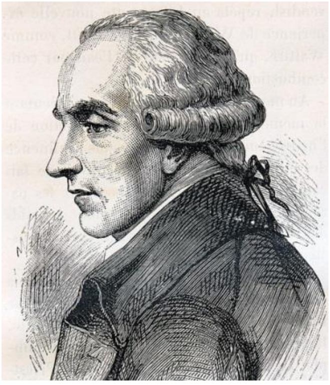
Chapter 4 Laplace Transforms



4-1 Laplace Transform $F(s)=L[f(t)]=\int_0^\infty e^{-st}\cdot f(t)dt$

Eg. Evaluate $L[\cos(at)]$ and $L[\sin(at)]$.

(Sol.) $e^{iat}=\cos(at)+i\sin(at)$

$$L[e^{iat}]=\int_0^\infty e^{-st}\cdot e^{iat}dt=\int_0^\infty e^{-(s-ia)t}dt$$

$$=\frac{-1}{s-ia}\cdot e^{-(s-ia)t}\Big|_0^\infty=\frac{1}{s-ia}$$

$$=\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}=L[\cos(at)]+iL[\sin(at)]$$

$$\therefore\ L[\cos(at)]=\frac{s}{s^2+a^2}\ \text{ and }\ L[\sin(at)]=\frac{a}{s^2+a^2}$$

Eg Find $\int_0^\infty e^{-ax}\cdot\cos(bx)dx$, with $a>0$. [2005 台大電研] (Ans.) $\frac{a}{a^2+b^2}$

Basic theorems of Laplace transforms $F(s)=L[f(t)]$ and $G(s)=L[g(t)]$:

1. $L[c_1f(t)+c_2g(t)]=c_1F(s)+c_2G(s)$

Eg. $L[-2\cos(2t)+3\sin(2t)]$. [2010台大光電所]

(Sol.) $L[-2\cos(2t)+3\sin(2t)]=-2L[\cos(2t)]+3L[\sin(2t)]=\frac{-2s}{s^2+4}+\frac{3\cdot 2}{s^2+4}=\frac{-2s+6}{s^2+4}$.

2. $L[f(t)e^{at}]=F(s-a)$, $s>a$

(Proof) For $s>a$, $L[f(t)e^{at}]=\int_0^\infty e^{-st}\cdot f(t)e^{at}dt=\int_0^\infty e^{-(s-a)t}\cdot f(t)dt=F(s-a)$.

Eg. Find $L[e^{at}\cos(kt)]$, $L[e^{at}\sin(kt)]$ and $L[e^{at}]$.

(Sol.) $L[\cos(kt)]=\frac{s}{s^2+k^2}$ and $L[\sin(kt)]=\frac{k}{s^2+k^2}$

According to $L[f(t)\cdot e^{at}]=F(s-a)$ and $L[1]=\int_0^\infty e^{-st}dt=\frac{1}{s}$,

$$\therefore\ L[e^{at}\cdot\cos(kt)]=\frac{s-a}{(s-a)^2+k^2},\ L[e^{at}\cdot\sin(kt)]=\frac{k}{(s-a)^2+k^2},\ \text{and } L[e^{at}]=\frac{1}{s-a}$$

3. $L[f'(t)]=sF(s)-f(0)$, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, $L[f'''(t)]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$, and $L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

(Proof) $L[f'(t)]=\int_0^\infty e^{-st}\cdot f'(t)dt=\int_0^\infty e^{-st}df(t)=e^{-st}f(t)|_0^\infty-\int_0^\infty(-s)e^{-st}\cdot f(t)dt$

$=-f(0)+s\int_0^\infty e^{-st}\cdot f(t)dt=sF(s)-f(0)$

Similarly, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, and by mathematical induction, we have
$L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

4. $L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}$

(Proof) $L\left[\int_0^t f(u)du\right] = \int_0^\infty e^{-st}\int_0^t f(u)dudt = \int_0^\infty\int_0^t e^{-st} f(u)dudt = \int_0^\infty\int_u^\infty e^{-st} f(u)dtdu$

$= \int_0^\infty f(u)du\int_u^\infty e^{-st}dt = -\frac{1}{s}\int_0^\infty f(u)[0-e^{-su}]du = \frac{1}{s}\int_0^\infty f(u)e^{-su}du = \frac{F(s)}{s}$

5. $L[t^n f(t)] = (-1)^n F^{(n)}(s)$

6. $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du$

(Proof of (5)) $L[tf(t)] = \int_0^\infty e^{-st}\cdot tf(t)dt = \int_0^\infty -\frac{de^{-st}}{ds}\cdot f(t)dt = -\frac{d}{ds}\int_0^\infty e^{-st}\cdot f(t)dt = -F'(s)$

By mathematical induction, we have $L[t^n f(t)] = (-1)^n F^{(n)}(s)$

Eg. Find $L[t^n]$.

