

Chapter 4 Laplace Transforms

4-1 Laplace Transform $F(s)=L[f(t)]=\int_0^{\infty} e^{-st} \cdot f(t)dt$

Eg. Evaluate $L[\cos(at)]$ and $L[\sin(at)]$.

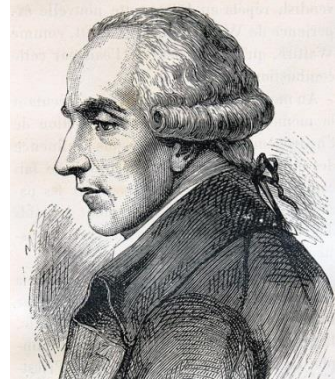
(Sol.) $e^{iat} = \cos(at) + i \sin(at)$

$$L[e^{iat}] = \int_0^{\infty} e^{-st} \cdot e^{iat} dt = \int_0^{\infty} e^{-(s-ia)t} dt$$

$$= \frac{-1}{s-ia} \cdot e^{-(s-ia)t} \Big|_0^{\infty} = \frac{1}{s-ia}$$

$$= \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2} = L[\cos(at)] + iL[\sin(at)]$$

$$\therefore L[\cos(at)] = \frac{s}{s^2+a^2} \quad \text{and} \quad L[\sin(at)] = \frac{a}{s^2+a^2}$$



Eg Find $\int_0^{\infty} e^{-ax} \cdot \cos(bx)dx$, with $a>0$. [2005 台大電研] (Ans.) $\frac{a}{a^2+b^2}$

Basic theorems of Laplace transforms $F(s)=L[f(t)]$ and $G(s)=L[g(t)]$:

1. $L[c_1f(t)+c_2g(t)]=c_1F(s)+c_2G(s)$

Eg. $L[-2\cos(2t)+3\sin(2t)]$. [2010 台大光電所]

(Sol.) $L[-2\cos(2t)+3\sin(2t)]=-2L[\cos(2t)]+3L[\sin(2t)]=\frac{-2s}{s^2+4} + \frac{3 \cdot 2}{s^2+4} = \frac{-2s+6}{s^2+4}$.

2. $L[f(t)e^{at}]=F(s-a)$, $s>a$

(Proof) For $s>a$, $L[f(t)e^{at}]=\int_0^{\infty} e^{-st} \cdot f(t)e^{at} dt = \int_0^{\infty} e^{-(s-a)t} \cdot f(t)dt = F(s-a)$.

Eg. Find $L[e^{at}\cos(kt)]$, $L[e^{at}\sin(kt)]$ and $L[e^{at}]$.

(Sol.) $L[\cos(kt)] = \frac{s}{s^2+k^2}$ and $L[\sin(kt)] = \frac{k}{s^2+k^2}$

According to $L[f(t) \cdot e^{at}] = F(s-a)$ and $L[1] = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$,

$$\therefore L[e^{at} \cdot \cos(kt)] = \frac{s-a}{(s-a)^2+k^2}, \quad L[e^{at} \cdot \sin(kt)] = \frac{k}{(s-a)^2+k^2}, \quad \text{and} \quad L[e^{at}] = \frac{1}{s-a}$$

3. $L[f'(t)]=sF(s)-f(0)$, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, $L[f'''(t)]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$, and $L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\dots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

(Proof) $L[f'(t)]=\int_0^{\infty} e^{-st} \cdot f'(t)dt = \int_0^{\infty} e^{-st} df(t) = e^{-st}f(t) \Big|_0^{\infty} - \int_0^{\infty} (-s)e^{-st} \cdot f(t)dt$

$$= -f(0) + s \int_0^{\infty} e^{-st} \cdot f(t)dt = sF(s) - f(0)$$

Similarly, $L[f''(t)]=s^2F(s)-sf(0)-f'(0)$, and by mathematical induction, we have

$$L[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\dots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

$$4. L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}$$

$$\begin{aligned} \text{(Proof)} \quad L\left[\int_0^t f(u)du\right] &= \int_0^\infty e^{-st} \int_0^t f(u)du dt = \int_0^\infty \int_0^t e^{-st} f(u)du dt = \int_0^\infty \int_u^\infty e^{-st} f(u)dt du \\ &= \int_0^\infty f(u)du \int_u^\infty e^{-st} dt = -\frac{1}{s} \int_0^\infty f(u)[0 - e^{-su}] du = \frac{1}{s} \int_0^\infty f(u)e^{-su} du = \frac{F(s)}{s} \end{aligned}$$

$$5. L[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$6. L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)du$$

$$\text{(Proof of (5)) } L[tf(t)] = \int_0^\infty e^{-st} \cdot tf(t) dt = \int_0^\infty -\frac{de^{-st}}{ds} \cdot f(t) dt = -\frac{d}{ds} \int_0^\infty e^{-st} \cdot f(t) dt = -F'(s)$$

