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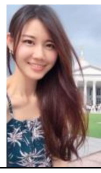
97	陳蕙質(F)	94245509	推甄上成大微電子研究所 
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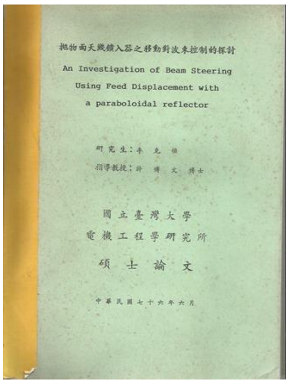

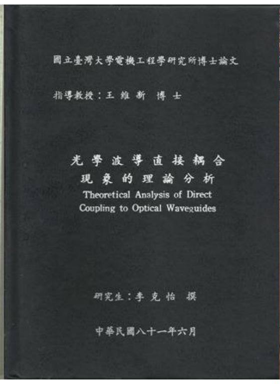

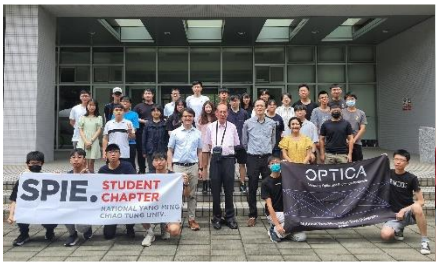

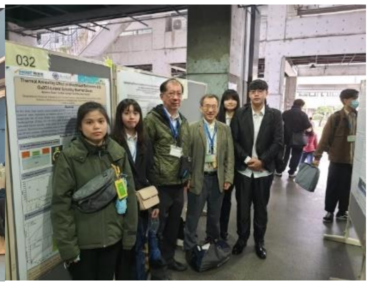

考試院聘書 考授題料三字第1111500684號

茲敦聘

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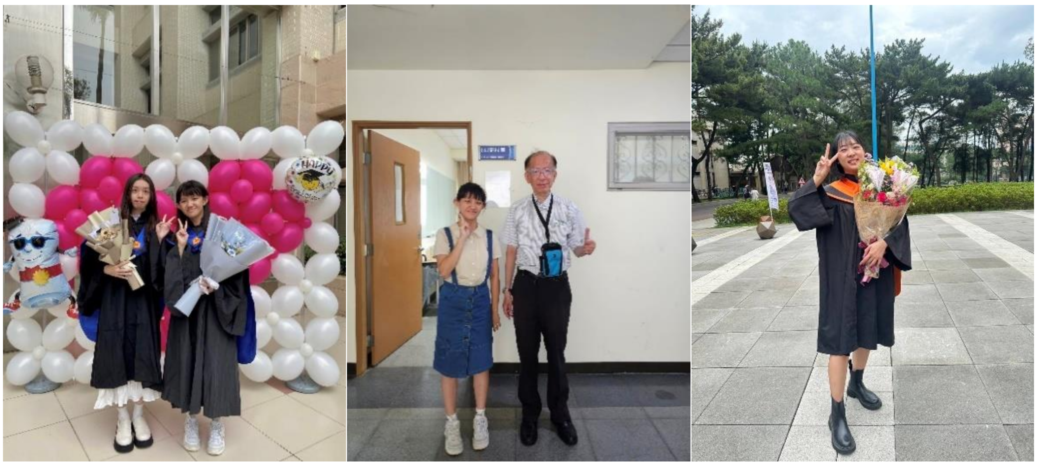
中華民國 111 年 5 月 14 日

優秀的學長姐獲得國立大學電資學院碩士



Chapter 1 The First-Order Ordinary Differential Equations (ODE)

1-1 Separable Differential Equation $A(x)dx=B(y)dy$

Solution: $\int A(x)dx=\int B(y)dy+C$

Eg. Solve (a) $y'=3x^2+1$, $y(1)=4$, (b) $6x-2yy'=0$, and (c) $2\cdot\frac{dy}{dx}-\frac{1}{y}=\frac{2x}{y}$, $y(0)=0$.

(Sol.) (a) $dy=(3x^2+1)dx$, $y=x^3+x+c$, $y(1)=4\Rightarrow c=2$, $\therefore$ $y=x^3+x+2$.

(b) $2ydy=6xdx\Rightarrow y^2=3x^2+c$.

(c) $2ydy=(1+2x)dx$, $y^2+c=x+x^2$, $y(0)=0\Rightarrow c=0$, $\therefore$ $y^2=x+x^2$.

