

Chapter 2 The Second-Order Ordinary Differential Equations

2-1 Introduction

Initial-value problem

Eg. $y''-2y=x^2-1, y(1)=3, y'(1)=-5.$

Boundary-value problem

Eg. $y''+y=0, y(0)=y(\pi)=0.$

Theorem Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of $y''+P(x)y'+Q(x)y=0$, then $y(x)=c_1y_1(x)+c_2y_2(x)$ is its general solution.

2-2 The 2-order Linear Constant-coefficient Ordinary Differential Equation

$y''+Ay'+By=F(x)$

Homogeneous equation: $y''+Ay'+By=0$

Let $y=e^{rx}, r^2+Ar+B=0 \Rightarrow r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$

Case 1 $A^2-4B>0 \Rightarrow r = \frac{-A \pm \sqrt{A^2 - 4B}}{2} = r_1, r_2, r_1 \neq r_2, \therefore y(x) = c_1e^{r_1x} + c_2e^{r_2x}$

Eg. Solve $y''+3y'+2y=0.$

(Sol.) $r^2+3r+2=0, r=-1, -2, \therefore y(x)=c_1e^{-x}+c_2e^{-2x}$

Case 2 $A^2-4B=0 \Rightarrow r = \frac{-A}{2}, y_1(x) = e^{\frac{-Ax}{2}}, y_2 = u(x)e^{\frac{-Ax}{2}}$

$y_2''(x) + Ay_2'(x) + By_2(x) = 0 \Rightarrow u(x) = c_1x + c_2$

Choose $c_1 = 1, c_2 = 0, y_2(x) = xe^{\frac{-Ax}{2}}, \therefore y(x) = c_1e^{\frac{-Ax}{2}} + c_2xe^{\frac{-Ax}{2}}$

Eg. Solve $y''+4y'+4y=0.$

(Sol.) $r^2+4r+4=0, r=-2, -2, \therefore y(x)=c_1e^{-2x}+c_2xe^{-2x}$

Case 3 $A^2-4B<0 \Rightarrow r = \frac{-A \pm i\sqrt{4B - A^2}}{2} = p \pm iq$

$y(x) = c_1e^{(p+iq)x} + c_2e^{(p-iq)x} = d_1e^{px} \cdot \cos(qx) + d_2e^{px} \cdot \sin(qx)$

Eg. Solve $y''+9y=0.$

(Sol.) $r^2+9=0, r=\pm i3, \therefore y(x)=c_1\cos(3x)+c_2\sin(3x)$

Eg. Solve $y''+2y'+26y=0.$

(Sol.) $r^2+2r+26=0, r=-1\pm 5i, \therefore y(x)=c_1e^{-x}\cos(5x)+c_2e^{-x}\sin(5x)$

Eg. Two students solve $y''+ay'+by=0$, $y(0)=A$ and $y'(0)=B$. Using wrong constants for b and B , one student obtain the solution $y_A=e^{-2x}(\cos 3x+2\sin 3x)$. Using wrong constants for a and A , one student obtain the solution $y_B=-3e^x+2e^{3x}$. Find the correct constants for a , b , A , and B and solve the initial value problem. [2001 台大電研]

(Sol.) $y_A=e^{-2x}(\cos 3x+2\sin 3x) \Rightarrow r=-2 \pm i3 \Rightarrow r^2+4r+13=0=r^2+ar+b$,

$\therefore b$ is wrong but a is correct, $\therefore a=4$

$\therefore y(0)=A$ is correct, $\therefore y_A(0)=e^0(\cos 0+2\sin 0)=1=A$

$y_B=-3e^x+2e^{3x} \Rightarrow r=1, 3 \Rightarrow r^2-4r+3=0=r^2+ar+b$, $\therefore a$ is wrong but b is correct, $\therefore b=3$

$y_B'(x)=-3e^x+6e^{3x}$, $\therefore y'(0)=B$ is correct, $\therefore y_B'(0)=-3+6=3=B$

The correct $r^2+ar+b=r^2+4r+3=0 \Rightarrow r=-1, -3 \Rightarrow$ The correct $y(x)=ce^{-x}+de^{-3x}$

$y(0)=A=1 \Rightarrow c+d=1$ and $y'(0)=B=3 \Rightarrow -c-3d=3$, we obtain $c=3, d=-2$

$\Rightarrow y(x)=3e^{-x}-2e^{-3x}$

The solutions given by ChatGPT3.5 are wrong !