(Sol.) According to $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$ and $L[1] = \int_0^\infty e^{-st}dt = \frac{1}{s}$

$$L[t^n] = L[t^n\cdot 1] = (-1)^n\frac{d^n}{ds^n}(s^{-1}) = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$$

7. $L[f(t)] = \frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}\cdot f(t)dt$ **if** $f(t+T)=f(t)$

(Proof) Let $t+T=u$, $t+2T=v$, …

$L[f(t)] = \int_0^\infty e^{-st}\cdot f(t)dt = \int_0^T e^{-st}\cdot f(t)dt + \int_T^{2T} e^{-st}\cdot f(t)dt + \int_{2T}^{3T} e^{-st}\cdot f(t)dt + \ldots$

$= \int_0^T e^{-st}\cdot f(t)dt + \int_T^{2T} e^{-st}\cdot f(t+T)dt + \int_{2T}^{3T} e^{-st}\cdot f(t+2T)dt + \ldots$

$= \int_0^T e^{-st}\cdot f(t)dt + e^{-sT}\int_T^{2T} e^{-s(t+T)}\cdot f(t+T)dt + e^{-2sT}\int_{2T}^{3T} e^{-s(t+2T)}\cdot f(t+2T)dt + \ldots$

$= \int_0^T e^{-st}\cdot f(t)dt + e^{-sT}\int_0^T e^{-su}\cdot f(u)du + e^{-2sT}\int_0^T e^{-sv}\cdot f(v)dv + \ldots$

$= (1+e^{-sT}+e^{-2sT}+\ldots)\int_0^T e^{-st}\cdot f(t)dt = \frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}\cdot f(t)dt$

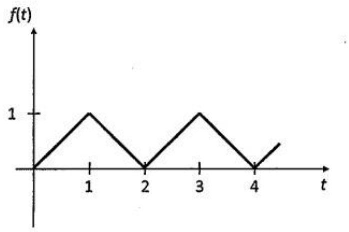
Eg. Find $L[f(t)]$ if $f(t+2)=f(t)$ and $f(t)=\begin{cases}1, & 0<t\le 1\\ -1, & 1<t\le 2\end{cases}$.

(Sol.) According to $L[f(t)] = \frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}\cdot f(t)dt$ if $f(t+T)=f(t)$ and $T=2$,

$$L[f(t)] = \frac{1}{1-e^{-2s}}\cdot\left[\int_0^1 e^{-st}dt + \int_1^2(-1)e^{-st}dt\right] = \frac{1}{1-e^{-2s}}\cdot\left[\frac{e^{-st}}{-s}\Big|_0^1 + \frac{e^{-st}}{s}\Big|_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\cdot\left[\frac{1-e^{-s}+e^{-2s}-e^{-s}}{s}\right] = \frac{1}{s}\cdot\frac{(1-e^{-s})^2}{(1+e^{-s})(1-e^{-s})} = \frac{1}{s}\cdot\frac{1-e^{-s}}{1+e^{-s}}$$

Eg. Find the Laplace transform of the given periodical function (triangular wave). [2022 台大電研工數 C]



(Sol.) Method 1: $f(t)=\begin{cases} t, & 0\le t<1 \\ 2-t, & 1<t\le 2 \end{cases}$ and $f(t+2)=f(t)$,

$$L[f(t)]=\frac{\int_0^2 f(t)e^{-st}dt}{1-e^{-2s}}=[\int_0^1 te^{-st}dt+\int_1^2(2-t)e^{-st}dt]\cdot\frac{1}{1-e^{-2s}}$$

It is known that $\int te^{-st}dt=\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}+C$ and $\int e^{-st}dt=-\frac{e^{-st}}{s}+C'$

$$\because\ \int_0^1 te^{-st}dt+\int_1^2(2-t)e^{-st}dt=-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}-\frac{2e^{-2s}}{s}+\frac{2e^{-s}}{s}+\frac{2e^{-2s}}{s}+\frac{e^{-2s}}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}$$

$$=-\frac{e^{-s}}{s^2}+\frac{1}{s^2}+\frac{e^{-2s}}{s^2}-\frac{e^{-s}}{s^2}=\frac{1-2e^{-s}+e^{-2s}}{s^2}=\frac{(1-e^{-s})^2}{s^2},$$

$$\therefore\ L[f(t)]=\frac{(1-e^{-s})^2}{s^2}\cdot\frac{1}{1-e^{-2s}}=\frac{(1-e^{-s})^2}{s^2(1-e^{-2s})}$$