By mathematical induction, we have $L[t^n f(t)] = (-1)^n F^{(n)}(s)$

Eg. Find $L[t^n]$.

$$\text{(Sol.) According to } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \text{ and } L[1] = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

$$L[t^n] = L[t^n \cdot 1] = (-1)^n \frac{d^n}{ds^n} (s^{-1}) = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$7. L[f(t)] = \frac{1}{1-e^{-sT}} \cdot \int_0^T e^{-st} \cdot f(t) dt \quad \text{if } f(t+T) = f(t)$$

(Proof) Let $t+T=u$, $t+2T=v$, ...

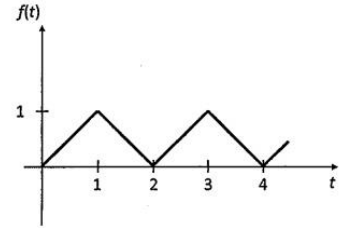
$$\begin{aligned} L[f(t)] &= \int_0^\infty e^{-st} \cdot f(t) dt = \int_0^T e^{-st} \cdot f(t) dt + \int_T^{2T} e^{-st} \cdot f(t) dt + \int_{2T}^{3T} e^{-st} \cdot f(t) dt + \dots \\ &= \int_0^T e^{-st} \cdot f(t) dt + \int_T^{2T} e^{-st} \cdot f(t+T) dt + \int_{2T}^{3T} e^{-st} \cdot f(t+2T) dt + \dots \\ &= \int_0^T e^{-st} \cdot f(t) dt + e^{-sT} \int_T^{2T} e^{-s(t+T)} \cdot f(t+T) dt + e^{-2sT} \int_{2T}^{3T} e^{-s(t+2T)} \cdot f(t+2T) dt + \dots \\ &= \int_0^T e^{-st} \cdot f(t) dt + e^{-sT} \int_0^T e^{-su} \cdot f(u) du + e^{-2sT} \int_0^T e^{-sv} \cdot f(v) dv + \dots \\ &= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T e^{-st} \cdot f(t) dt = \frac{1}{1-e^{-sT}} \cdot \int_0^T e^{-st} \cdot f(t) dt \end{aligned}$$

$$\text{Eg. Find } L[f(t)] \text{ if } f(t+2) = f(t) \text{ and } f(t) = \begin{cases} 1, & 0 < t \leq 1 \\ -1, & 1 < t \leq 2 \end{cases}$$

$$\text{(Sol.) According to } L[f(t)] = \frac{1}{1-e^{-sT}} \cdot \int_0^T e^{-st} \cdot f(t) dt \quad \text{if } f(t+T) = f(t) \text{ and } T=2,$$

$$\begin{aligned} L[f(t)] &= \frac{1}{1-e^{-2s}} \cdot \left[\int_0^1 e^{-st} dt + \int_1^2 (-1)e^{-st} dt \right] = \frac{1}{1-e^{-2s}} \cdot \left[\frac{e^{-st}}{-s} \Big|_0^1 + \frac{e^{-st}}{s} \Big|_1^2 \right] \\ &= \frac{1}{1-e^{-2s}} \cdot \left[\frac{1-e^{-s} + e^{-2s} - e^{-s}}{s} \right] = \frac{1}{s} \cdot \frac{(1-e^{-s})^2}{(1+e^{-s})(1-e^{-s})} = \frac{1}{s} \cdot \frac{1-e^{-s}}{1+e^{-s}} \end{aligned}$$

Eg. Find the Laplace transform of the given periodical function (triangular wave). [2022 台大電研工數 C]



(Sol.) Method 1: $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2-t, & 1 < t \leq 2 \end{cases}$ and $f(t+2) = f(t)$,

$$L[f(t)] = \frac{\int_0^2 f(t)e^{-st} dt}{1 - e^{-2s}} = \left[\int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt \right] \cdot \frac{1}{1 - e^{-2s}}$$

It is known that $\int te^{-st} dt = \frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} + C$ and $\int e^{-st} dt = -\frac{e^{-st}}{s} + C'$

$$\begin{aligned} \therefore \int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt &= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{2e^{-2s}}{s} + \frac{2e^{-s}}{s} + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \\ &= -\frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} = \frac{(1 - e^{-s})^2}{s^2}, \end{aligned}$$

$$\therefore L[f(t)] = \frac{(1 - e^{-s})^2}{s^2} \cdot \frac{1}{1 - e^{-2s}} = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-2s})}$$