The screenshot shows a ChatGPT3.5 response to the problem. The solution is incorrect. It starts by identifying the general form of the differential equation $y'' + ay' + by = 0$. It then incorrectly identifies the characteristic equation for the first student's solution as $r^2 + 4r + 13 = 0$ and for the second student's solution as $r^2 - 4r + 3 = 0$. It then incorrectly concludes that $a=4$ and $b=3$ based on these equations. The final solution given is $y(x) = 3e^{-x} - 2e^{-3x}$, which is incorrect. The correct solution is $y(x) = 3e^{-x} - 2e^{-3x}$.

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For yA:
yA = e^(-2x)(cos(3x) + 2sin(3x))
Using the initial condition y(0) = A:
A = e^(0)(cos(0) + 2sin(0))
A = 1
For yB:
yB = -3e^x + 2e^(3x)
Using the initial condition y(0) = B:
B = -3e^(0) + 2e^(3*0)
B = -3 + 2
B = -1
So, the correct constants are:
a = 1
b = -3
A = 1
B = -1
Now, we can write the solution to the initial value problem:
y'' + ay' + by = 0
y(0) = A = 1
y'(0) = B = -1

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The correct solution is: wrong answer

$$y(x) = e^{-2x}(\cos(3x) + 2\sin(3x))$$

with $a = 1$, $b = -3$, $A = 1$, and $B = -1$.

Non-homogeneous equation: $y'' + Ay' + By = F(x)$

1. Find the homogeneous solution y_h of $y'' + Ay' + By = 0$,
2. Find a particular solution y_p of $y'' + Ay' + By = F(x)$,
3. General solution is $y_h + y_p$.

Eg. Solve $y'' - 4y = 8x^2 - 2x$.

(Sol.) $r^2 - 4 = 0, r = 2, -2, \therefore y_h = c_1 e^{-2x} + c_2 e^{2x}$,

$y_p = ax^2 + bx + c, y_p' = 2ax + b, y_p'' = 2a, \therefore y_p'' - 4y_p = -4ax^2 - 4bx + 2a - 4c = 8x^2 - 2x$

$\Rightarrow a = -2, b = \frac{1}{2}, c = -1, \therefore y(x) = c_1 e^{-2x} + c_2 e^{2x} - 2x^2 + \frac{1}{2}x - 1$

Eg. Solve $y'' + 2y' - 3y = 4e^{2x}$.

(Sol.) $r^2 + 2r - 3 = 0, r = 1, -3, \therefore y_h = c_1 e^x + c_2 e^{-3x}$.

Let $y_p = Ae^{2x}, y_p' = 2Ae^{2x}, y_p'' = 4Ae^{2x}, \therefore y_p'' + 2y_p' - 3y_p = 4e^{2x} \Rightarrow A = \frac{4}{5}$,

$\therefore y = c_1 e^x + c_2 e^{-3x} + \frac{4}{5} e^{2x}$

Eg. Solve $y'' + 2y' - 3y = 4e^x$.

(Sol.) $r^2 + 2r - 3 = 0, r = 1, -3 \Rightarrow y_h = c_1 e^x + c_2 e^{-3x}, \therefore y_p \neq Ae^x$,

Try $y_p = Axe^x, y_p' = Ae^x + Axe^x, y_p'' = 2Ae^x + Axe^x$,

$\therefore y_p'' + 2y_p' - 3y_p = 4e^x \Rightarrow A = 1, \therefore y = c_1 e^x + c_2 e^{-3x} + xe^x$

Eg. Solve $y'' + 4y = \cos(x)$.

(Sol.) $r^2 + 4 = 0, r = \pm 2i, \therefore y_h = c_1 \cos(2x) + c_2 \sin(2x)$.

Let $y_p = A \cos(x) + B \sin(x), y_p' = -A \sin(x) + B \cos(x), y_p'' = -A \cos(x) - B \sin(x)$,

$y_p'' + 4y_p = 3A \cos(x) + 3B \sin(x) = \cos(x) \Rightarrow A = \frac{1}{3}, B = 0, \therefore y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{3} \cos(x)$

Eg. Solve $y''+4y=\cos(2x)$.