Method 2: $f(t)=\int_0^t f(u)du$ is the definite integral for $g(t)=\begin{cases} 1, & 0<t\le 1 \\ -1, & 1<t\le 2 \end{cases}$ and $g(t+2)=g(t)$. It is known $L[g(t)]=\frac{1}{s}\cdot\frac{1-e^{-s}}{1+e^{-s}}$. By $L\left[\int_0^t f(u)du\right]=\frac{F(s)}{s}$, we have

$$L[f(t)]=\frac{1}{s^2}\cdot\frac{1-e^{-s}}{1+e^{-s}}=\frac{1}{s^2}\cdot\frac{(1-e^{-s})^2}{1-e^{-2s}}.$$

8. $L[f(at)]=[F(s/a)]/a$ and $L[f(t/a)]=aF(as)$, $a>0$

(Proof) For $a>0$, let $at=u$

$$L[f(at)]=\int_0^\infty e^{-st}\cdot f(at)dt=\frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})at}\cdot f(at)d(at)=\frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})u}\cdot f(u)du=\frac{1}{a}F[(\frac{s}{a})]$$

Let $b=\frac{1}{a}\Rightarrow L[f(\frac{t}{a})]=L[f(bt)]=\frac{1}{b}F[(\frac{s}{b})]=aF(as)$ and then $L^{-1}[F(as)]=\frac{1}{a}[f(\frac{t}{a})]$

9. $L[f(t-a)u(t-a)]=e^{-as}F(s)$

(Proof) For $t>a$, let $t-a=u$

$$L[f(t\text{-}a)u(t\text{-}a)]=\int_0^\infty e^{-st}\cdot f(t-a)u(t-a)dt=\int_a^\infty e^{-st}\cdot f(t-a)d(t-a)$$

$$=e^{-as}\int_a^\infty e^{-s(t-a)}\cdot f(t-a)d(t-a)$$

$=e^{-as}\int_0^\infty e^{-su}\cdot f(u)du=e^{-as}F(s)$, and then we have $L^{-1}[F(s)e^{-as}]=f(t-a)u(t-a)$

10. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ if (a) all nonzero roots of the denominator of $F(s)$ must have negative real parts, or (b) $F(s)$ must not have more than one pole at the origin.

11. $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Eg. Find $L[3t\text{-}5\sin(2t)]$. [2001台大電研]

(Sol.) $L[3t\text{-}5\sin(2t)]=3L[t]\text{-}5L[\sin(2t)]=\frac{3}{s^2}-\frac{10}{s^2+4}$

Eg. Find $L[\frac{1}{2}te^{2t}\sin(2t)]$. [2015中興電研乙丙丁組、中興光電所]

(Sol.) $L[\sin(2t)]=\frac{2}{s^2+4}$, $L[e^{2t}\sin(2t)]=\frac{2}{(s-2)^2+4}$, $\because$ $L[tf(t)]= -F'(s)$,

$\therefore$ $L[te^{2t}\sin(2t)]=\frac{2\cdot 2(s-2)}{[(s-2)^2+4]^2}$, $L[\frac{1}{2}te^{2t}\sin(2t)]=\frac{2(s-2)}{[(s-2)^2+4]^2}$

Eg. Find $L[e^{-t}f(3t)]$ in case of $L[f(t)]=e^{-1/s}$.

(Sol.) 1. $L[f(3t)]=\frac{1}{3}e^{-1/(s/3)}=\frac{1}{3}e^{-3/s}$, $L[e^{-t}\cdot f(3t)]=\frac{1}{3}e^{\frac{-3}{s+1}}$. The result is correct!

2. $L[e^{-t}\cdot f(t)]=e^{-\frac{1}{s+1}}$, $L[e^{-t}\cdot f(3t)]=\frac{1}{3}e^{\frac{-1}{(s/3)+1}}=\frac{1}{3}e^{\frac{-3}{s+3}}$. The result is wrong!

