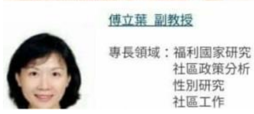
Chapter 5 Fourier Analysis

5-1 Fourier Transforms and Inverse Fourier Transforms



Fourier Transform pair defined in Engineering: $i=j=\sqrt{-1}$ for EE

$$\begin{cases} F(\omega)=\Im\left[f(x)\right]\equiv\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\ f(x)=\Im^{-1}\left[F(\omega)\right]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \end{cases}$$

, where $\omega=2\pi\nu$.

Note: $\Im\{\Im[f(x)]\}=2\pi f(-x)$.

Fourier Transform pair defined in Mathematics:

$$\begin{cases} F(\omega)=\Im\left[f(x)\right]\equiv\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\ f(x)=\Im^{-1}\left[F(\omega)\right]=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \end{cases}$$

, where $\omega=2\pi\nu$.

Fourier Transform pair defined in Physics/Optics:

$$\begin{cases} G(f)=\Im\left[g(x)\right]\equiv\int_{-\infty}^{\infty} g(x)e^{-i2\pi fx}dx \\ g(x)=\Im^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty} G(f)e^{i2\pi fx}df \end{cases}$$

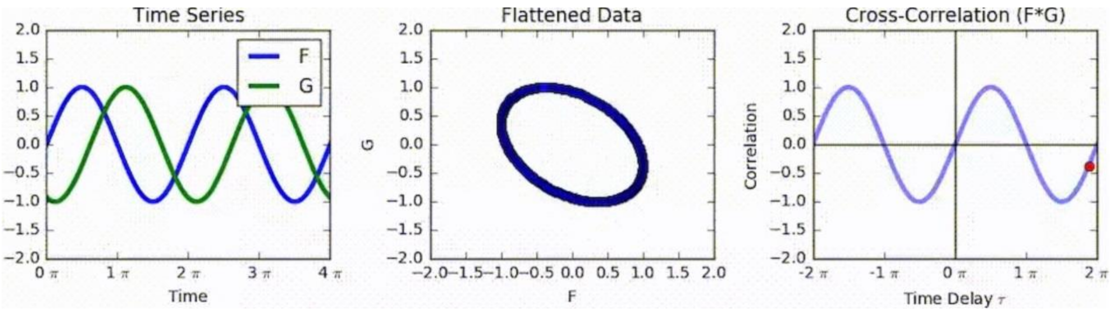
Parseval's Identities for Fourier Transform pairs $\Im[f(x)]=F(\omega)$ and $\Im[g(x)]=G(\omega)$:

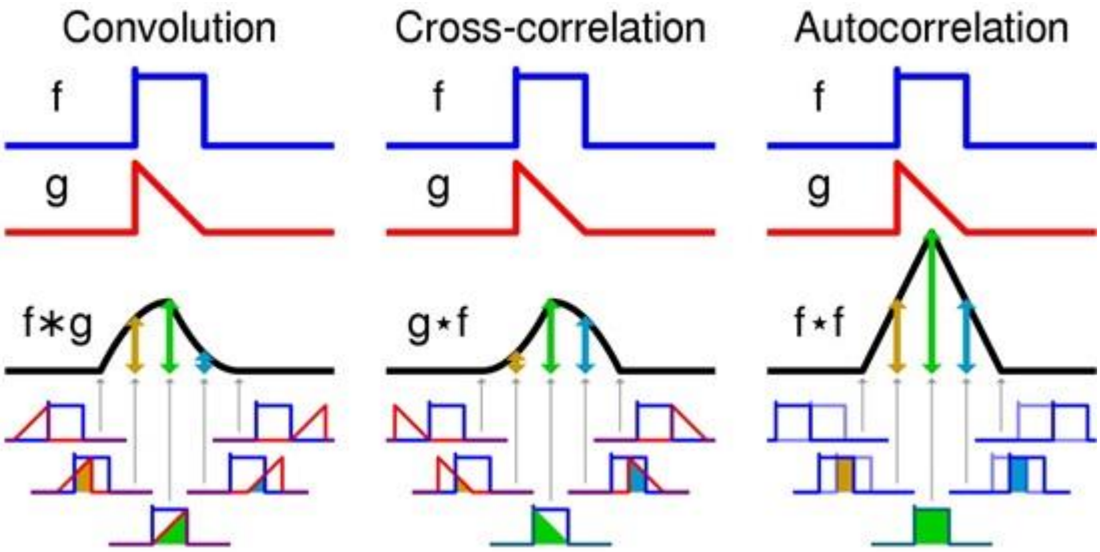
$$\int_{-\infty}^{\infty}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega \quad \textbf{and} \quad \int_{-\infty}^{\infty} f(x)g^*(x)dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)G^*(\omega)d\omega$$

