Chapter 3 Series Solutions of Differential Equations

3-1 Simple Power Series Solutions of Ordinary Differential Equations

For $y'+g(x)y=r(x)$ or $y''+P(x)y'+Q(x)y=F(x)$, if $g(x)$, $r(x)$, $P(x)$, $Q(x)$, and $F(x)$ are analytical at zero, then $y(x)=\sum_{n=0}^{\infty}a_n x^n$,

$$y'=\sum_{n=0}^{\infty}na_n x^{n-1}=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n ,$$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n .$$

Eg. Solve $(1+x^2)y''+2xy'=0$, $y(0)=0$, $y'(0)=1$. [2004 台大電研]

(Sol.) Let $y=\sum_{n=0}^{\infty}a_n x^n$, $y'=\sum_{n=0}^{\infty}na_n x^{n-1}$, $y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$

$$(1+x^2)y''+2xy'=y''+x^2y''+2xy'=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2na_n x^n$$

$$=\sum_{n=0}^{\infty}[n(n+1)a_n+(n+2)(n+1)a_{n+2}]x^n=0\Rightarrow a_{n+2}=-\frac{n}{(n+2)}a_n$$

$$\Rightarrow a_2=a_4=a_6=\ldots=0 \text{ and } a_{2n+1}=\frac{(-1)^n}{2n+1}a_1$$

$$\Rightarrow y(x)=\sum_{n=0}^{\infty}a_n x^n=a_0+a_1(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+-\ldots)=a_0+a_1\tan^{-1}(x)$$

$y(0)=0$, $y'(0)=1\Rightarrow a_0=0$ and $a_1=1\Rightarrow y(x)=\tan^{-1}(x)$

Eg. Solve $(1+x^2)y''-2xy'+2y=0$. [1990 台大土木所]

(Sol.) Let $y=\sum_{n=0}^{\infty}a_n x^n$, $y'=\sum_{n=0}^{\infty}na_n x^{n-1}$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1+x^2)y''-2xy'+2y=y''+x^2y''-2xy'+2y$$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}2na_n x^n+\sum_{n=0}^{\infty}2a_n x^n$$

$$=\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n-2)(n-1)a_n]x^n=0\Rightarrow a_{n+2}=-\frac{(n-1)(n-2)}{(n+2)(n+1)}a_n$$

$a_2=-\frac{(-1)(-2)}{1\cdot 2}a_0=-a_0$, $a_3=0$, $a_4=0$, $a_5=0$, $a_6=0$, $a_7=0$, …

$$\therefore\ y(x)=a_0+a_1x-a_0x^2=a_0(1-x^2)+a_1x$$

Eg. Solve $(1-x^2)y''+2xy'-2y=0$. [1990 台大土木所類似題]

(Sol.) Let $y=\sum_{n=0}^{\infty}a_n x^n$, $y'=\sum_{n=0}^{\infty}na_n x^{n-1}$

$$y''=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1-x^2)y''+2xy'-2y=y''-x^2y''+2xy'-2y$$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2na_n x^n-\sum_{n=0}^{\infty}2a_n x^n$$

$$=\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-(n-2)(n-1)a_n]x^n=0\Rightarrow a_{n+2}=\frac{(n-1)(n-2)}{(n+2)(n+1)}a_n$$

$a_2=\frac{(-1)(-2)}{1\cdot 2}a_0=a_0$, $a_3=0$, $a_4=0$, $a_5=0$, $a_6=0$, $a_7=0$, …

$$\therefore\ y(x)=a_0+a_1x+a_0x^2=a_0(1+x^2)+a_1x$$

Eg. Solve $y''+x^2y=0$ by series expansions. [2015 師大電研]

(Sol.) $y=\sum_{n=0}^{\infty}a_n x^n$, $y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$, $x^2y=\sum_{n=0}^{\infty}a_n x^{n+2}=\sum_{m=2}^{\infty}a_{m-2}x^m=\sum_{n=2}^{\infty}a_{n-2}x^n$

$$y''+x^2y=\sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2}+a_{n-2}]x^n+2a_2+6a_3x=0$$

$$\Rightarrow a_{n+2}=\frac{-a_{n-2}}{(n+2)(n+1)},\quad n=2,3,\ldots\ldots,\quad a_2=a_3=0$$

$$\Rightarrow a_4=\frac{-a_0}{4\cdot 3},\quad a_8=\frac{a_0}{8\cdot 7\cdot 4\cdot 3}\qquad\qquad a_5=\frac{-a_1}{5\cdot 4},\quad a_9=\frac{a_1}{9\cdot 8\cdot 5\cdot 4}$$

⋮ ⋮

$$\therefore\ y=a_0\left(1-\frac{x^4}{4\cdot 3}+\frac{x^8}{8\cdot 7\cdot 4\cdot 3}-+\ldots\ldots\right)+a_1\left(x-\frac{x^5}{5\cdot 4}+\frac{x^9}{9\cdot 8\cdot 5\cdot 4}-+\ldots\ldots\right)$$

Eg. Solve $y'+ky=0$ by series expansions.