Another method: $L[e^{-t}\cdot f(3t)]=\int_0^\infty e^{-st}\cdot e^{-t}\cdot f(3t)dt=\int_0^\infty e^{-(s+1)t}\cdot f(3t)dt$

$=\frac{1}{3}\int_0^\infty e^{-\left(\frac{s+1}{3}\right)\cdot(3t)}f(3t)d(3t)=\frac{1}{3}\int_0^\infty e^{-\left(\frac{s+1}{3}\right)u}f(u)du=\frac{1}{3}F\left(\frac{s+1}{3}\right)=\frac{1}{3}e^{\frac{-3}{s+1}}$

Eg. Find $\int_0^\infty \frac{\sin(x)}{x}dx$. [2003 中央光電所、1993 交大應數研]

(Sol.) $L\left[\frac{\sin(t)}{t}\right]=\int_0^\infty e^{-st}\cdot\frac{\sin(t)}{t}dt=\int_s^\infty L[\sin(t)]ds=\int_s^\infty\frac{1}{s^2+1}ds=\tan^{-1}(s)\Big|_s^\infty=\frac{\pi}{2}-\tan^{-1}(s)$

Set s=0, $\int_0^\infty\frac{\sin(x)}{x}dx=\frac{\pi}{2}$

Eg. Determine $f(0)$ if the Laplace transform $F(s)$ of $f(t)$ is given as below: $F(s)=\frac{s}{(s-1)(s+2)}$. [2017 台師大電研]

(Sol.) $f(0)=\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{s^2}{(s-1)(s+2)}=1$

4-2 Inverse Laplace Transform $L^{-1}[F(s)]=f(t)$

Basic theorems of the inverse Laplace Transforms:

1. $L^{-1}[c_1F(s)+c_2G(s)]=c_1f(t)+c_2g(t)$

2. $L^{-1}[F(s+a)]=f(t)e^{-at}$

3. $L^{-1}[F(s)\cdot e^{-as}]=f(t-a)\cdot u(t-a)=\begin{cases} f(t-a), t\ge a \\ 0, t<a \end{cases}$

4. $L^{-1}[F(as)]=[f(t/a)]/a$

5. $L^{-1}[F^{(n)}(s)]=(-1)^n t^n f(t)$

6. $L^{-1}[\int_s^{\infty} F(u)du]=\dfrac{f(t)}{t}$

7. $L^{-1}[sF(s)]=f'(t)+f(0)\delta(t)$

8. $L^{-1}\left[\dfrac{F(s)}{s}\right]=\int_0^t f(u)du$

9. $L^{-1}[1/s^n]=t^{n-1}/(n-1)!=t^{n-1}/\Gamma(n)$

Heaviside's formulae: $L^{-1}[\dfrac{A}{s-a}]=Ae^{at}$, $L^{-1}[\dfrac{B}{(s-a)^m}]=\dfrac{Bt^{m-1}e^{at}}{(m-1)!}$,

$L^{-1}[\dfrac{Cs}{s^2+\omega^2}]=C\cos(\omega t)$, $L^{-1}[\dfrac{D\omega}{s^2+\omega^2}]=D\sin(\omega t)$

$L^{-1}[\dfrac{\alpha(s-r)}{(s-r)^2+\omega^2}]=\alpha\cos(\omega t)\cdot e^{rt}$, $L^{-1}[\dfrac{\beta\omega}{(s-r)^2+\omega^2}]=\beta\sin(\omega t)\cdot e^{rt}$

Eg. Find $L^{-1}\left[\dfrac{1}{s}\right]$, $L^{-1}\left[\dfrac{1}{s^2}\right]$, $L^{-1}\left[\dfrac{1}{s^3}\right]$, $L^{-1}\left[\dfrac{1}{s^4}\right]$, **and** $L^{-1}[1]$.

(Sol.) $L^{-1}[1/s^n]=t^{n-1}/(n-1)!$, $L^{-1}\left[\dfrac{1}{s}\right]=\dfrac{t^0}{0!}=1$, $L^{-1}\left[\dfrac{1}{s^2}\right]=\dfrac{t}{1!}=t$, $L^{-1}\left[\dfrac{1}{s^3}\right]=\dfrac{t^2}{2!}=\dfrac{t^2}{2}$,

$L^{-1}\left[\dfrac{1}{s^4}\right]=\dfrac{t^3}{3!}=\dfrac{t^3}{6}$. By $L^{-1}[sF(s)]=f'(t)+f(0)\delta(t)$, $L^{-1}[1]=L^{-1}\left[s\cdot\dfrac{1}{s}\right]=0+1\cdot\delta(t)=\delta(t)$.

Eg. Find $L^{-1}\left[\dfrac{2s-1}{s(s-1)}\right]$.