Formulae:

$$\int x^m dx=\begin{cases}\frac{x^{m+1}}{m+1}+c, & m\neq -1\\ \ln|x|+c, & m=-1\end{cases} \qquad x^{-n}=\frac{1}{x^n} \qquad \int e^{ax}dx=\frac{e^{ax}}{a}+c$$

$$\int \sin(ax)dx=-\frac{\cos(ax)}{a}+c \qquad \int \cos(ax)dx=\frac{\sin(ax)}{a}+c$$

Eg. Solve $y'=xe^{x-y}$ with the boundary condition: $y=\ln 2$ at $x=0$. [2005 台大電研]

(Sol.) $e^y dy=xe^x dx\Rightarrow e^y=xe^x-e^x+c$, $y(x=0)=\ln2\Rightarrow c=3$, $\therefore$ $e^y=xe^x-e^x+3$.

Eg. Solve $y'=y^2e^{2t}$ with $y(0)=2$. [2010 台大生醫電資所]

(Sol.) $\frac{1}{y^2}dy=e^{2t}dt\Rightarrow \frac{-1}{y}=\frac{e^{2t}}{2}+c$, $y(t=0)=2\Rightarrow c=-1$, $\therefore$ $\frac{-1}{y}=\frac{e^{2t}}{2}-1$.

1-2 The first-order Linear Differential Equation $y'+p(x)y=q(x)$

Solution: $y'\cdot e^{\int p(x)dx}+p(x)y\cdot e^{\int p(x)dx}=q(x)\cdot e^{\int p(x)dx}$

$$\Rightarrow\left[y\cdot e^{\int p(x)dx}\right]'=q(x)\cdot e^{\int p(x)dx}\Rightarrow ye^{\int p(x)dx}=\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c$$

$$\Rightarrow y(x)=e^{-\int p(x)dx}\cdot\left\{\int\left[q(x)\cdot e^{\int p(x)dx}\right]dx+c\right\}$$

Formulae:

$$A^B=e^{B\ln(A)} \qquad e^{\pm n\ln(x)}=x^{\pm n}$$

Eg. Solve $xy'+2y=3x^3$.

(Sol.) $y'+\frac{2}{x}y=3x^2$, $p(x)=\frac{2}{x}$, $\int p(x)dx=2ln(x)$, $e^{2ln(x)}=x^2\Rightarrow x^2y'+2xy=3x^4$

$$\Rightarrow\left(x^2y\right)'=3x^4,\ x^2y=\frac{3x^5}{5}+c\Rightarrow y(x)=\frac{3x^3}{5}+\frac{c}{x^2}$$

Eg. Solve $y'+y=\sin(x)$.

(Sol.) $p(x)=1$, $\int p(x)dx=x \Rightarrow y'\cdot e^x+e^x\cdot y=e^x\cdot\sin(x) \Rightarrow (e^x\cdot y)'=e^x\cdot\sin(x)$,

$e^x\cdot y=\dfrac{e^x[\sin(x)-\cos(x)]}{2}+c \Rightarrow y(x)=\dfrac{\sin(x)-\cos(x)}{2}+c\cdot e^{-x}$

Eg. Solve (a) $y'=\dfrac{2}{x}y+x^2e^x$ and (b) $y'+3x^2y=xe^{-x^3}$. [2013 成大電研]

(Sol.) (a) $y'-\dfrac{2}{x}y=x^2e^x$, $p(x)=-\dfrac{2}{x}$, $\int p(x)dx=-2ln(x)$, $e^{-2ln(x)}=x^{-2}$

$\Rightarrow x^{-2}y'-2x^{-3}y=e^x$, $[x^{-2}y]'=e^x$, $x^{-2}y=e^x+c \Rightarrow y=x^2e^x+cx^2$

(b) $y'+3x^2y=xe^{-x^3}$, $p(x)=3x^2$, $\int p(x)dx=x^3 \Rightarrow e^{x^3}y'+3x^2e^{x^3}y=x$, $[e^{x^3}y]'=x$,

$e^{x^3}y=\dfrac{x^2}{2}+c \Rightarrow y=(\dfrac{x^2}{2}+c)\cdot e^{-x^3}$

Eg. Solve $y'+y\cot(x)=5e^{\cos(x)}$. [2013 師大應用電子所]

(Sol.) $p(x)=\cot(x)$, $\int p(x)dx=ln|\sin(x)|$, $e^{ln|\sin(x)|}=\sin(x) \Rightarrow \sin(x)y'+y\cos(x)=5\sin(x)e^{\cos(x)}$

$\Rightarrow [y\sin(x)]'=5\sin(x)e^{\cos(x)} \Rightarrow y\sin(x)=-5e^{\cos(x)}+c$, $\therefore y(x)=\dfrac{-5e^{\cos(x)}+c}{\sin(x)}$

Eg. Solve $(x^2+1)y'+(x^2+2x+1)y=x^2+1$, $y(0)=1$. [2017 台大生醫電資所]