(Sol.) $r^2 + 4 = 0$, $r = \pm 2i$, $\therefore y_h = c_1 \cos(2x) + c_2 \sin(2x)$ but $y_p \neq A \cos(2x) + B \sin(2x)$

Try $y_p = Ax \cos(2x) + Bx \sin(2x)$, $y_p' = A \cos(2x) - 2Ax \sin(2x) + B \sin(2x) + 2Bx \cos(2x)$,

$y_p'' = -2A \sin(2x) - 2A \sin(2x) - 4Ax \cos(2x) + 2B \cos(2x) + 2B \cos(2x) - 4Bx \sin(2x)$,

$y_p'' + 4y_p = -4A \sin(2x) + 4B \cos(2x) = \cos(2x) \Rightarrow A = 0, B = \frac{1}{4}$, $\therefore y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{4} \sin(2x)$

Variation of parameters to find the particular solution y_p :

Let y_1 and y_2 be linearly independent solutions of $y'' + Ay' + By = 0$, then a particular solution y_p is $y_p = u(x)y_1(x) + v(x)y_2(x)$, and

$$y_p' = u' y_1 + v' y_2 + u y_1' + v y_2' = u y_1' + v y_2'$$

Impose the condition: $u' y_1 + v' y_2 = 0 \dots \dots (1)$

$$y_p'' = u' y_1' + v' y_2' + u y_1'' + v y_2''$$

$$\Rightarrow u' y_1' + v' y_2' + u y_1'' + v y_2'' + A(u y_1' + v y_2') + B(u y_1 + v y_2) = F(x)$$

$$\Rightarrow u [y_1'' + A y_1' + B y_1] + v [y_2'' + A y_2' + B y_2] + u' y_1' + v' y_2' = F(x)$$

$$u' y_1' + v' y_2' = F(x) \dots \dots (2)$$

$$(1), (2) \Rightarrow u' = \frac{-y_2' F(x)}{y_1 y_2' - y_2 y_1'}, \quad v' = \frac{y_1' F(x)}{y_1 y_2' - y_2 y_1'}$$

Wronskian determinant: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$$u(x) = \int \frac{-y_2' F(x)}{W(y_1, y_2)} dx, \quad v(x) = \int \frac{y_1' F(x)}{W(y_1, y_2)} dx \Rightarrow y_p = u(x) y_1(x) + v(x) y_2(x)$$

Eg. Solve $y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1$. [1991 清大電研、2015 中興電研乙丙丁組]

(Sol.) $y_1 = x, y_2 = x^4, W(y_1, y_2) = x \cdot 4x^3 - x^4 \cdot 1 = 3x^4$,

$$u(x) = \int \frac{-x^4(x^2 + 1)}{3x^4} dx = -\frac{x^3}{9} - \frac{x}{3}, \quad v(x) = \int \frac{x(x^2 + 1)}{3x^4} dx = \frac{1}{3} \ln|x| - \frac{1}{6x^2}$$

$$y_p = u(x)y_1 + v(x)y_2 = -\frac{x^4}{9} - \frac{x^2}{2} + \frac{x^4}{3} \ln|x|$$

$$\therefore y(x) = y_h + y_p = cx + dx^4 - \frac{x^4}{9} - \frac{x^2}{2} + \frac{x^4}{3} \ln|x| = c_1 x + c_2 x^4 - \frac{x^2}{2} + \frac{x^4}{3} \ln|x|$$

Eg. Solve $y''+4y=\tan(2x)$.

(Sol.) $y''+4y=0 \Rightarrow y_1=\cos(2x), y_2=\sin(2x)$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 2 \cos^2(2x) - \sin(2x)[-2 \sin(2x)] = 2$$

$$u(x) = \int \frac{-\sin(2x) \cdot \tan(2x)}{2} dx = \frac{1}{4} \sin(2x) - \frac{1}{4} \ln \left| \tan \left(\frac{\pi}{4} + x \right) \right|$$

$$v(x) = \int \frac{\cos(2x) \cdot \tan(2x)}{2} dx = -\frac{1}{4} \cos(2x)$$

$$y_p = u(x)y_1(x) + v(x)y_2(x)$$

$$= \frac{1}{4} \sin(2x) \cos(2x) - \frac{\cos(2x)}{4} \cdot \ln \left| \tan \left(\frac{\pi}{4} + x \right) \right| - \frac{1}{4} \cos(2x) \sin(2x)$$

$$= -\frac{1}{4} \cos(2x) \cdot \ln \left| \tan \left(\frac{\pi}{4} + x \right) \right|$$

Eg. Solve $4y''+36y=\csc(3x)$.