(Sol.) $\dfrac{2s-1}{s(s-1)}=\dfrac{A}{s}+\dfrac{B}{s-1}\Rightarrow A(s-1)+Bs=2s-1\Rightarrow\begin{cases} A+B=2 \\ -A=-1 \end{cases}$ or $\begin{cases} s=1\Rightarrow B=1 \\ s=0\Rightarrow -A=-1 \end{cases}$

$\Rightarrow A=1, B=1\Rightarrow L^{-1}\left[\dfrac{2s-1}{s(s-1)}\right]=L^{-1}[\dfrac{1}{s}]+L^{-1}[\dfrac{1}{s-1}]=1+e^t$

Eg. Find $L^{-1}\left[\dfrac{1}{(s-2)^3}\right]$. **[2013 成大電研]**

(Sol.) $L^{-1}[1/s^n]=t^{n-1}/(n-1)!$, $L^{-1}\left[\dfrac{1}{s^3}\right]=\dfrac{t^2}{2!}=\dfrac{t^2}{2}$, and $L^{-1}[F(s+a)]=f(t)e^{-at}$,

$\because$ $a=-2$, $\therefore$ $L^{-1}\left[\dfrac{1}{(s-2)^3}\right]=\dfrac{t^2e^{2t}}{2}$

Eg. Find $L^{-1}\left[\dfrac{e^{-2s}}{(s-2)^4}\right]$**. [1993 中山電研]**

(Sol.) According to $L^{-1}[F(s)\cdot e^{-as}] = f(t-a)\cdot u(t-a)$, $L^{-1}[\dfrac{1}{(s-2)^4}] = \dfrac{e^{2t}\cdot t^3}{3!}$

$$L^{-1}[\frac{e^{-2s}}{(s-2)^4}] = \frac{e^{2(t-2)}\cdot(t-2)^3}{3!}\cdot u(t-2)$$

Eg. Find $L^{-1}\left[\dfrac{2s^2-9s+19}{(s-1)^2(s+3)}\right]$**.**

(Sol.) $\dfrac{2s^2-9s+19}{(s-1)^2(s+3)} = \dfrac{A_1}{(s-1)} + \dfrac{A_2}{(s-1)^2} + \dfrac{B}{s+3}$

$$\Rightarrow \begin{cases} A_1 + B = 2 \\ 2A_1 + A_2 - 2B = -9 \\ -3A_1 + 3A_2 + B = 19 \end{cases} \Rightarrow \begin{cases} A_1 = -2 \\ A_2 = 3 \\ B = 4 \end{cases},\quad L^{-1}\left[\frac{2s^2-9s+19}{(s-1)^2(s+3)}\right] = -2e^t + 3te^t + 4e^{-3t}$$

Eg. Find $L^{-1}\left[\ell n\left(\dfrac{s+2}{s+1}\right)\right]$**. [2005 北科大電研]**

(Sol.) According to $L^{-1}[\int_s^\infty F(u)du] = \dfrac{f(t)}{t}$ and $L^{-1}[F(s+a)] = e^{-at}\cdot f(t)$

$$L^{-1}\left[\ell n\left(\frac{s+2}{s+1}\right)\right] = L^{-1}\left[\int_s^\infty\left(\frac{1}{s+1} - \frac{1}{s+2}\right)ds\right] = \frac{1}{t}\cdot[e^{-t} - e^{-2t}]$$

Eg. Find $L^{-1}\left[\dfrac{s-2}{s^2-4s+4+\pi^2}\right]$**. [2021 台大電研]**

(Sol.) $L^{-1}\left[\dfrac{s-2}{s^2-4s+4+\pi^2}\right] = L^{-1}\left[\dfrac{s-2}{(s-2)^2+\pi^2}\right] = e^{2t}\cdot L^{-1}\left[\dfrac{s}{s^2+\pi^2}\right] = e^{2t}\cdot\cos(\pi t)$

Eg. Find $L^{-1}\left[\dfrac{s^2}{s^2-2s+3}\right]$**. [2015 台大電研]**

(Sol.) According to $L^{-1}[F(s-a)] = e^{at}\cdot f(t)$ and $L^{-1}[1] = \delta(t)$

$$L^{-1}\left[\frac{s^2}{s^2-2s+3}\right] = L^{-1}\left[1 + \frac{2s-3}{s^2-2s+3}\right] = L^{-1}\left[1 + \frac{2(s-1)}{(s-1)^2+2} - \frac{1}{(s-1)^2+2}\right]$$

$$= L^{-1}\left[1 + \frac{2(s-1)}{(s-1)^2+(\sqrt{2})^2} - \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s-1)^2+(\sqrt{2})^2}\right] = \delta(t) + 2\cos(\sqrt{2}t)e^t - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)e^t.$$