(Sol.) $y'+\dfrac{x^2+2x+1}{x^2+1}y=1$, $y'+(1+\dfrac{2x}{x^2+1})y=1$, $\int(1+\dfrac{2x}{x^2+1})dx=x+\ln(x^2+1)$

$\exp[x+\ln(x^2+1)]=(x^2+1)e^x$, $(x^2+1)e^x\cdot[y'+\dfrac{x^2+2x+1}{x^2+1}y=1]$,

$(x^2+1)e^x\cdot y'+(x^2+2x+1)\cdot e^x\cdot y=(x^2+1)e^x$, $[(x^2+1)e^x\cdot y]'=(x^2+1)e^x$,

$(x^2+1)e^x\cdot y=(x^2-2x+3)e^x+C$, $y(0)=1 \Rightarrow C=-2$, $\therefore y=\dfrac{x^2-2x+3-2e^{-x}}{x^2+1}$

1-3 Bernoulli Differential Equations $y'+p(x)y=r(x)y^{\alpha}$ (It is nonlinear if $\alpha\neq1$)

Solution: Set $z=y^{1-\alpha}$, $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}=\dfrac{y^{\alpha}}{1-\alpha}\cdot\dfrac{dz}{dx} \Rightarrow \dfrac{y^{\alpha}}{1-\alpha}\cdot\dfrac{dz}{dx}+p(x)\cdot zy^{\alpha}=r(x)y^{\alpha}$

$\Rightarrow \dfrac{dz}{dx}+(1-\alpha)p(x)\cdot z=r(x)\cdot(1-\alpha)$

$\Rightarrow z(x)=e^{-\int(1-\alpha)p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)\cdot e^{\int(1-\alpha)p(x)dx}dx+c\right]$

$\Rightarrow [y(x)]^{1-\alpha}=e^{-(1-\alpha)\int p(x)dx}\cdot\left[(1-\alpha)\cdot\int r(x)e^{(1-\alpha)\int p(x)dx}dx+c\right]$

Eg. Solve $y'=y(xy^3-1)$. [2004 台大電研]

(Sol.) $y'+y=xy^4$. Set $z=y^{1-4}=y^{-3}$, $y=z^{-1/3}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{3}z^{-4/3}\cdot z'$

$-\frac{1}{3}z^{-4/3}\cdot z'+z^{-1/3}=xz^{-4/3}$, $(-3z^{4/3})\times[-\frac{1}{3}z^{-4/3}\cdot z'+z^{-1/3}=xz^{-4/3}]$

$\Rightarrow z'-3z=-3x\Rightarrow\left[e^{-3x}\cdot z\right]'=-3x\cdot e^{-3x}\Rightarrow z=x+\frac{1}{3}+ce^{3x}\Rightarrow y^{-3}=x+\frac{1}{3}+ce^{3x}$

Eg. Solve $\frac{dy(x)}{dx}-y(x)+e^{2x}y^2(x)=0$. [2015 台大電研]

(Sol.) $y'-y=-e^{2x}y^2$. Set $z=y^{1-2}=y^{-1}$, $y=z^{-1}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{z^2}\cdot z'$

$-z^{-2}\cdot z'-z^{-1}=-e^{2x}z^{-2}$, $(-z^2)\times[-z^{-2}\cdot z'-z^{-1}=-e^{2x}z^{-2}]$

$\Rightarrow z'+z=e^{2x}\Rightarrow\left[e^{x}\cdot z\right]'=e^{3x}\Rightarrow z=\frac{e^{2x}}{3}+ce^{-x}\Rightarrow y^{-1}=\frac{e^{2x}}{3}+ce^{-x}$

Eg. Solve $xy'+y=4x^2y^2$. [2015 師大電研]

(Sol.) $y'+\frac{1}{x}y=4xy^2$. Set $z=y^{1-2}=y^{-1}$, $y=z^{-1}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{z^2}\cdot z'$

$-\frac{1}{z^2}\cdot z'+\frac{1}{xz}=\frac{4x}{z^2}$, $-z'+\frac{z}{x}=4x$, $z'-\frac{z}{x}=-4x$, $p(x)=-\frac{1}{x}$, $\int p(x)dx=-ln(x)$, $e^{-ln(x)}=x^{-1}$

$x^{-1}z'-\frac{z}{x^2}=-4$, $[x^{-1}z]'=-4$, $x^{-1}z=-4x+c$, $z=-4x^2+cx\Rightarrow\frac{1}{y}=-4x^2+cx$

Eg. Solve $y'+\frac{1}{3}y=\frac{(1-2x)}{3}\cdot y^4$. [2013 師大應用電子所]

(Sol.) Set $z=y^{1-4}=y^{-3}$, $y=z^{-1/3}$, $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{1}{3}z^{-4/3}\cdot z'$,