Method 2: $f(t) = \int_0^t g(u) du$ is the definite integral for $g(t) = \begin{cases} 1, & 0 < t \leq 1 \\ -1, & 1 < t \leq 2 \end{cases}$ and

$g(t+2) = g(t)$. It is known $L[g(t)] = \frac{1}{s} \cdot \frac{1 - e^{-s}}{1 + e^{-s}}$. By $L\left[\int_0^t f(u) du\right] = \frac{F(s)}{s}$, we have

$$L[f(t)] = \frac{1}{s^2} \cdot \frac{1 - e^{-s}}{1 + e^{-s}} = \frac{1}{s^2} \cdot \frac{(1 - e^{-s})^2}{1 - e^{-2s}}$$

8. $L[f(at)] = [F(s/a)]/a$ and $L[f(t/a)] = aF(as)$, $a > 0$

(Proof) For $a > 0$, let $at = u$

$$L[f(at)] = \int_0^\infty e^{-st} \cdot f(at) dt = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)at} \cdot f(at) d(at) = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} \cdot f(u) du = \frac{1}{a} F\left[\left(\frac{s}{a}\right)\right]$$

Let $b = \frac{1}{a} \Rightarrow L\left[f\left(\frac{t}{a}\right)\right] = L[f(bt)] = \frac{1}{b} F\left[\left(\frac{s}{b}\right)\right] = aF(as)$ and then $L^{-1}[F(as)] = \frac{1}{a} \left[f\left(\frac{t}{a}\right)\right]$

9. $L[f(t-a)u(t-a)] = e^{-as}F(s)$

(Proof) For $t > a$, let $t-a = u$

$$L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} \cdot f(t-a)u(t-a) dt = \int_a^\infty e^{-st} \cdot f(t-a) d(t-a)$$

$$= e^{-as} \int_a^\infty e^{-s(t-a)} \cdot f(t-a) d(t-a)$$

$$= e^{-as} \int_0^\infty e^{-su} \cdot f(u) du = e^{-as}F(s), \text{ and then we have } L^{-1}[F(s)e^{-as}] = f(t-a)u(t-a)$$

10. $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$ if (a) all nonzero roots of the denominator of $F(s)$ must have negative real parts, or (b) $F(s)$ must not have more than one pole at the origin.

11. $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

Eg. Find $L[3t-5\sin(2t)]$. [2001 台大電研]

$$\text{(Sol.) } L[3t-5\sin(2t)] = 3L[t] - 5L[\sin(2t)] = \frac{3}{s^2} - \frac{10}{s^2 + 4}$$

Eg. Find $L[\frac{1}{2}te^{2t}\sin(2t)]$. [2015 中興電研乙丙丁組、中興光電所]

$$\text{(Sol.) } L[\sin(2t)] = \frac{2}{s^2 + 4}, L[e^{2t}\sin(2t)] = \frac{2}{(s-2)^2 + 4}, \because L[tf(t)] = -F'(s),$$

$$\therefore L[te^{2t}\sin(2t)] = \frac{2 \cdot 2(s-2)}{[(s-2)^2 + 4]^2}, L[\frac{1}{2}te^{2t}\sin(2t)] = \frac{2(s-2)}{[(s-2)^2 + 4]^2}$$

Eg. Find $L[e^{-t}f(3t)]$ in case of $L[f(t)] = e^{-1/s}$.

$$\text{(Sol.) } 1. L[f(3t)] = \frac{1}{3}e^{-1/(s/3)} = \frac{1}{3}e^{-3/s}, L[e^{-t} \cdot f(3t)] = \frac{1}{3}e^{-\frac{3}{s+1}}. \text{ The result is correct!}$$

$$2. L[e^{-t} \cdot f(t)] = e^{-\frac{1}{s+1}}, L[e^{-t} \cdot f(3t)] = \frac{1}{3}e^{-\frac{1}{(s/3)+1}} = \frac{1}{3}e^{-\frac{3}{s+3}}. \text{ The result is wrong!}$$

$$\text{Another method: } L[e^{-t} \cdot f(3t)] = \int_0^{\infty} e^{-st} \cdot e^{-t} \cdot f(3t) dt = \int_0^{\infty} e^{-(s+1)t} \cdot f(3t) dt$$

$$= \frac{1}{3} \int_0^{\infty} e^{-\left(\frac{s+1}{3}\right) \cdot (3t)} f(3t) d(3t) = \frac{1}{3} \int_0^{\infty} e^{-\left(\frac{s+1}{3}\right)u} f(u) du = \frac{1}{3} F\left(\frac{s+1}{3}\right) = \frac{1}{3} e^{-\frac{3}{s+1}}$$