(Sol.) $y=\sum_{n=0}^{\infty}a_n x^n$, $y'=\sum_{n=0}^{\infty}na_n x^{n-1}=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

$$\Rightarrow\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+k\sum_{n=0}^{\infty}a_n x^n=0\Rightarrow\sum_{n=0}^{\infty}[(n+1)a_{n+1}+ka_n]x^n=0$$

$\Rightarrow a_{n+1}=-\frac{ka_n}{n+1}$, $n=0, 1, 2, 3, \ldots$ $\Rightarrow a_n=\frac{(-1)^n k^n a_0}{n!}$, $n=1,2,3,\ldots\ldots$

$$\therefore\ y=\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}\frac{a_0(-1)^n(kx)^n}{n!}=a_0\cdot\sum_{n=0}^{\infty}\frac{(-kx)^n}{n!}=a_0e^{-kx}$$

Eg. Solve $(x-1)y''-xy'+y=0$. [1990 台大應力所] (Ans.) $y(x)=c_1e^x+c_2x$

Eg. Solve $(x+1)y''-(x+2)y'+y=0$. [1991 台大土木所] (Ans.) $y(x)=c_1e^x+c_2(x+2)$

Eg. Solve $y''+k^2y=0$ by series expansions.

(Sol.) $y=\sum_{n=0}^{\infty} a_n x^n$, $y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}=\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$

$$y''+k^2y=\sum_{n=0}^{\infty}[a_{n+2}(n+2)(n+1)+k^2a_n]x^n=0$$

$$\Rightarrow a_{n+2}=\frac{-k^2a_n}{(n+2)(n+1)}; \quad n=0,1,2,\cdots\cdots$$

$$\Rightarrow a_2=\frac{-k^2a_0}{2\cdot 1} \qquad a_3=\frac{-k^2a_1}{3\cdot 2}$$

$$a_4=\frac{k^4a_0}{4\cdot 3\cdot 2\cdot 1} \qquad a_5=\frac{k^4a_1}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$\vdots \qquad \vdots$$

$$a_{2n}=\frac{(-1)^n k^{2n}\cdot a_0}{(2n)!} \qquad a_{2n+1}=\frac{(-1)^n k^{2n}a_1}{(2n+1)!}$$

$$y=\sum_{n=0}^{\infty}a_nx^n=\sum_{m=0}^{\infty}a_{2m}x^{2m}+\sum_{m=0}^{\infty}a_{2m+1}x^{2m+1}=a_0\sum_{m=0}^{\infty}\frac{(-1)^m(kx)^{2m}}{(2m)!}+\frac{a_1}{k}\sum_{m=0}^{\infty}\frac{(-1)^m(kx)^{2m+1}}{(2m+1)!}$$

$$=a_0\cos(kx)+\frac{a_1}{k}\cdot\sin(kx)$$

Eg. Solve $y''-e^xy=0$ by series expansions.

(Sol.) $y=\sum_{n=0}^{\infty}a_nx^n$, $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right)$

$$y''-e^xy=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\cdot\left(a_0+a_1x+a_2x^2+\ldots\right)$$

$$=(2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots\ldots)$$

$$-\left[a_0+(a_0+a_1)x+\left(\frac{a_0}{2}+a_1+a_2\right)x^2+\left(\frac{a_0}{6}+\frac{a_1}{2}+a_2+a_3\right)x^3+\ldots\right]=0$$

$$\Rightarrow 2a_2-a_0=0,\ 6a_3-a_0-a_1=0,\ 12a_4-\frac{a_0}{2}-a_1-a_2=0$$

$$\Rightarrow a_2=\frac{a_0}{2},\ a_3=\frac{a_0+a_1}{6},\ a_4=\frac{a_0+a_1}{12},\ \ldots$$

$$\Rightarrow y=a_0+a_1x+\frac{a_0}{2}x^2+\left(\frac{a_0+a_1}{6}\right)x^3+\left(\frac{a_0+a_1}{12}\right)x^4+\ldots\ldots$$

$$=a_0\left(1+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+\ldots\right)+a_1\left(x+\frac{x^3}{6}+\frac{x^4}{12}+\ldots\right)$$

$$=a_0y_0(x)+a_1y_1(x)$$

Eg. Solve $y''-e^xy=\sin(x)+1$. [1990 台大化工所]

3-2 Method of Frobenius

For $P(x)y''+Q(x)y'+R(x)y=0$. If $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ have a regular singular point at $x_0=0$, and $\begin{cases} x\frac{Q(x)}{P(x)} \\ x^2\frac{R(x)}{P(x)} \end{cases}$ are analytical at $x_0=0$, then $y=\sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution $\Rightarrow$ $r^2+Ar+B=0 \Rightarrow r=r_1, r_2$

Case 1 $r_1 \neq r_2$ and r_1-r_2 is not an integer, then

$$y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2=\sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Case 2 r_1-r_2 is a positive integer, then $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ and

$y_2=\sum_{n=0}^{\infty} b_n x^{n+r_2}+Ay_1\ell n(x)$, where A may be 0.