$-\frac{1}{3}z^{-4/3}\cdot z'+\frac{1}{3}z^{-1/3}=\frac{(1-2x)}{3}\cdot z^{-4/3}$, $(-3z^{4/3})\times[-\frac{1}{3}z^{-4/3}\cdot z'+\frac{1}{3}z^{-1/3}=\frac{(1-2x)}{3}\cdot z^{-4/3}]$

$\Rightarrow z'-z=2x-1$, $[e^{-x}\cdot z]'=(2x-1)e^{-x}\Rightarrow e^{-x}\cdot z=(-2x-1)e^{-x}+c$,

$z=-2x-1+c\,e^{x}$, $\therefore y^{-3}=-2x-1+c\,e^{x}$

Eg. Solve $\frac{dP(t)}{dt}=P(t)\cdot[c_1-c_2P(t)]$. [2003 台科大電研]

(Sol.) $\frac{dP}{dt}-c_1P=-c_2P^2$. Set $z=P^{1-2}=P^{-1}$, $P=z^{-1}$, $\frac{dP}{dt}=\frac{dP}{dz}\cdot\frac{dz}{dt}=-z^{-2}\cdot z'$,

$-z^{-2}\cdot z'-c_1z^{-1}=-c_2z^{-2}$, $(-z^2)\times[-z^{-2}\cdot z'-c_1z^{-1}=-c_2z^{-2}]$

$z'+c_1z=c_2\Rightarrow\left[e^{c_1t}\cdot z\right]'=c_2e^{c_1t}\Rightarrow z=\frac{c_2}{c_1}+De^{-c_1t}\Rightarrow P(t)^{-1}=\frac{c_2}{c_1}+De^{-c_1t}$

1-4 Homogeneous & Quasi-homogeneous Differential Equations

The first-order homogeneous differential equation: $y'=f(y/x)$

Solution: Set $u=\dfrac{y}{x}$, $y=ux$, $\dfrac{dy}{dx}=u+x\dfrac{du}{dx}\Rightarrow x\dfrac{du}{dx}+u=f(u)\Rightarrow\dfrac{du}{f(u)-u}=\dfrac{dx}{x}$ is a separable differential equation.

Eg. Solve $x\dfrac{dy}{dx}=\dfrac{y^2}{x}+y$.

(Sol.) Set $u=\dfrac{y}{x}$, $y=ux$, $\dfrac{dy}{dx}=u+x\dfrac{du}{dx}\Rightarrow\dfrac{dy}{dx}=\left(\dfrac{y}{x}\right)^2+\dfrac{y}{x}\Rightarrow u^2+u=u+x\dfrac{du}{dx}$

$$\Rightarrow u^2=x\frac{du}{dx},\ \frac{dx}{x}=\frac{du}{u^2},\ \ell n|x|+c=-\frac{1}{u}$$

$$\Rightarrow u=\frac{-1}{\ell n|x|+c},\ y=xu=\frac{-x}{\ell n|x|+c}$$

Eg. Solve $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$.

(Sol.) $$\frac{dy}{dx}=\frac{1+\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)}\Rightarrow\frac{1+u}{1-u}=u+x\frac{du}{dx}\Rightarrow x\frac{du}{dx}=\frac{1+u-u+u^2}{1-u}=\frac{1+u^2}{1-u}$$

$$\Rightarrow\left(\frac{1-u}{1+u^2}\right)du=\frac{dx}{x},\ \tan^{-1}(u)-\frac{1}{2}\ell n|1+u^2|=\ell n|x|+c$$

$$\Rightarrow\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ell n\left|1+\left(\frac{y}{x}\right)^2\right|=\ell n|x|+c\Rightarrow\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ell n\left|x^2+y^2\right|=c$$

Eg. Solve $\left(x-\sqrt{xy}\right)y'=y$. **[1990 中山電研]**

(Sol.) $(1-\sqrt{\dfrac{y}{x}})\dfrac{dy}{dx}=\dfrac{y}{x}$. Let $u=\dfrac{y}{x}$, $y=ux$, $\dfrac{dy}{dx}=u+x\dfrac{du}{dx}$

$$\Rightarrow\frac{dx}{x}=\left(\frac{1-\sqrt{u}}{u\sqrt{u}}\right)du\Rightarrow\ell n|x|+C=\int\frac{1}{u\sqrt{u}}du-\int\frac{1}{u}du=\int u^{\frac{-3}{2}}du-\ell n|u|=-2u^{\frac{-1}{2}}-\ell n|u|$$

$$\Rightarrow\ell n|x|+2\sqrt{\frac{x}{y}}+\ell n\left(\frac{y}{x}\right)=C\Rightarrow\ell n|y|+2\sqrt{\frac{x}{y}}=C$$