Continuous Spectrum of $f(t)$: $f(t)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega$

***Convolution* in Fourier Transform:** $f(t)*g(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau$

***Correlation* in Fourier Transform:** $f(t)\star g(t)=\int_{-\infty}^{\infty}f(\tau)g(\tau-t)d\tau$





Basic theorems of Fourier Transforms $\Im[f(x)]=F(\omega)$ **and** $\Im[g(x)]=G(\omega)$:

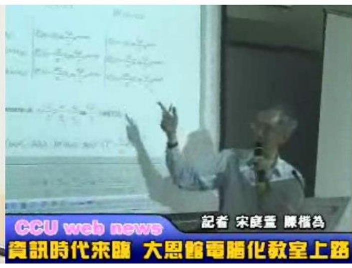
1. $\Im[af(x)+bg(x)]=aF(\omega)+bG(\omega)$
2. $\Im[f(ax)]=[F(\omega/a)]/a$ **and** $\Im^{-1}[F(a\omega)]=[f(x/a)]/a,\ a>0$

(Proof) For $a>0$, let $ax=u$, $\Im[f(ax)]=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(ax)dx=\frac{1}{a}\int_{-\infty}^{\infty} e^{-i(\frac{\omega}{a})ax}\cdot f(ax)d(ax)=$ $\frac{1}{a}\int_{-\infty}^{\infty} e^{-i(\frac{\omega}{a})u}\cdot f(u)du=\frac{1}{a}F[(\frac{\omega}{a})]$

3. $\Im[f(x)e^{iax}]=F(\omega-a)$ **and** $\Im^{-1}[F(\omega-a)]=f(x)e^{iax}$

(Proof) $\Im[f(x)e^{iax}]=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(x)e^{iax}dx=\int_{-\infty}^{\infty} e^{-i(\omega-a)x}\cdot f(x)dx=F(\omega-a)$



4. $\Im[f'(x)]=i\omega F(\omega)$, $\Im[f^{(n)}(x)]=(i\omega)^n F(\omega)$ in case of $f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\ldots=0$

(Proof) $\Im[f'(x)]=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f'(x)dx=\int_{-\infty}^{\infty} e^{-i\omega x}df(x)$

$= e^{-i\omega x}f(x)|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}(-i\omega)e^{-i\omega x}\cdot f(x)dx= e^{-i\omega\infty}f(\infty)- e^{i\omega\infty}f(-\infty)+i\omega\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(x)dx= i\omega F(\omega)$

By mathematical induction, we have $\Im[f^{(n)}(x)]=(i\omega)^n F(\omega)$ if $f(\pm\infty)=f'(\pm\infty)=f''(\pm\infty)=\ldots=0$.

5. $\Im[f(x)*g(x)]=F(\omega)G(\omega)$

6. $\Im[g^*(x)\bigstar f(x)]=F(\omega)G^*(\omega)$, where $g^*(x)$ and $G^*(\omega)$ are the conjugate complexes of $g(x)$ and $G(\omega)$, respectively.