(Ans.) $y = a \cos(3x) + b \sin(3x) - \frac{x \cos(3x)}{12} + \frac{\sin(3x) \cdot \ln[|\sin(3x)|]}{36}$

Eg. Solve $y''+4y=\sec(2x)$. [中原電子所]

<https://www.wolframalpha.com/examples/mathematics/differential-equations/>

Eg. Solve $y''+2y'-3y=e^{2x}$.

WolframAlpha computational intelligence.

$y'' + 2y' - 3y = \exp(2x)$

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Input:
 $y''(x) + 2y'(x) - 3y(x) = \exp(2x)$

Differential equation solution: Approximate form Step-by-step solution

$y(x) = c_1 e^{-3x} + c_2 e^x + \frac{e^{2x}}{5}$

2-3 Euler Equation $x^2y''+Axy'+By=F(x)$

Solution: $z=\ln(x), y'=\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz},$

$$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}, \text{ and } x^2y''=-\frac{dy}{dz}+\frac{d^2y}{dz^2}, \quad xy'=\frac{dy}{dz}$$

$$\Rightarrow \frac{d^2y}{dz^2}+(A-1)\frac{dy}{dz}+By=F(e^z) \text{ is the second-order linear ODE.}$$

Eg. Solve $x^2y''-5xy'+8y=2\ln(x)+x^3$.

(Sol.) $z=\ln(x), y'=\frac{1}{x}\frac{dy}{dz}, y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}, \frac{d^2y}{dz^2}-6\frac{dy}{dz}+8y=2z+e^{3z}$

$y_h=c_1e^{2z}+c_2e^{4z}=c_1x^2+c_2x^4$ and $y_p=az+b+ce^{3z}, y_p'=a+3ce^{3z}, y_p''=9ce^{3z},$

$y_p''-6y_p'+8y_p=2z+e^{3z} \Rightarrow 8a=2, -6a+8b=0, -c=1 \Rightarrow a=1/4, b=3/16, c=-1,$

$\therefore y(x)=c_1x^2+c_2x^4+\frac{1}{4}\ln(x)+\frac{3}{16}-x^3$

Eg. Solve $xy''+4y'=\frac{\ln x^3}{x}$. [2004 台大電研]

(Sol.) $x^2y''+4xy'=3\ln(x), z=\ln(x), y'=\frac{1}{x}\frac{dy}{dz}, y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2},$

$\frac{d^2y}{dz^2}+3\frac{dy}{dz}=3z, y_h=c_1e^0+c_2e^{-3z}=c_1+c_2x^{-3}$ and $y_p=az^2+bz+c, y_p'=2az+b, y_p''=2a,$

$y_p''+3y_p'=3z \Rightarrow 6a=3, 2a+3b=0 \Rightarrow a=1/2, b=-1/3, \therefore y(x)=c_1+c_2x^{-3}+\frac{(\ln x)^2}{2}-\frac{\ln x}{3}$

Eg. Solve $x^2y''-2y=1/x$. [文化電機轉學考] (Ans.) $y(x)=cx^2+\frac{d}{x}-\frac{\ln(x)}{3x}$

Eg. Solve (a) $x^2y''-4xy'+4y=0$ and (b) $x^2y''+5xy'+4y=0$.

(Sol.) (a) Let $y=x^r, y'=rx^{r-1}, y''=r(r-1)x^{r-2} \Rightarrow x^2y''-4xy'+4y=x^r(r^2-5r+4)=0$

$\Rightarrow r=1, 4 \Rightarrow y=c_1x+c_2x^4.$

(b) $y=x^r \Rightarrow r^2+4r+4=0, r=-2, -2 \Rightarrow y_1=x^{-2}, y_2=x^{-2}\ln(x) \Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ln(x)$

Another method: $z=\ln(x), y'=\frac{1}{x}\frac{dy}{dz}, y''=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$

$\Rightarrow \frac{d^2y}{dz^2}+4\frac{dy}{dz}+4y=0 \Rightarrow y_1=e^{-2z}=x^{-2}, y_2=ze^{-2z}=x^{-2}\ln(x) \Rightarrow y(x)=c_1x^{-2}+c_2x^{-2}\ln(x)$