Eg. Solve $y'=4y/(4x-y)$. [文化電機轉學考]

(Ans.) $\ln(|y|)=-\dfrac{4x}{y}+c$

Quasi-homogeneous differential equation: $\frac{dy}{dx}=f\left(\frac{ax+by+c}{dx+ey+h}\right)$

Case 1 ae-bd≠0

Solution: Let A and B fulfill $\begin{cases}aA+bB+c=0\\dA+eB+h=0\end{cases}$, and set $x=X+A,\ y=Y+B$

$$\frac{dy}{dx}=\frac{dY}{dX}=f\left(\frac{ax+by+c}{dx+ey+h}\right)=f\left(\frac{a(X+A)+b(Y+b)+c}{d(X+A)+e(Y+B)+h}\right)$$

$$=f\left(\frac{aX+bY+aA+bB+c}{dX+eY+dA+eB+h}\right)\Rightarrow\frac{dY}{dX}=f\left(\frac{aX+bY}{dX+eY}\right)$$ is a homogeneous equation.

Eg. Solve $\frac{dy}{dx}=\frac{2x+y-1}{x-2}$.

(Sol.) $a=2,\ b=1,\ c=-1,\ d=1,\ e=0,\ h=-2,\ ae-bd=0-1=-1\neq 0$

$$\begin{cases}2A+B-1=0\\A-2=0\end{cases}\Rightarrow\begin{array}{l}A=2\\B=-3\end{array}\Rightarrow x=X+2,\ y=Y-3$$

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{2(X+2)+(Y-3)-1}{X}=\frac{2X+Y}{X}=2+\left(\frac{Y}{X}\right)=u+2$$

$$X\frac{du}{dX}+u=u+2,\quad u=\ln(cX^2)\Rightarrow\frac{y+3}{x-2}=\ln[c(x-2)^2]$$

Case 2 ae-bd=0

Solution: Set $v=\frac{ax+by}{a}=\frac{dx+ey}{d}$, $y=\frac{a}{b}(v-x)\Rightarrow\frac{dy}{dx}=\frac{a}{b}\left(\frac{dv}{dx}-1\right)$

$\therefore\ \frac{dy}{dx}=f\left(\frac{ax+by+c}{dx+ey+h}\right)\Rightarrow\frac{a}{b}\left(\frac{dv}{dx}-1\right)=f\left(\frac{av+c}{dv+h}\right)\Rightarrow\frac{dv}{dx}=1+\frac{b}{a}f\left(\frac{av+c}{dv+h}\right)$ is a separable equation for v and x

Eg. Solve $\frac{dy}{dx}=\frac{2x+y-1}{4x+2y-4}$.

(Sol.) $a=2,\ b=1,\ c=-1,\ d=4,\ e=2,\ h=-4$

$$\because\ ae-bd=0,\ \therefore\ v=\frac{2x+y}{2}=\frac{4x+2y}{4}$$

$$\Rightarrow\frac{dv}{dx}=1+\frac{1}{2}\left(\frac{2v-1}{4v-4}\right)\Rightarrow\left(\frac{8v-8}{10v-9}\right)dv=dx\Rightarrow\frac{4v}{5}-\frac{2}{25}\ell n|10v-9|+c=x$$

$$\frac{2}{5}(2x+y)-\frac{2}{25}\ell n|10x+5y-9|+c=x$$

1-5 Exact Differential Equations and Integrating Factors

Exact differential equation: $M(x,y)dx+N(x,y)dy=0$ **if** $\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

Solution: $\exists F(x,y)$ fulfills $\frac{\partial^2 F(x,y)}{\partial x\partial y}=\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}$

$$\Rightarrow dF=\frac{\partial F(x,y)}{\partial x}\cdot dx+\frac{\partial F(x,y)}{\partial y}dy=M(x,y)dx+N(x,y)dy=0$$

Solve $\frac{\partial F(x,y)}{\partial x}=M(x,y)$ and $\frac{\partial F(x,y)}{\partial y}=N(x,y)$

$\Rightarrow$ $F(x,y)=C$ is its solution.