Eg. Find $\int_0^{\infty} \frac{\sin(x)}{x} dx$. [2003 中央光電所、1993 交大應數研]

$$\text{(Sol.) } L\left[\frac{\sin(t)}{t}\right] = \int_0^{\infty} e^{-st} \cdot \frac{\sin(t)}{t} dt = \int_s^{\infty} L[\sin(t)] ds = \int_s^{\infty} \frac{1}{s^2 + 1} ds = \tan^{-1}(s) \Big|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s)$$

$$\text{Set } s=0, \int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

**Eg. Determine $f(0)$ if the Laplace transform $F(s)$ of $f(t)$ is given as below:
 $F(s) = \frac{s}{(s-1)(s+2)}$. [2017 台師大電研]**

$$\text{(Sol.) } f(0) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{s^2}{(s-1)(s+2)} = 1$$

4-2 Inverse Laplace Transform $L^{-1}[F(s)]=f(t)$

Basic theorems of the inverse Laplace Transforms:

1. $L^{-1}[c_1F(s)+c_2G(s)]=c_1f(t)+c_2g(t)$
2. $L^{-1}[F(s+a)] = f(t)e^{-at}$
3. $L^{-1}[F(s) \cdot e^{-as}] = f(t-a) \cdot u(t-a) = \begin{cases} f(t-a), t \geq a \\ 0, t < a \end{cases}$
4. $L^{-1}[F(as)] = [f(t/a)]/a$
5. $L^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$
6. $L^{-1}[\int_s^\infty F(u)du] = \frac{f(t)}{t}$
7. $L^{-1}[sF(s)] = f'(t) + f(0)\delta(t)$
8. $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(u)du$
9. $L^{-1}[1/s^n] = t^{n-1}/(n-1)! = t^{n-1}/\Gamma(n)$

Heaviside's formulae: $L^{-1}\left[\frac{A}{s-a}\right] = Ae^{at}$, $L^{-1}\left[\frac{B}{(s-a)^m}\right] = \frac{Bt^{m-1}e^{at}}{(m-1)!}$,

$L^{-1}\left[\frac{Cs}{s^2+\omega^2}\right] = C \cos(\omega t)$, $L^{-1}\left[\frac{D\omega}{s^2+\omega^2}\right] = D \sin(\omega t)$

$L^{-1}\left[\frac{\alpha(s-r)}{(s-r)^2+\omega^2}\right] = \alpha \cos(\omega t) \cdot e^{rt}$, $L^{-1}\left[\frac{\beta\omega}{(s-r)^2+\omega^2}\right] = \beta \sin(\omega t) \cdot e^{rt}$

Eg. Find $L^{-1}\left[\frac{1}{s}\right]$, $L^{-1}\left[\frac{1}{s^2}\right]$, $L^{-1}\left[\frac{1}{s^3}\right]$, $L^{-1}\left[\frac{1}{s^4}\right]$, and $L^{-1}[1]$.

(Sol.) $L^{-1}[1/s^n] = t^{n-1}/(n-1)!$, $L^{-1}\left[\frac{1}{s}\right] = \frac{t^0}{0!} = 1$, $L^{-1}\left[\frac{1}{s^2}\right] = \frac{t}{1!} = t$, $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!} = \frac{t^2}{2}$,

$L^{-1}\left[\frac{1}{s^4}\right] = \frac{t^3}{3!} = \frac{t^3}{6}$. By $L^{-1}[sF(s)] = f'(t) + f(0)\delta(t)$, $L^{-1}[1] = L^{-1}\left[s \cdot \frac{1}{s}\right] = 0 + 1 \cdot \delta(t) = \delta(t)$.

Eg. Find $L^{-1}\left[\frac{2s-1}{s(s-1)}\right]$.

(Sol.) $\frac{2s-1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \Rightarrow A(s-1) + Bs = 2s-1 \Rightarrow \begin{cases} A+B=2 \\ -A=-1 \end{cases}$ or $\begin{cases} s=1 \Rightarrow B=1 \\ s=0 \Rightarrow -A=-1 \end{cases} \Rightarrow$

$A=1, B=1 \Rightarrow L^{-1}\left[\frac{2s-1}{s(s-1)}\right] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] = 1 + e^t$

Eg. Find $L^{-1}\left[\frac{1}{(s-2)^3}\right]$. [2013 成大電研]

(Sol.) $L^{-1}[1/s^n] = t^{n-1}/(n-1)!$, $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!} = \frac{t^2}{2}$, and $L^{-1}[F(s+a)] = f(t)e^{-at}$,

$\therefore a=-2, \therefore L^{-1}\left[\frac{1}{(s-2)^3}\right] = \frac{t^2 e^{2t}}{2}$