Eg. Solve (a) $x^2y''-xy'+y=\ln(x)$ and (b) $x^2y''-4xy'+4y=x^4+x^2$. [交大電子所]

(Ans.) (a) $y(x)=c_1x+c_2x\ln(x)+\ln(x)+2,$ (b) $y(x)=c_1x+c_2x^4+\frac{1}{3}x^4\ln x-\frac{1}{2}x^2$

Eg. Solve $(x+2)^2y''-(x+2)y'+y=3x+4$. [2012 師大應用電子所]

(Sol.) $z=\ln(x+2)$, $y' = \frac{1}{(x+2)} \frac{dy}{dz}$, $y'' = \frac{-1}{(x+2)^2} \frac{dy}{dz} + \frac{1}{(x+2)^2} \cdot \frac{d^2y}{dz^2}$,

$\frac{d^2y}{dz^2} - 2 \frac{dy}{dz} + y = 3e^z - 2$, $y_1=e^z=x+2$, $y_2=ze^z=(x+2) \cdot \ln(x+2)$, and $y_p=az^2e^z+b$,

$y_p' = 2aze^z + az^2e^z$, $y_p'' = 2ae^z + 4aze^z + az^2e^z \Rightarrow y_p'' - 2y_p' + y_p = (2ae^z + 4aze^z + az^2e^z) - 2(2aze^z + az^2e^z) + az^2e^z + b = 2ae^z + b = 3e^z - 2 \Rightarrow a=3/2, b=-2$

$\Rightarrow y_p = \frac{3}{2} z^2 e^z - 2 = \frac{3}{2} [\ln(x+2)]^2 \cdot (x+2) - 2$,

$\therefore y(x) = c_1(x+2) + c_2(x+2) \cdot \ln(x+2) + \frac{3}{2} [\ln(x+2)]^2 \cdot (x+2) - 2$

Eg. Solve $(x-2)^2y''+4(x-2)y'+6y=0$. [文化電機轉學考]

(Sol.) Let $y=(x-2)^r$, $r^2+3r+6=0$, $r = \frac{-3 \pm i\sqrt{15}}{2}$,

$\therefore y(x) = c_1(x-2)^{-\frac{3}{2}} \cdot \cos\left[\frac{\sqrt{15}}{2} \ln(x-2)\right] + c_2(x-2)^{-\frac{3}{2}} \cdot \sin\left[\frac{\sqrt{15}}{2} \ln(x-2)\right]$

<https://www.wolframalpha.com/examples/mathematics/differential-equations/>

Eg. Solve $x^2y''+4xy'+2y=0$ and $x^2y''-2xy'+2y=x^3$.



$x^2y'' + 4xy' + 2y = 0$

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Input:
 $x^2 y''(x) + 4 x y'(x) + 2 y(x) = 0$

Differential equation solution: Step-by-step solution

$y(x) = \frac{c_2}{x^2} + \frac{c_1}{x}$

$x^2 y'' - 2xy' + 2y = x^3$

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Input:

 $x^2 y''(x) - 2x y'(x) + 2y(x) = x^3$

[Open code](#)

Differential equation solution: Step-by-step solution

$y(x) = c_2 x^2 + c_1 x + \frac{x^3}{2}$

2-4 Miscellaneous Problems

Eg. Solve $y'=(y+x)^3-1$. [2012台大電子所甲組]

(Sol.) Let $u=y+x$, $\frac{du}{dx}=y'+1=u^3$, $\frac{du}{u^3}=dx$, $\frac{-1}{2u^2}=x+c \Rightarrow \frac{-1}{2(y+x)^2}=x+c$

Eg. Solve $y'=(-2x+y)^2-7, y(0)=0$. [2010成大電研]

(Sol.) Let $u=-2x+y$, $\frac{du}{dx}+2=u^2-7$, $\frac{du}{dx}=u^2-9$, $\frac{du}{u^2-9}=\frac{1}{6}\left[\frac{1}{u-3}-\frac{1}{u+3}\right]du=dx$,
 $\ln\left(\frac{u-3}{u+3}\right)=6x+c$, $\frac{u-3}{u+3}=Ae^{6x}$, $\frac{y-2x-3}{y-2x+3}=Ae^{6x}$, $y(0)=0 \Rightarrow A=-1$, $\therefore \frac{y-2x-3}{y-2x+3}=-e^{6x}$