7. $\Im[f(x-a)]= e^{-ia\omega}F(\omega)$ and $\Im^{-1}[e^{-ia\omega}F(\omega)]=f(x-a)$

(Proof) Let $x-a=u$, $\Im[f(x-a)]=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(x-a)dx=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(x-a)d(x-a)$

$=e^{-ia\omega}\int_{-\infty}^{\infty} e^{-i\omega(x-a)}\cdot f(x-a)d(x-a)=e^{-ia\omega}\int_{-\infty}^{\infty} e^{-i\omega u}\cdot f(u)du = e^{-ia\omega}F(\omega)$

8. $\Im[x^n f(x)]=(i)^n F^{(n)}(\omega)$

(Proof) $\Im[xf(x)]=\int_{-\infty}^{\infty} e^{-i\omega x}\cdot xf(x)dx=\int_{-\infty}^{\infty} i\frac{d}{d\omega}e^{-i\omega x}\cdot f(x)dx=i\frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-i\omega x}\cdot f(x)dx$

$= iF'(\omega)$

By mathematical induction, we have $\Im[x^n f(x)]=(i)^n F^{(n)}(\omega)$.

9. $\Im[\delta(x)]=1$, $\Im[1]=2\pi\delta(\omega)$, and $\Im[e^{iax}]=2\pi\delta(\omega-a)$

Eg. Find (a) $\Im[1]$, (b) $\Im[e^{jax}]$, (c) $\Im[\sin(ax)]$, and (d) $\Im[\cos(ax)]$.

(Sol.) (a) $\Im[\delta(x)]=1$, $\Im^{-1}[1]=\delta(x)$, $\frac{1}{2\pi}\int_{-\infty}^{\infty}1\cdot e^{i\omega x}d\omega=\delta(x)$,

$\int_{-\infty}^{\infty}1\cdot e^{i\omega x}d\omega=2\pi\delta(x)$, $\int_{-\infty}^{\infty}1\cdot e^{i\omega x}d\omega=\int_{-\infty}^{\infty}1\cdot e^{-i\omega(-x)}d\omega=\int_{-\infty}^{\infty}1\cdot e^{-i(-x)\omega}d\omega$ $=2\pi\delta(x)$.

Let ω be replaced by u, and $-x$ be replaced by v, we have $\int_{-\infty}^{\infty}1\cdot e^{-ivu}du=2\pi\delta(-v)=2\pi\delta(v)$ because $\delta(x)$ is an even function.

Let u be replaced by x and v be replaced by ω, we have

$\int_{-\infty}^{\infty}1\cdot e^{-i\omega x}dx=\Im[1]=2\pi\delta(\omega)$

(b) $\because$ $\Im[f(x)e^{iax}]=F(\omega-a)$, $\therefore$ $\Im[e^{iax}]=2\pi\delta(\omega-a)$

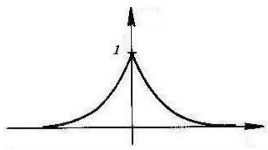
(c) $\sin(ax)=\frac{e^{iax}-e^{-iax}}{2i}$, $\Im[\sin(ax)]=\Im[\frac{e^{iax}-e^{-iax}}{2i}]=-i\pi[\delta(\omega-a)-\delta(\omega+a)]$ $=i\pi[\delta(\omega+a)-\delta(\omega-a)]$

(d) $\cos(ax)=\frac{e^{iax}+e^{-iax}}{2}$, $\Im[\cos(ax)]=\Im[\frac{e^{iax}+e^{-iax}}{2}]=\pi[\delta(\omega-a)+\delta(\omega+a)]$.

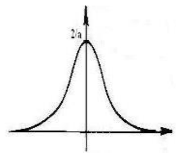
Eg. Find $\Im[e^{-a|x|}]$ and $\Im[e^{-|x|}]$.