Eg. Solve $(6xy-y^3)dx+(4y+3x^2-3xy^2)dy=0$. [2011 中正電研]

(Sol.) $\frac{\partial(6xy-y^3)}{\partial y}=6x-3y^2=\frac{\partial(4y+3x^2-3xy^2)}{\partial x}$

$$\frac{\partial F(x,y)}{\partial x}=6xy-y^3\Rightarrow F(x,y)=3x^2y-xy^3+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=4y+3x^2-3xy^2\Rightarrow F(x,y)=2y^2+3x^2y-xy^3+c_2(x)$$

$$\Rightarrow F(x,y)=2y^2+3x^2y-xy^3+c\,,\ \therefore 2y^2+3x^2y-xy^3=C$$

Eg. Solve $y'[\sinh(3y)-2xy]=y^2$. [2013 中央電研固態組、生醫電子組]

(Sol.) $y^2dx+[2xy-\sinh(3y)]dy=0$

$$\frac{\partial(y^2)}{\partial y}=2y=\frac{\partial(2xy-\sinh(3y))}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x}=y^2\Rightarrow F(x,y)=xy^2+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=2xy-\sinh(3y)\Rightarrow F(x,y)=xy^2-\frac{\cosh(3y)}{3}+c_2(x)$$

$$\Rightarrow F(x,y)=xy^2-\frac{\cosh(3y)}{3}+c\,,\ \therefore\ xy^2-\frac{\cosh(3y)}{3}=C$$

Eg. Solve $y=(y^2-x)y'$.

(Sol.) $y\,dx+(x-y^2)dy=0$

$$\Rightarrow \frac{\partial(y)}{\partial y}=\frac{\partial(x-y^2)}{\partial x}=1\,,\ F(x,y)=\int y\,dx+c_1=\int(x-y^2)dy+c_2=xy-\frac{y^3}{3}+c_2\,,$$

$$\therefore\ xy-\frac{y^3}{3}=C$$

Eg. Solve $\dfrac{dy}{dx}=\dfrac{-2xy^3-2}{3x^2y^2+e^y}$.

(Sol.) $(2xy^3+2)dx+(3x^2y^2+e^y)dy=0$, $\dfrac{\partial(2xy^3+2)}{\partial y}=6xy^2=\dfrac{\partial(3x^2y^2+e^y)}{\partial x}$

$$\frac{\partial F(x,y)}{\partial x}=2xy^3+2\Rightarrow F(x,y)=x^2y^3+2x+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=3x^2y^2+e^y\Rightarrow F(x,y)=x^2y^3+e^y+c_2(x)$$

$$\Rightarrow F(x,y)=x^2y^3+2x+e^y+c,\ \therefore\ x^2y^3+2x+e^y=C$$

Integrating factor *u(x,y)*: For $M(x,y)dx+N(x,y)dy=0$, in case $\dfrac{\partial M(x,y)}{\partial y}\neq\dfrac{\partial N(x,y)}{\partial x}$ but $\dfrac{\partial[u(x,y)M(x,y)]}{\partial y}=\dfrac{\partial[u(x,y)N(x,y)]}{\partial x}$, and then $u(x,y)$ is called the integrating factor.

Eg. Solve $(y^2-6xy)dx+(3xy-6x^2)dy=0$.

(Sol.) $\dfrac{\partial(y^2-6xy)}{\partial y}=2y-6x\neq3y-12x=\dfrac{\partial(3xy-6x^2)}{\partial x}$

Choose $u(x,y)=y\Rightarrow(y^3-6xy^2)dx+(3xy^2-6x^2y)dy=0$

$$\frac{\partial(y^3-6xy^2)}{\partial y}=3y^2-12xy=\frac{\partial(3xy^2-6x^2y)}{\partial x}$$

$$\frac{\partial F(x,y)}{\partial x}=y^3-6xy^2\Rightarrow F(x,y)=xy^3-3x^2y^2+c_1(y)$$

$$\frac{\partial F(x,y)}{\partial y}=3xy^2-6x^2y\Rightarrow F(x,y)=xy^3-3x^2y^2+c_2(x).\ \therefore\ xy^3-3x^2y^2=C$$

Another Method: $\dfrac{dy}{dx}=-\dfrac{y^2-6xy}{3xy-6x^2}$ is the *first-order homogeneous equation*.

$\dfrac{dy}{dx}=-\dfrac{y^2-6xy}{3xy-6x^2}=-\dfrac{\left(\frac{y}{x}\right)^2-6\left(\frac{y}{x}\right)}{3\left(\frac{y}{x}\right)-6}$. Let $u=\dfrac{y}{x}\Rightarrow y'=u+x\dfrac{du}{dx}$

$$\frac{3u-6}{-4u^2+12u}du=\frac{dx}{x}\Rightarrow-\frac{3}{4}\int\frac{u-2}{u(u-3)}du=\ell n|x|+c$$

$$\Rightarrow-\frac{1}{4}\left[\int\frac{2du}{u}+\int\frac{du}{u-3}\right]=\ln|x|+c\Rightarrow\ell n\left|u^2(u-3)\right|=-4\ell n|x|-4c$$

$$\Rightarrow\left(\frac{y}{x}\right)^2\left(\frac{y}{x}-3\right)=\frac{A}{x^4},\ xy^3-3x^2y^2=A,\ \ xy^3-3x^2y^2=A$$

Eg. Solve $(x+e^y)dy-dx=0$.