Eg. Find $L^{-1}\left[\frac{e^{-2s}}{(s-2)^4}\right]$. [1993 中山電研]

(Sol.) According to $L^{-1}[F(s) \cdot e^{-as}] = f(t-a) \cdot u(t-a)$, $L^{-1}\left[\frac{1}{(s-2)^4}\right] = \frac{e^{2t} \cdot t^3}{3!}$

$$L^{-1}\left[\frac{e^{-2s}}{(s-2)^4}\right] = \frac{e^{2(t-2)} \cdot (t-2)^3}{3!} \cdot u(t-2)$$

Eg. Find $L^{-1}\left[\frac{2s^2 - 9s + 19}{(s-1)^2(s+3)}\right]$.

$$(Sol.) \frac{2s^2 - 9s + 19}{(s-1)^2(s+3)} = \frac{A_1}{s-1} + \frac{A_2}{(s-1)^2} + \frac{B}{s+3}$$

$$\Rightarrow \begin{cases} A_1 + B = 2 \\ 2A_1 + A_2 - 2B = -9 \\ -3A_1 + 3A_2 + B = 19 \end{cases} \Rightarrow \begin{cases} A_1 = -2 \\ A_2 = 3 \\ B = 4 \end{cases}, L^{-1}\left[\frac{2s^2 - 9s + 19}{(s-1)^2(s+3)}\right] = -2e^t + 3te^t + 4e^{-3t}$$

Eg. Find $L^{-1}\left[\ln\left(\frac{s+2}{s+1}\right)\right]$. [2005 北科大電研]

(Sol.) According to $L^{-1}\left[\int_s^\infty F(u)du\right] = \frac{f(t)}{t}$ and $L^{-1}[F(s+a)] = e^{-at} \cdot f(t)$

$$L^{-1}\left[\ln\left(\frac{s+2}{s+1}\right)\right] = L^{-1}\left[\int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+2}\right)ds\right] = \frac{1}{t} \cdot [e^{-t} - e^{-2t}]$$

Eg. Find $L^{-1}\left[\frac{s+5}{s^2+4s+5}\right]$. [2025 台大電研]

(Sol.)

$$L^{-1}\left[\frac{s+5}{s^2+4s+5}\right] = L^{-1}\left[\frac{s+2+3}{s^2+4s+4+1}\right] = L^{-1}\left[\frac{(s+2)+3}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] + L^{-1}\left[\frac{3}{(s+2)^2+1}\right]$$

$$= \cos(t)e^{-2t} + 3\sin(t)e^{-2t}$$

Eg. Find $L^{-1}\left[\frac{\frac{s}{2} + \frac{7}{3}}{s^2+4s+6}\right]$. [2018 台大電研]

$$(Sol.) \frac{\frac{s}{2} + \frac{7}{3}}{s^2+4s+6} = \frac{1}{2} \cdot \frac{s+2}{(s+2)^2+(\sqrt{2})^2} + \frac{4}{3} \cdot \frac{1}{(s+2)^2+(\sqrt{2})^2} = \frac{1}{2} \cdot \frac{s+2}{(s+2)^2+(\sqrt{2})^2} + \frac{2\sqrt{2}}{3} \cdot \frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2},$$

$$L^{-1}\left[\frac{\frac{s}{2} + \frac{7}{3}}{s^2+4s+6}\right] = \frac{e^{-2t}\cos(\sqrt{2}t)}{2} + \frac{2\sqrt{2}e^{-2t}\sin(\sqrt{2}t)}{3}.$$

Eg. Find $L^{-1}\left[\frac{s^2}{s^2-2s+3}\right]$. [2015 台大電研]

(Sol.) According to $L^{-1}[F(s-a)] = e^{at} \cdot f(t)$ and $L^{-1}[1] = \delta(t)$

$$\begin{aligned} L^{-1}\left[\frac{s^2}{s^2-2s+3}\right] &= L^{-1}\left[1 + \frac{2s-3}{s^2-2s+3}\right] = L^{-1}\left[1 + \frac{2(s-1)}{(s-1)^2+2} - \frac{1}{(s-1)^2+2}\right] \\ &= L^{-1}\left[1 + \frac{2(s-1)}{(s-1)^2+(\sqrt{2})^2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s-1)^2+(\sqrt{2})^2}\right] = \delta(t) + 2\cos(\sqrt{2}t)e^t - \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)e^t. \end{aligned}$$

Eg. Determine $f(0)$ and $f(\infty)$ if the Laplace transform $F(s)$ of $f(t)$ is given as below:
 $F(s) = \frac{s}{(s-1)(s+2)}$. [2017 台師大電研]