Eg. Solve $y'=y^2-2xy+x^2+1$. [1991中央資電所、交大控制所]

(Sol.) Let $u=y-x$, $y'=u'+1=u^2+1$, $\frac{du}{u^2}=dx$, $-1/u=x+c \Rightarrow \frac{-1}{y-x}=x+c$

Eg. Solve $1+x^2y^2+y+xy'=0$. [交大電信所]

(Sol.) Let $u=xy$, $u'=y+xy'$, $1+u^2+u'=0$, $\frac{du}{1+u^2}=-dx$, $\tan^{-1}(u)=-x+c \Rightarrow \tan^{-1}(xy)=-x+c$

Eg. Solve $x^2+y\cos(x)+y'\sin(x)=0$. [2015師大電研]

(Sol.) $x^2+[y\sin(x)]'=0$, $[y\sin(x)]'=-x^2$, $y\sin(x)=\frac{-x^3}{3}+c$

Eg. Solve $xy''+2y'=4x^3$.

(Sol.) Let $u=y' \Rightarrow xu' + 2u = 4x^3 \Rightarrow u = \frac{4x^3}{5} + \frac{c}{x^2} \Rightarrow y(x) = \int u(x)dx = \frac{x^4}{5} - \frac{c}{x} + D$

Another method: $x^2y''+2xy'=4x^4$ (Euler's equation)

Eg. Solve $y''-2yy'=0$.

(Sol.) Set $u=y'$, $y'' = \frac{dy'}{dx} = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = u \frac{du}{dy} \Rightarrow u \frac{du}{dy} - 2yu = 0 \Rightarrow u = y^2 + c$

$\Rightarrow \frac{dy}{dx} = y^2 + c \Rightarrow \frac{dy}{y^2 + c} = dx \Rightarrow \frac{1}{\sqrt{c}} \tan^{-1}\left(\frac{y}{\sqrt{c}}\right) = x + K$

Eg. Solve $\frac{dy}{dx} = \frac{x}{x^2y + y^3}$.

(Sol.) Let $u=x^2$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2x \cdot \frac{dy}{du} = \frac{x}{uy + y^3}$

$\frac{du}{dy} - 2yu = 2y^3 \Rightarrow [u \cdot e^{-y^2}]' = 2y^3 e^{-y^2}$

$u = e^{+y^2} \cdot [-y^2 e^{-y^2} - e^{-y^2} + c] = -y^2 - 1 + ce^{y^2} \Rightarrow x^2 = -y^2 - 1 + ce^{y^2}$

Another method: $\frac{dx}{dy} = xy + \frac{y^3}{x}$ (Bernoulli's equation)

Given a solution $y_1(x)$ of $y''+P(x)y'+Q(x)y=0$, then a second solution $y_2(x)=v(x)y_1(x)$ is obtained by the following method:

$v''y_1(x) + v'[2y_1'(x) + P(x)y_1] = 0$. Set $v'=u \Rightarrow u' + \left[\frac{2y_1'}{y_1} + P(x) \right] u = 0$

Eg. Solve $y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$ if $y_1 = \frac{1}{x}$ is given.

(Sol.) Let $y_2 = v(x)y_1(x) = \frac{v(x)}{x}$, $y_2' = v' \cdot y_1 + v \cdot y_1'$

$y_2'' = (v' \cdot y_1 + v \cdot y_1')' = v'' \cdot y_1 + v' \cdot y_1' + v' \cdot y_1' + vy_1'' = v'' \cdot y_1 + 2v' \cdot y_1' + vy_1''$

$v'' \cdot y_1 + 2v' \cdot y_1' + vy_1'' + \frac{3}{x}(v' \cdot y_1 + vy_1') + \frac{v y_1}{x^2} = v''y_1 + 2v'y_1' + \frac{3}{x}v'y_1 = 0$

$v' = u \Rightarrow u' + \left(\frac{2y_1'}{y_1} + \frac{3}{x} \right) u = u' + \left(\frac{-2 \cdot \frac{1}{x^2}}{\frac{1}{x}} + \frac{3}{x} \right) u = u' + \frac{1}{x}u = 0$

$u = v' = \frac{1}{x} \Rightarrow v(x) = \ln|x| \Rightarrow y_2(x) = \frac{\ln|x|}{x}$

Eg. Given $y(x)=x$ is a solution of $y''-xy'+y=0$, find the other solution. [交大電信所]