Case 3 $r_1=r_2$, then $y_1=\sum_{n=0}^{\infty} a_n x^{n+r_1}$ and $y_2=y_1\ell n(x)+\sum_{n=1}^{\infty} b_n x^{n+r_2}$

Eg. Solve $3xy''+y'-y=0$.

(Sol.) $y''+\frac{1}{3x}y'-\frac{1}{3x}y=0$, $\frac{1}{3x}$ and $-\frac{1}{3x}$ have a regular singular point at 0, but $\frac{x}{3x}=\frac{1}{3}$ and $-\frac{x^2}{3x}=-\frac{x}{3}$ are analytic at 0.

$$\therefore\ y=\sum_{n=0}^{\infty}C_n x^{n+r},\quad y'=\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1},\quad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2},$$

$$3xy''+y'-y=3\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1}-\sum_{n=0}^{\infty}C_n x^{n+r}$$

$$=\sum_{n=0}^{\infty}(n+r)(3n+3r-2)C_n x^{n+r-1}-\sum_{n=0}^{\infty}C_n x^{n+r}$$

$$=r(3r-2)C_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(3n+3r-2)C_n x^{n+r-1}-\sum_{n=0}^{\infty}C_n x^{n+r}$$

$$=r(3r-2)C_0x^{r-1}+\sum_{m=0}^{\infty}(m+r+1)(3m+3r+1)C_{m+1} x^{m+r}-\sum_{n=0}^{\infty}C_n x^{n+r}$$

$$=r(3r-2)C_0x^{r-1}+\sum_{n=0}^{\infty}[(n+r+1)(3n+3r+1)C_{n+1}-C_n]x^{n+r}$$

$$C_{n+1}=\frac{C_n}{(n+r+1)(3n+3r+1)}\quad \left(\text{or}\quad D_{n+1}=\frac{D_n}{(n+r+1)(3n+3r+1)}\right)$$

$\therefore\ r=\frac{2}{3},\ 0\Rightarrow r_1-r_2=\frac{2}{3}$ is not an integer: *Case 1*

$$r_1=\frac{2}{3}\Rightarrow C_{n+1}=\frac{C_n}{(3n+5)(n+1)}\Rightarrow C_1=\frac{C_0}{5},\quad C_2=\frac{C_1}{8\cdot 2}=\frac{C_0}{2!5\cdot 8},\ \ldots,$$

$$C_n=\frac{C_0}{n!5\cdot 8\cdot 11\ldots\ldots(3n+2)}$$

$$r_2=0\Rightarrow D_{n+1}=\frac{D_n}{(n+1)(3n+1)}\Rightarrow D_1=\frac{D_0}{1\cdot 1},\quad D_2=\frac{D_0}{2!1\cdot 4},\ \ldots,$$

$$D_n=\frac{D_0}{n!1\cdot 4\cdot 7\ldots\ldots(3n-2)}$$

$$\therefore\ y(x)=C_0\sum_{n=0}^{\infty}\frac{1}{n!5\cdot 8\cdot 11\ldots\ldots(3n+2)}x^{n+\frac{2}{3}}+D_0\sum_{n=0}^{\infty}\frac{1}{n!1\cdot 4\cdot 7\ldots\ldots(3n-2)}x^n$$

$$=C_0y_1(x)+D_0y_2(x)$$

Eg. Solve $x^2y''+ x^2y'-2y=0$.