(Sol.) $\int_{-\infty}^{\infty} e^{-a|x|}\cdot e^{-i\omega x}dx=\int_{-\infty}^{0} e^{+ax}\cdot e^{-i\omega x}dx+\int_{0}^{\infty} e^{-ax}\cdot e^{-i\omega x}dx$

$$=\frac{1}{a-i\omega}+\frac{1}{a+i\omega}=\frac{2a}{a^2+\omega^2}=\Im\left[e^{-a|x|}\right].\ \text{For}\ \ a=1,\ \Im\left[e^{-|x|}\right]=\frac{2}{1+\omega^2}$$



$f(x)=e^{-a|x|}$



$F(\omega)=\dfrac{2a}{a^2+\omega^2}$

Eg. Determine $\Im^{-1}\left\{\dfrac{1}{\omega^2+1}\right\}$**. [2013 成大電研]**

(Sol.) $\Im^{-1}\left[\dfrac{2a}{\omega^2+a^2}\right]=e^{-a|x|}$ and $a=1$, $\Im^{-1}\left\{\dfrac{1}{\omega^2+1}\right\}=\dfrac{1}{2}\Im^{-1}\left\{\dfrac{2}{\omega^2+1}\right\}=\dfrac{1}{2}e^{-|x|}$

Eg. Find (a) $\Im\,[xe^{-|x|}]$**, (b)** $\Im\,[e^{-3|x|}]$**, (c)** $\Im^{-1}\left\{\dfrac{4}{4+\omega^2}\right\}$**, (d)** $\displaystyle\int_{-\infty}^{\infty}\frac{\cos\omega}{\omega^2+4}d\omega$**, (e)** $f(x)$ **if** $\displaystyle\int_0^{\infty}f(x)\cos(2x)dx=e^{-2}$**. [文化電機轉學考]**

(Sol.) $\Im\left[e^{-a|x|}\right]=\dfrac{2a}{a^2+\omega^2}$ (a) According to $\Im\left[x^n f(x)\right]=(i)^n\dfrac{d^n}{d\omega^n}F(\omega)$,

$$\Im\left[x\cdot e^{-a|x|}\right]=i\frac{d}{d\omega}\Im\left[e^{-a|x|}\right]=i\frac{d}{d\omega}\left(\frac{2a}{a^2+\omega^2}\right)=\frac{-4ai\omega}{\left(a^2+\omega^2\right)^2},\quad a=1,\ \Im\left[xe^{-|x|}\right]=\frac{-i4\omega}{\left(\omega^2+1\right)^2}$$

(b) $a=3$, $\Im\left[e^{-3|x|}\right]=\dfrac{6}{\omega^2+9}$, (c) $a=2$, $\Im^{-1}\left[\dfrac{4}{4+\omega^2}\right]=e^{-2|x|}$

(d) $\Im^{-1}\left[\dfrac{4}{4+\omega^2}\right]=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\frac{4}{4+\omega^2}e^{i\omega x}d\omega=e^{-2|x|}$

$$x=1,\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{4+\omega^2}e^{i\omega}d\omega=e^{-2},\quad \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{4+\omega^2}\left[\cos\omega+i\sin\omega\right]d\omega=e^{-2}$$

$$I_m(\cdots\cdots)=0,\ R_e(\cdots\cdots)\neq 0,\ \therefore\ \int_{-\infty}^{\infty}\frac{\cos\omega}{4+\omega^2}d\omega=\frac{\pi}{2}e^{-2}$$

(e) $\displaystyle\int_{-\infty}^{\infty}\frac{2}{\pi}\cdot\frac{\cos\omega}{\omega^2+4}d\omega=e^{-2}=2\int_0^{\infty}\frac{2}{\pi}\cdot\frac{\cos\omega}{\omega^2+4}d\omega=\int_0^{\infty}\frac{4}{\pi}\frac{\cos\omega}{\omega^2+4}d\omega$

Set $\omega=2x$, $\displaystyle\int_0^{\infty}\frac{4}{\pi}\cdot\frac{\cos 2x}{4x^2+4}\cdot 2dx=e^{-2}=\int_0^{\infty}f(x)\cos(2x)dx$, $\therefore\ f(x)=\dfrac{2}{\pi}\cdot\dfrac{1}{(x^2+1)}$

Eg. Find the Fourier transform of $f(t)=\begin{cases}e^{-at}, & t\geq 0\\ 0, & t<0\end{cases}$**. [2017 台聯大電研]**