(Sol.) $F(s) = \frac{s}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{\frac{2}{3}}{s+2}$, $f(t) = \frac{e^t}{3} + \frac{2e^{-2t}}{3}$, $\therefore f(0) = 1$ and $f(\infty) = \infty$

Eg. Solve $y' + y + \int_0^t y(u)du = 1, y(0) = 0$. [2011 中正電研]

(Sol.) $sY(s) - y(0) + Y(s) + \frac{Y(s)}{s} = \frac{1}{s}$,

$$Y(s) = \frac{1}{s^2 + s + 1} = \frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \frac{2}{\sqrt{3}} \cdot \frac{\frac{\sqrt{3}}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \Rightarrow y(t) = \frac{2}{\sqrt{3}} \sin(\frac{\sqrt{3}t}{2}) e^{-\frac{t}{2}}$$

Eg. Solve $y' + 2y + \int_0^t y(u)du = u(t-1), y(0) = 0$. [2013 台聯大系統電機類聯招]

(Sol.) $Y(s) = \frac{e^{-s}}{(s+1)^2}$, $L^{-1}\left[\frac{1}{(s+1)^2}\right] = te^{-t}$ and $L^{-1}[F(s)e^{-as}] = f(t-a)u(t-a)$,

$\therefore y(t) = (t-1)e^{-(t-1)} \cdot u(t-1)$

4-3 Laplace Transform Solutions of Differential Equations with Polynomial Coefficients

Eg. Solve $xy''-xy'-y=0, y(0)=0$ and $y'(0)=3$. [1991 成大電研]

(Sol.) $L[y(x)] = \int_0^{\infty} y(x)e^{-sx} dx = Y(s)$, $L[x^n y(x)] = (-1)^n \frac{d^n}{ds^n} Y(s)$, and

$$L[y^{(n)}(x)] = s^n Y(s) - s^{n-1} \cdot y(0) - s^{n-2} \cdot y'(0) - \dots - s \cdot y^{(n-2)}(0) - y^{(n-1)}(0)$$

$$-\frac{d}{ds}[s^2 Y(s) - sy(0) - y'(0)] - (-\frac{d}{ds})[sY(s) - y(0)] - Y(s) = 0$$

$$-2sY(s) - s^2 Y'(s) + Y(s) + sY'(s) - Y(s) = 0$$

$$(-s^2 + s)Y'(s) - 2sY(s) = 0, \quad Y'(s) + \frac{2}{s-1} Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{A}{(s-1)^2} \Rightarrow y(x) = Axe^x, y'(0)=3 \Rightarrow A=3, \therefore y(x)=3xe^x$$

Eg. Solve $2y''+ty'-2y=10, y(0)=y'(0)=0$. [2011 台大電子所甲組]

(Sol.) $L[2y''+ty'-2y]=L(10)=\frac{10}{s}$,

$$2[s^2 Y(s) - sy(0) - y'(0)] + (-1) \frac{d}{ds} [sY(s) - y(0)] - 2Y(s) = \frac{10}{s},$$

$$-sY'(s) + (2s^2 - 3)Y(s) = \frac{10}{s}, \quad Y'(s) + (-2s + \frac{3}{s})Y(s) = -\frac{10}{s^2}, \quad \int (-2s + \frac{3}{s}) ds = -s^2 + 3 \ln(s),$$

$$\exp[-s^2 + 3 \ln(s)] = s^3 e^{-s^2}, \quad s^3 e^{-s^2} Y'(s) + [-2s^4 e^{-s^2} + 3s^2 e^{-s^2}] Y(s) = -10s e^{-s^2},$$

$$[s^3 e^{-s^2} Y(s)]' = -10s e^{-s^2}, \quad s^3 e^{-s^2} Y(s) = 5e^{-s^2} + C, \quad Y(s) = \frac{5}{s^3} + C s^{-3} e^{s^2},$$

$$\lim_{t \rightarrow 0} y(t) = \lim_{s \rightarrow \infty} sY(s) = 0 \Rightarrow y(t) = \frac{5}{2} t^2$$

Eg. Solve $(1-t)y''+ty'-y=0, y(0)=3, y'(0)=-1$. [2025 成大電研]