Eg. Find $L^{-1}[\frac{\frac{s}{2}+\frac{7}{3}}{s^2+4s+6}]$. [2018 台大電研]

(Sol.) $\frac{\frac{s}{2}+\frac{7}{3}}{s^2+4s+6}=\frac{1}{2}\cdot\frac{s+2}{(s+2)^2+(\sqrt{2})^2}+\frac{4}{3}\cdot\frac{1}{(s+2)^2+(\sqrt{2})^2}=\frac{1}{2}\cdot\frac{s+2}{(s+2)^2+(\sqrt{2})^2}+\frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}$,

$L^{-1}[\frac{\frac{s}{2}+\frac{7}{3}}{s^2+4s+6}]=\frac{e^{-2t}cos(\sqrt{2}t)}{2}+\frac{2\sqrt{2}e^{-2t}sin(\sqrt{2}t)}{3}$.

Eg. Determine $f(0)$ and $f(\infty)$ if the Laplace transform $F(s)$ of $f(t)$ is given as below: $F(s)=\frac{s}{(s-1)(s+2)}$. [2017 台師大電研]

(Sol.) $F(s)=\frac{s}{(s-1)(s+2)}=\frac{\frac{1}{3}}{s-1}+\frac{\frac{2}{3}}{s+2}$, $f(t)=\frac{e^t}{3}+\frac{2e^{-2t}}{3}$, $\therefore f(0)=1$ and $f(\infty)=\infty$

Eg. Solve $y'+y+\int_0^t y(u)du=1$, $y(0)=0$. [2011中正電研]

(Sol.) $sY(s)-y(0)+Y(s)+\frac{Y(s)}{s}=\frac{1}{s}$,

$Y(s)=\frac{1}{s^2+s+1}=\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}=\frac{2}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\Rightarrow y(t)=\frac{2}{\sqrt{3}}\sin(\frac{\sqrt{3}t}{2})e^{-\frac{t}{2}}$

Eg. Solve $y'+2y+\int_0^t y(u)du=u(t-1)$, $y(0)=0$. [2013台聯大系統電機類聯招]

(Sol.) $Y(s)=\frac{e^{-s}}{(s+1)^2}$, $L^{-1}[\frac{1}{(s+1)^2}]=te^{-t}$ and $L^{-1}[F(s)e^{-as}]=f(t-a)u(t-a)$,

$\therefore y(t)=(t-1)e^{-(t-1)}\cdot u(t-1)$

4-3 Laplace Transform Solutions of Differential Equations with Polynomial Coefficients

Eg. Solve $xy''-xy'-y=0$, $y(0)=0$ and $y'(0)=3$. [1991 成大電研]

(Sol.) $L[y(x)]=\int_0^\infty y(x)e^{-sx}dx=Y(s)$, $L[x^n y(x)]=(-1)^n\frac{d^n}{ds^n}Y(s)$, and

$$L[y^{(n)}(x)]=s^nY(s)-s^{n-1}\cdot y(0)-s^{n-2}\cdot y'(0)-\cdots\cdots-s\cdot y^{(n-2)}(0)-y^{(n-1)}(0)$$

$$-\frac{d}{ds}[s^2Y(s)-sy(0)-y'(0)]-(-\frac{d}{ds})[sY(s)-y(0)]-Y(s)=0$$

$$-2sY(s)-s^2Y'(s)+Y(s)+sY'(s)-Y(s)=0$$

$$(-s^2+s)Y'(s)-2sY(s)=0,\quad Y'(s)+\frac{2}{s-1}Y(s)=0$$

$$\Rightarrow Y(s)=\frac{A}{(s-1)^2}\Rightarrow y(x)=Axe^x,\ y'(0)=3\Rightarrow A=3,\ \therefore y(x)=3xe^x$$

Eg. Solve $2y''+ty'-2y=10$, $y(0)=y'(0)=0$. [2011 台大電子所甲組]

(Sol). $L[2y''+ty'-2y]=L(10)=\frac{10}{s}$,

$$2[s^2Y(s)-sy(0)-y'(0)]+(-1)\frac{d}{ds}[sY(s)-y(0)]-2Y(s)=\frac{10}{s},$$

$-sY'(s)+(2s^2-3)Y(s)=\frac{10}{s}$, $Y'(s)+(-2s+\frac{3}{s})Y(s)=-\frac{10}{s^2}$, $\int(-2s+\frac{3}{s})ds=-s^2+3\ln(s)$,