(Sol.) $M(x,y)=-1,\ \ N(x,y)=x+e^y,\ \ \dfrac{\partial M(x,y)}{\partial y}\neq\dfrac{\partial N(x,y)}{\partial x}$

Choose integrating factor: $e^{-y}\Rightarrow -e^{-y}dx+\left(xe^{-y}+1\right)dy=0$

$$\frac{\partial\left(-e^{-y}\right)}{\partial y}=e^{-y}=\frac{\partial\left(xe^{-y}+1\right)}{\partial x}$$

$F(x,y)=-xe^{-y}+c_1(x,y)=-xe^{-y}+y+c_2(x,y)$, $\therefore$ $-xe^{-y}+y=C$

Another method: $\frac{dx}{dy}-x=e^y$ is the *first-order linear differential equation* for $x(y)$.

$$\frac{dx}{dy}-x=e^y,\ \ p(y)=-1,\ \ e^{\int p(y)dy}=e^{-y},\ \ e^{-y}\frac{dx}{dy}-e^{-y}x=e^{-y}e^y=1$$

$$\left(e^{-y}x\right)'=1,\ \ e^{-y}x=y+c,\ \ -xe^{-y}+y=C$$

1-6 Riccati's Equation $y'=P(x)y^2+Q(x)y+R(x)$

Suppose that there exists one specific solution $y=S(x)$, then a general solution can be obtained as follows

$$y=S(x)+\frac{1}{z},\ \ y'=S'(x)-\frac{1}{z^2}\cdot z'$$

$$S'(x)-\frac{1}{z^2}\cdot z'=P(x)\cdot\left[S^2(x)+\frac{2S(x)}{z}+\frac{1}{z^2}\right]+Q(x)\cdot\left[S(x)+\frac{1}{z}\right]+R(x)$$

$$-\frac{1}{z^2}\cdot z'=P(x)\cdot\frac{1}{z^2}+2P(x)S(x)\frac{1}{z}+Q(x)\cdot\frac{1}{z}$$

$z'+\left[2P(x)S(x)+Q(x)\right]z=-P(x)$ is the 1st-order linear differential equation.

Eg. Solve $y'=e^{-3x}y^2-y+3e^{3x}$.

(Sol.) $y=e^{3x}$ is a solution $\Rightarrow y=e^{3x}+\dfrac{1}{z},y'=3e^{3x}-\dfrac{z'}{z^2}$

$$3e^{3x}-\frac{z'}{z^2}=e^{-3x}\cdot\left(e^{6x}+\frac{2e^{3x}}{z}+\frac{1}{z^2}\right)-\left(e^{3x}+\frac{1}{z}\right)+3e^{3x}$$

$$z'=-2z-e^{-3x}+z,z'+z=-e^{-3x},\ \ z'e^x+e^x\cdot z=-e^{-2x}$$

$$\left(z\cdot e^x\right)'=-e^{-2x},\ \ ze^x=\frac{1}{2}e^{-2x}+c,\ \therefore\ \ y=e^{3x}+\frac{2}{e^{-3x}+2ce^{-x}}$$

1-7 Some Theorems on the First-order Ordinary Differential Equations

A family of curves $F(x,y,k)=0$ is a solution of $y'=f(x,y)$.

Eg. A family of circles $x^2+y^2-k^2=0$ is a solution of $y'=-x/y$.

Theorem An oblique trajectory intersecting $y'=f(x,y)$ at an angle α is $y'=\dfrac{f(x,y)+\tan(\alpha)}{1-f(x,y)\tan(\alpha)}$; particularly, if $\alpha=\pi/2$, then the orthogonal trajectory is $y'=-1/f(x,y)$.

Eg. Find the families of oblique trajectories intersecting the circle $x^2+y^2=k^2$ at angles of 45° and 90°.

(Sol.) $x^2+y^2=k^2 \leftrightarrow y'=-\dfrac{x}{y}=f(x,y)$

1. $\tan(45°)=1,\ y'=\dfrac{-\dfrac{x}{y}+1}{1+\dfrac{x}{y}}=\dfrac{y-x}{y+x}=\dfrac{\left(\dfrac{y}{x}\right)-1}{\left(\dfrac{y}{x}\right)+1}=\dfrac{v-1}{v+1}$

$$v+xv'=\frac{v-1}{v+1},\quad v'=\frac{v-1-v^2-v}{(v+1)x}=-\frac{1+v^2}{v+1}\cdot\frac{1}{x}$$