(Sol.) $y''-ty''+ty'-y=0, \quad L[y''-ty''+ty'-y]=0$,

$$s^2 Y(s) - sy(0) - y'(0) - (-1) \frac{d[s^2 Y(s) - sy(0) - y'(0)]}{ds} + (-1) \frac{d[sY(s) - y(0)]}{ds} - Y(s) = 0$$

$$s^2 Y(s) - 3s + 1 + \frac{d[s^2 Y(s) - 3s + 1]}{ds} - \frac{d[sY(s) - 3]}{ds} - Y(s) = 0,$$

$$s^2 Y(s) - 3s + 1 + 2sY(s) + s^2 Y'(s) - 3 - Y(s) - sY'(s) - Y(s) = 0,$$

$$(s^2 - s)Y'(s) + [s^2 + 2s - 2]Y(s) = 3s + 2$$

$$Y'(s) + \frac{s^2 + 2s - 2}{s(s-1)} Y(s) = \frac{3s + 2}{s(s-1)}, \quad Y'(s) + (1 + \frac{2}{s} + \frac{1}{s-1})Y(s) = \frac{-2}{s} + \frac{5}{s-1}$$

$$\int \left(1 + \frac{2}{s} + \frac{1}{s-1}\right) ds = s + 2\ln(s) + \ln(s-1), \quad e^{s+2\ln(s)+\ln(s-1)} = s^2(s-1)e^s$$

$$[s^2(s-1)e^s \cdot Y(s)]' = -2s(s-1)e^s + 5s^2e^s = 3s^2e^s + 2se^s,$$

$$s^2(s-1)e^s \cdot Y(s) = 3s^2e^s - 4se^s + 4e^s, \quad Y(s) = \frac{3}{s-1} - \frac{4}{s^2}, \quad \therefore y(t) = 3e^t - 4t$$

4-4 Convolution and Dirac Delta Function

Convolution in Laplace transform: $f(t)*g(t) = \int_0^t f(t-\alpha)g(\alpha)d\alpha$

Theorem $L[f(t)*g(t)] = L\left[\int_0^t f(t-\alpha)g(\alpha)d\alpha\right] = F(s)G(s)$

Eg. Solve $2\sin(at)*\cos(at)$. [2024 成大電研]

$$(\text{Sol.}) \quad L[2\sin(at)*\cos(at)] = \frac{2a}{s^2+a^2} \cdot \frac{s}{s^2+a^2} = \frac{2as}{(s^2+a^2)^2} = a \cdot (-1) \frac{d}{ds} \left[\frac{1}{s^2+a^2} \right]$$

$$\text{By } L^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t), \quad 2\sin(at)*\cos(at) = L^{-1}\left\{a \cdot (-1) \frac{d}{ds} \left[\frac{1}{s^2+a^2} \right]\right\} = t \cdot \sin(at)$$

Eg. Given $f(t) = 2t*[\sin(t) - e^t]$, find $f(t)$? [2024 台大電研]

$$(\text{Sol.}) \quad L[f(t)] = L\{2t*[\sin(t) - e^t]\} = \frac{2}{s^2} \cdot \left[\frac{1}{s^2+1} - \frac{1}{s-1} \right] = \frac{2}{s^2(s^2+1)} - \frac{2}{s^2(s-1)}$$

$$= \frac{2}{s^2} - \frac{2}{s^2+1} + \frac{2}{s} - \frac{2}{s-1} = \frac{4}{s^2} - \frac{2}{s^2+1} + \frac{2}{s} - \frac{2}{s-1}$$

$$\therefore f(t) = L^{-1}\left[\frac{4}{s^2} - \frac{2}{s^2+1} + \frac{2}{s} - \frac{2}{s-1}\right] = 4t - 2\sin(t) + 2 - 2e^t$$

Eg. Solve $\int_0^t f(\tau)f(t-\tau)d\tau = 6t^3$. [2006 台大電研]

$$(\text{Sol.}) \quad \text{According to } L[t^n] = \frac{n!}{s^{n+1}} \text{ and } L^{-1}[1/s^n] = t^{n-1}/(n-1)!, \quad [F(s)]^2 = 6L[t^3] = \frac{36}{s^4},$$

$$F(s) = \frac{6}{s^2} \Rightarrow f(t) = 6t$$

Eg. Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\rho)e^{t-\rho}d\rho$. [2011 台大電研]

$$(\text{Sol.}) \quad F(s) = \frac{6}{s^3} - \frac{1}{s+1} - \frac{1}{s-1} \cdot F(s), \quad \frac{s}{s-1} \cdot F(s) = \frac{6}{s^3} - \frac{1}{s+1}, \quad F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1},$$

$$\Rightarrow f(t) = 3t^2 - t^3 + 1 - 2e^{-t}$$

Eg. Solve $y''+y-4\int_0^t y(\tau)\sin(t-\tau)d\tau=e^{-2t}$, $y(0)=1$ and $y'(0)=0$. [1990 交大電信所]

$$\text{(Sol.) } s^2Y(s) - sy(0) - y'(0) + Y(s) - 4Y(s) \frac{1}{s^2+1} = \frac{1}{s+2}$$

