of the first-order ordinary differential equation:

```
>> soln=dsolve('Dy=3*y+exp(2*x)', 'y(0) = 3') % solve y'=3y+exp(2x), y(0)=3
```

```
ans=-exp(2*x)+4*exp(3*x)
```

Wolfram Alpha Address : <https://www.wolframalpha.com/examples/mathematics/>

Eg. Solve $y'+xy=0$.



$y'+xy=0$ ☆

Extended Keyboard

Upload

Examples

Random

Input:

$$y'(x) + x y(x) = 0$$

Open code

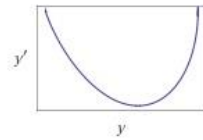
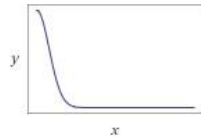
Differential equation solution:

Approximate form

Step-by-step solution

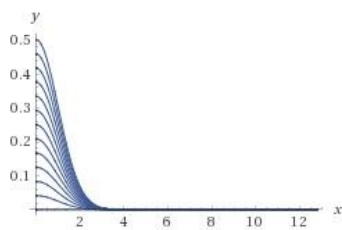
$$y(x) = c_1 e^{-x^2/2}$$

Plots of sample individual solution:



$$y(0) = 1$$

Sample solution family:



(sampling $y(0)$)