$\exp[-s^2+3\ln(s)]=s^3e^{-s^2}$, $s^3e^{-s^2}Y'(s)+[-2s^4e^{-s^2}+3s^2e^{-s^2}]Y(s)=-10se^{-s^2}$,

$[s^3e^{-s^2}Y(s)]'=-10se^{-s^2}$, $s^3e^{-s^2}Y(s)=5e^{-s^2}+C$, $Y(s)=\frac{5}{s^3}+Cs^{-3}e^{s^2}$,

$$\lim_{t\to 0}y(t)=\lim_{s\to\infty}sY(s)=0\Rightarrow y(t)=\frac{5}{2}t^2$$

4-4 Convolution and Dirac Delta Function

Convolution in Laplace transform: $f(t)*g(t)=\int_0^t f(t-\alpha)g(\alpha)d\alpha$

Theorem $L[f(t)*g(t)]=L[\int_0^t f(t-\alpha)g(\alpha)d\alpha]=F(s)G(s)$

Eg. Find $L[e^t\int_0^t e^{-2\tau}\sinh(t-\tau)\cos(\tau)d\tau]$. [2015 台大電研]

(Sol.) $L[e^t\int_0^t e^{-2\tau}\sinh(t-\tau)\cos(\tau)d\tau]=L[\int_0^t e^{t-2\tau}\sinh(t-\tau)\cos(\tau)d\tau]$

$$=L[\int_0^t\frac{e^{2(t-\tau)}-1}{2}\cdot e^{-\tau}\cos(\tau)d\tau]=L\{(\frac{e^{2t}-1}{2})*[e^{-t}\cos(t)]\}=\frac{1}{2}(\frac{1}{s-2}-\frac{1}{s})\cdot[\frac{s+1}{(s+1)^2+1}]$$

Eg. Given $f(t)=2t*[\sin(t)-e^t]$**, find** $f(t)$**=? [2024 台大電研]**

(Sol.) $L[f(t)]=L\{2t*[\sin(t)-e^t]\}$, $F(s)=\frac{2}{s^2}\cdot[\frac{1}{s^2+1}-\frac{1}{s-1}]=\frac{2}{s^2(s^2+1)}-\frac{2}{s^2(s-1)}$

$$F(s)=\frac{2}{s^2}-\frac{2}{s^2+1}+\frac{2}{s}+\frac{2}{s^2}-\frac{2}{s-1}=\frac{2}{s}+\frac{4}{s^2}-\frac{2}{s-1}-\frac{2}{s^2+1}$$

$\therefore\quad f(t)=2+4t-2e^t-2\sin(t)$

Eg. Solve $\int_0^t f(\tau)f(t-\tau)d\tau=6t^3$**. [2006 台大電研]**

(Sol.) According to $L[t^n]=\frac{n!}{s^{n+1}}$ and $L^{-1}[1/s^n]=t^{n-1}/(n-1)!$, $[F(s)]^2=6L[t^3]=\frac{36}{s^4}$,

$F(s)=\frac{6}{s^2}\Rightarrow f(t)=6t$

Eg. Solve $y''+y-4\int_0^t y(\tau)\sin(t-\tau)d\tau=e^{-2t}$**,** $y(0)=1$ **and** $y'(0)=0$**. [1990 交大電信所]**

(Sol.) $s^2Y(s)-sy(0)-y'(0)+Y(s)-4Y(s)\frac{1}{s^2+1}=\frac{1}{s+2}$

$$\Rightarrow Y(s)=\frac{(s^2+1)(s+1)}{(s+2)(s^2+3)(s-1)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{Cs+D}{s^2+3}$$

$(s^2+1)(s+1)=A(s+2)(s^2+3)+B(s-1)(s^2+3)+(Cs+D)(s-1)(s+2)$

Let $s=1\Rightarrow 4=A\cdot3\cdot4\Rightarrow A=\frac{1}{3}$.

Let $s=-2\Rightarrow -5=B(-3)\cdot7\Rightarrow B=\frac{5}{21}$.

Let $s=0\Rightarrow 1=2-\frac{5}{7}-2D\Rightarrow D=\frac{1}{7}$.