(Sol.) $y''+y'-\frac{2}{x^2}y=0$, $-\frac{2}{x^2}$ has a regular singular point at 0, but $-\frac{2x^2}{x^2}$=-2 is analytic at 0.

$$\therefore\quad y=\sum_{n=0}^{\infty}a_n x^{n+r},\quad x^2y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$x^2y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}=\sum_{m=1}^{\infty}(m+r-1)a_{m-1}x^{m+r}=\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}$$

$$x^2y''+x^2y'-2y$$

$$=[r(r-1)a_0-2a_0]x^r+\sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n+(n+r-1)a_{n-1}-2a_n]\cdot x^{n+r}=0$$

$\Rightarrow r=2\,,-1\Rightarrow r_1-r_2=3$ is a positive integer: *Case 2*

$$[(n+r)(n+r-1)-2]a_n+(n+r-1)a_{n-1}=0$$

$$r_1=2\Rightarrow a_n=-\frac{n+1}{n(n+3)}a_{n-1}\Rightarrow a_n=(-1)^n\cdot\frac{6a_0}{n!(n+2)(n+3)}$$

$$\therefore\quad y_1(x)=6a_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!(n+1)(n+3)}$$

Let $y_2(x)=Ay_1\ell n(x)+\sum_{n=0}^{\infty}b_n x^{n+r_2}=Ay_1\ell n(x)+\sum_{n=0}^{\infty}b_n x^{n-1}$

$$\Rightarrow Ax^2y_1''\cdot\ell n(x)+2Axy_1'-Ay_1+\sum_{n=0}^{\infty}(n-1)(n-2)b_n x^{n-1}+Ax^2y_1'\ell n(x)$$

$$+Axy_1+\sum_{n=1}^{\infty}(n-2)b_{n-1}x^{n-1}-2\sum_{n=0}^{\infty}b_n x^{n-1}-2Ay_1\ell n(x)=0$$

$$\Rightarrow A(2xy_1'+xy_1-y_1)+2b_0x^{-1}-2b_0x^{-1}+\sum_{n=1}^{\infty}[n(n-3)b_n+(n-2)b_{n-1}]x^{n-1}=0$$

$$\Rightarrow A=0,\ b_n=-\frac{n-2}{n(n-3)}b_{n-1}\Rightarrow b_1=-\frac{1}{2}b_0,\ b_2=0,\ b_3=0,\ldots$$

$$\therefore\quad y_2(x)=b_0\left(\frac{1}{x}-\frac{1}{2}\right)$$

$$\Rightarrow\begin{cases}y_1(x)=6\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!(n+1)(n+3)}\\ y_2(x)=\frac{1}{x}-\frac{1}{2}\end{cases}\Rightarrow y(x)=c_1y_1(x)+c_2y_2(x)$$

Eg. Solve $xy''+(1-2x)y'+(x-1)y=0$. [1990 交大資訊所]

(Ans.) $y(x)=c_1e^x+c_2e^x\cdot\ell nx$

Eg. Solve $x^2y''+5xy'+(x+4)y=0$.

(Sol.) $y''+\frac{5}{x}y'+\frac{x+4}{x^2}y=0$, $\frac{5}{x}$ and $\frac{x+4}{x^2}$ have a regular singular point at 0, but $\frac{5x}{x}$=5 and $x^2\cdot\frac{x+4}{x^2}$=x+4 are analytic at 0.

$\therefore y=\sum_{n=0}^{\infty}a_n x^{n+r}$, $x^2y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$

$xy'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$, $xy=\sum_{n=0}^{\infty}a_n x^{n+r+1}=\sum_{m=1}^{\infty}a_{m-1}x^{m+r}=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}$

$$\therefore\ x^2y''+5xy'+(x+4)y=\sum_{n=1}^{\infty}[(n+r)(n+r-1)a_n+5(n+r)a_n+a_{n-1}+4a_n]x^{n+r}$$

$$+[r(r-1)+5r+4]a_0x^r=0$$

$\Rightarrow r^2+4r+4=0, r=-2,-2:$ *Case 3*

$[(n+r)(n+r+4)+4]a_n+a_{n-1}=0$

$r=-2\Rightarrow a_n=-\frac{a_{n-1}}{n^2}\Rightarrow a_n=\frac{(-1)^n a_0}{(n!)^2}$

$\therefore\ a_0y_1(x)=a_0\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\cdot x^{n-2}$

Let $y_2=y_1\ell n(x)+\sum_{n=1}^{\infty}b_n x^{n-2}$

$$\Rightarrow 4y_1+2xy_1'+\sum_{n=1}^{\infty}(n-2)(n-3)b_n x^{n-2}+\sum_{n=1}^{\infty}5(n-2)b_n x^{n-2}+\sum_{n=1}^{\infty}b_n x^{n-1}$$

$$+\sum_{n=1}^{\infty}4b_n x^{n-2}+\ell n(x)\cdot[x^2y_1''+5xy_1'+(x+4)y_1]=0$$

$y_1=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\cdot x^{n-2}\Rightarrow b_1=2$, $b_n=\frac{-b_{n-1}}{n^2}-\frac{2(-1)^n}{n(n!)^2}$

$$\Rightarrow y_2(x)=y_1\ell n(x)+\frac{2}{x}-\frac{3}{4}+\frac{11}{108}x-\frac{25}{576}x^2+-......$$