$$\Rightarrow Y(s) = \frac{(s^2+1)(s+1)}{(s+2)(s^2+3)(s-1)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+3}$$

$$(s^2+1)(s+1) = A(s+2)(s^2+3) + B(s-1)(s^2+3) + (Cs+D)(s-1)(s+2)$$

$$\text{Let } s=1 \Rightarrow 4 = A \cdot 3 \cdot 4 \Rightarrow A = \frac{1}{3}$$

$$\text{Let } s=-2 \Rightarrow -5 = B(-3) \cdot 7 \Rightarrow B = \frac{5}{21}$$

$$\text{Let } s=0 \Rightarrow 1 = 2 \cdot \frac{5}{7} - 2D \Rightarrow D = \frac{1}{7}$$

$$\text{Let } s=-1 \Rightarrow 0 = \frac{1}{3} \cdot 1 \cdot 4 + \frac{5}{21} \cdot (-2) \cdot 4 + (-C + \frac{1}{7})(-1)(2) \Rightarrow C = \frac{3}{7}$$

$$\Rightarrow Y(s) = \frac{1/3}{s-1} + \frac{5/21}{s+2} + \frac{3s/7}{s^2+3} + \frac{1}{7\sqrt{3}} \cdot \frac{\sqrt{3}}{s^2+3}$$

$$\Rightarrow y(t) = \frac{1}{3}e^t + \frac{5}{21}e^{-2t} + \frac{3}{7}\cos(\sqrt{3}t) + \frac{1}{7\sqrt{3}}\sin(\sqrt{3}t)$$

Eg. Solve $f(t) + 2\int_0^t f(\tau)\cos(t-\tau)d\tau = 4e^{-t} + \sin(t)$. [2022 台大電研工數 C]

$$\text{(Sol.) } F(s) + 2F(s) \cdot \frac{s}{s^2+1} = \frac{4}{s+1} + \frac{1}{s^2+1}, F(s) = \frac{4s^2+s+5}{(s+1)^3} = \frac{4}{s+1} - \frac{7}{(s+1)^2} + \frac{8}{(s+1)^3}$$

$$\therefore f(t) = 4e^{-t} - 7te^{-t} + 4t^2e^{-t}$$

Eg. Solve $y + e^t \int_0^t y(\tau)e^{-\tau}d\tau = \cos(t)$. [2015 中興電研乙丙丁組、中興光電所]

$$\text{(Sol.) } y + \int_0^t y(\tau)e^{t-\tau}d\tau = \cos(t), y + y * e^t = \cos(t), Y(s) + Y(s) \frac{1}{s-1} = \frac{s}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2+1} = \frac{s}{s^2+1} - \frac{1}{s^2+1} \Rightarrow y(t) = \cos(t) - \sin(t)$$

Dirac delta function:

$$\delta(t-a) = \delta_a(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [u(t-a) - u(t-a-\varepsilon)] = \begin{cases} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}, & a \leq t \leq a + \varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

$$\text{Kronecker delta: } \delta_{ab} = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$$

Characteristics of Dirac's delta function:

1. $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$. 2. $L[\delta(t-a)] = e^{-as}$. 3. $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$.

4. $f(t) * \delta(t) = \int_0^t f(x)\delta(t-x)dx = f(t)$ 5. $f(t)\delta(t-a) = f(a)\delta(t-a)$

Eg. Solve $y' + 7y = 10\delta(t-2)$, $y(0)=10$. [2024 台大電研]

(Sol.) $sY(s) - 10 + 7Y(s) = 10e^{-2s}$, $Y(s) = \frac{10 + 10e^{-2s}}{s + 7}$,

$\therefore y(t) = 10e^{-7t} + 10e^{-7(t-2)} \cdot u(t-2)$

Eg. Solve $y'' + y = \delta(t-\pi)$, $y(0)=y'(0)=0$. [2017 台聯大電研]

(Sol.) $s^2Y(s) - sy(0) - y'(0) + Y(s) = s^2Y(s) + Y(s) = e^{-\pi s}$, $Y(s) = \frac{e^{-\pi s}}{s^2 + 1}$,

$\therefore L^{-1}\left[\frac{1}{s^2+1}\right] = \sin(t)$ and $L^{-1}[F(s) \cdot e^{-as}] = f(t-a) \cdot u(t-a) = \begin{cases} f(t-a), t \geq a \\ 0, t < a \end{cases}$

$\therefore y(t) = L^{-1}[Y(s)] = \sin[(t-\pi)] \cdot u(t-\pi)$

Eg. Solve $y'' + 5y' + 4y = 3 + 2\delta(t)$. 【北科大土木所】