Let $s=-1\Rightarrow 0=\frac{1}{3}\cdot1\cdot4+\frac{5}{21}\cdot(-2)\cdot4+(-C+\frac{1}{7})(-1)(2)\Rightarrow C=\frac{3}{7}$

$$\Rightarrow Y(s)=\frac{1/3}{s-1}+\frac{5/21}{s+2}+\frac{3s/7}{s^2+3}+\frac{1}{7\sqrt{3}}\cdot\frac{\sqrt{3}}{s^2+3}$$

$$\Rightarrow y(t)=\frac{1}{3}e^t+\frac{5}{21}e^{-2t}+\frac{3}{7}\cos\left(\sqrt{3}t\right)+\frac{1}{7\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

Eg. Solve $f(t)+2\int_0^t f(\tau)\cos(t-\tau)d\tau=4e^{-t}+\sin(t)$**. [2022 台大電研工數 C]**

(Sol.) $F(s)+2F(s)\cdot\frac{s}{s^2+1}=\frac{4}{s+1}+\frac{1}{s^2+1}$, $F(s)=\frac{4s^2+s+5}{(s+1)^3}=\frac{4}{s+1}-\frac{7}{(s+1)^2}+\frac{8}{(s+1)^3}$

$\therefore f(t)=4e^{-t}-7te^{-t}+4t^2e^{-t}$.

Eg. Solve $f(t)=3t^2-e^{-t}-\int_0^t f(\rho)e^{t-\rho}d\rho$. [2011 台大電研]

(Sol.) $F(s)=\frac{6}{s^3}-\frac{1}{s+1}-\frac{1}{s-1}\cdot F(s)$, $\frac{s}{s-1}\cdot F(s)=\frac{6}{s^3}-\frac{1}{s+1}$, $F(s)=\frac{6}{s^3}-\frac{6}{s^4}+\frac{1}{s}-\frac{2}{s+1}$,

$\Rightarrow f(t)=3t^2-t^3+1-2e^{-t}$

Eg. Solve $y+e^t\int_0^t y(\tau)e^{-\tau}d\tau=\cos(t)$. [2015 中興電研乙丙丁組、中興光電所]

(Sol.) $y+\int_0^t y(\tau)e^{t-\tau}d\tau=\cos(t)$, $y+y*e^t=\cos(t)$, $Y(s)+Y(s)\frac{1}{s-1}=\frac{s}{s^2+1}$

$\Rightarrow Y(s)=\frac{s-1}{s^2+1}=\frac{s}{s^2+1}-\frac{1}{s^2+1}\Rightarrow y(t)=\cos(t)-\sin(t)$

Dirac delta function:

$$\delta(t-a)=\delta_a(t)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[u(t-a)-u(t-a-\varepsilon)]=\begin{cases}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}, a\le t\le a+\varepsilon\\ 0, elsewhere\end{cases}$$

Kronecker delta: $\delta_{ab}=\begin{cases}1, a=b\\ 0, a\ne b\end{cases}$

Characteristics of Dirac's delta function:

1. $\int_{-\infty}^{\infty}\delta(t-a)dt=1$. 2. $L[\delta(t-a)]=e^{-as}$. 3. $\int_{-\infty}^{\infty}f(t)\delta(t-a)dt=f(a)$.

4. $f(t)*\delta(t)=\int_0^t f(x)\delta(t-x)dx=f(t)$ 5. $f(t)\delta(t-a)=f(a)\delta(t-a)$

Eg. Solve $y'+7y=10\delta(t-2)$. [2024 台大電研]

(Sol.) $sY(s)-10+7Y(s)=10e^{-2s}$, $Y(s)=\frac{10+10e^{-2s}}{s+7}$,

$\therefore\ y(t)=10e^{-7t}+10e^{-7(t-2)}\cdot u(t-2)$

Eg. Solve $y''+y=\delta(t-\pi)$, $y(0)=y'(0)=0$. [2017 台聯大電研]

(Sol.) $s^2Y(s)-sy(0)-y'(0)+Y(s)=s^2Y(s)+Y(s)=e^{-\pi s}$, $Y(s)=\frac{e^{-\pi s}}{s^2+1}$,

$\because L^{-1}[\frac{1}{s^2+1}]=\sin(t)$ and $L^{-1}[F(s)\cdot e^{-as}]=f(t-a)\cdot u(t-a)=\begin{cases}f(t-a), t\ge a\\ 0, t<a\end{cases}$

$\therefore y(t)=L^{-1}[Y(s)]=\sin[(t-\pi)]\cdot u(t-\pi)$

Eg. Solve $y''+5y'+4y=3+2\delta(t)$.【北科大土木所】