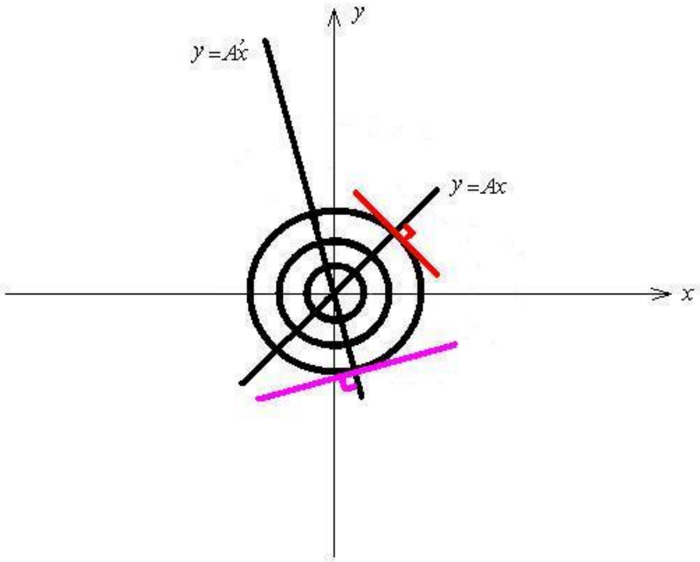
$$\left(\frac{1+v}{1+v^2}\right)dv=-\frac{dx}{x}\Rightarrow\frac{1}{2}\ell n\,|1+v^2|+\tan^{-1}(v)=-\ell n\,|x|+c$$

$$\therefore\ \frac{1}{2}\ell n\left|1+\left(\frac{y}{x}\right)^2\right|+\tan^{-1}\left(\frac{y}{x}\right)=-\ell n\,|x|+c$$

2.

$$y'=-\frac{1}{\left(-\dfrac{x}{y}\right)}=\frac{y}{x},\quad \frac{dy}{y}=\frac{dx}{x},$$

$\therefore\ y=Ax$

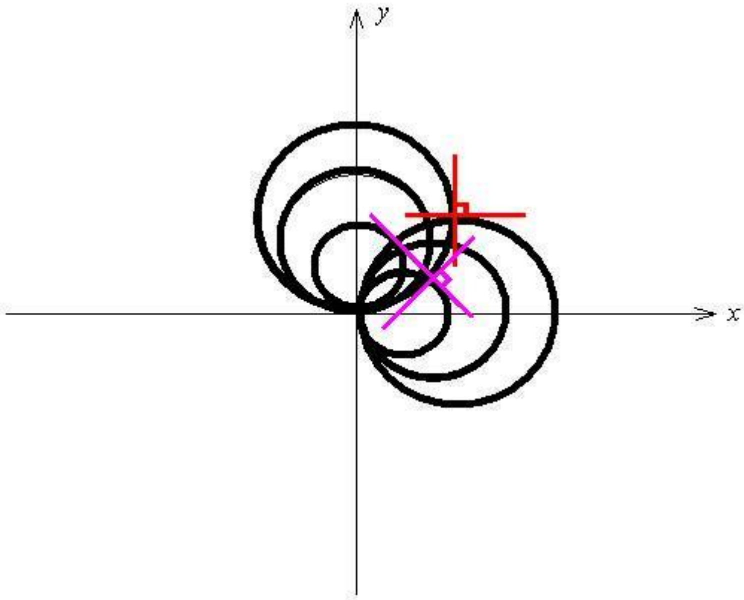


Theorem **A family of curves $F(\theta,r,k)=0$, of which differential equation is $f(\theta,r,r')=0$. Then the family of orthogonal trajectories has differential equation $f(\theta,r,-r^2/r')=0$, where $r'=dr/d\theta$.**

Eg. Find the family of trajectories orthogonal to $r=k\cos(\theta)$.

(Sol.) $r=k\cos(\theta)$, $r'=-k\sin(\theta) \Rightarrow r=-\cot(\theta)r'$

Family of orthogonal trajectories: $r=-\cot(\theta)\cdot\left(-\frac{r^2}{r'}\right) \Rightarrow \frac{r}{r'}=\tan(\theta) \Rightarrow r=k'\sin(\theta)$



1-8 Solutions of the First-order Ordinary Differential Equations by Matlab language and Wolfram Alpha

In **Matlab** language, we can use the following instructions to obtain the solution of the first-order ordinary differential equation:

```
>>soln=dsolve('Dy=3*y+exp(2*x)','y(0) = 3') % solve y'=3y+exp(2x), y(0)=3
```

ans=-exp(2*x)+4*exp(3*x)

Wolfram Alpha Address : https://www.wolframalpha.com/examples/mathematics/

Eg. Solve *y'*+*xy*=0.