(Sol.) $\displaystyle\int_{-\infty}^{\infty}f(t)\cdot e^{-i\omega t}dt=\int_{-\infty}^{0}0\cdot e^{-i\omega t}dt+\int_0^{\infty}e^{-at}\cdot e^{-i\omega t}dt=0+\frac{1}{a+i\omega}=\frac{a-i\omega}{a^2+\omega^2}$

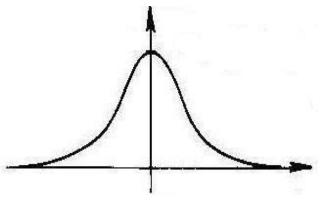
Eg. Find $\Im\left[\dfrac{1}{a^2+x^2}\right]$ **and** $\Im\left[\dfrac{1}{a^2+(x+b)^2}\right]$.

(Sol.) $\Im\left[e^{-a|x|}\right]=\dfrac{2a}{a^2+\omega^2}$, $e^{-a|x|}=\Im^{-1}\left[\dfrac{2a}{a^2+\omega^2}\right]=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{2a}{a^2+\omega^2}\cdot e^{i\omega x}d\omega$

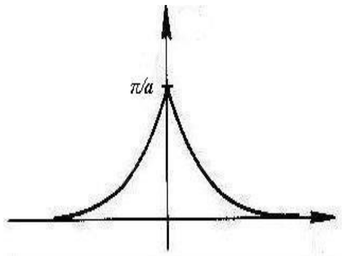
$$=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}e^{-i(-x)\omega}d\omega$$

$\therefore\ \int_{-\infty}^{\infty}\dfrac{1}{a^2+\omega^2}e^{-i(-x)\omega}d\omega=\dfrac{\pi}{a}e^{-a|x|}$. Set u=-$x\Rightarrow\int_{-\infty}^{\infty}\dfrac{1}{a^2+\omega^2}\cdot e^{-iu\omega}d\omega=\dfrac{\pi}{a}e^{-a|u|}$

Set x=ω, ω=u, $\Im\left[\dfrac{1}{a^2+x^2}\right]=\int_{-\infty}^{\infty}\dfrac{1}{a^2+x^2}e^{-i\omega x}dx=\dfrac{\pi}{a}\cdot e^{-a|\omega|}$



$f(x)=\dfrac{1}{a^2+x^2}$



$F(\omega)=\dfrac{\pi}{a}e^{-a|\omega|}$

$\because\ \Im\left[f(x-a)\right]=e^{-i\omega a}F(\omega)$, $\therefore\ \Im\left[\dfrac{1}{a^2+(x+b)^2}\right]=e^{i\omega b}\cdot\dfrac{\pi}{a}e^{-a|\omega|}$

Eg. Find $\Im\left[\dfrac{1}{9+x^2}\right]$ **and** $\int_{-\infty}^{\infty}e^{-3|x|}\cdot\cos(x)dx$.

(Sol.) $\Im\left[\dfrac{1}{a^2+x^2}\right]=\dfrac{\pi}{a}\cdot e^{-a|\omega|}$, a=3, $\therefore\ \Im\left[\dfrac{1}{9+x^2}\right]=\dfrac{\pi}{3}\cdot e^{-3|\omega|}$

$$\frac{1}{9+x^2}=\Im^{-1}[\frac{\pi}{3}\cdot e^{-3|\omega|}]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{3}\cdot e^{-3|\omega|}\cdot e^{i\omega x}d\omega=\frac{1}{6}\int_{-\infty}^{\infty}e^{-3|\omega|}\cdot[\cos(\omega x)+i\sin(\omega x)]d\omega$$

$I_m(...)=0$, $R_e(...)\neq 0$. Set x=1, we have $\int_{-\infty}^{\infty}e^{-3|\omega|}\cdot\cos(\omega)d\omega=\dfrac{3}{5}$

Eg. Determine $\Im[\dfrac{1}{a+it}]$. **[2003 台科大電研]**

(Sol.) $\because\ \Im\left[\dfrac{1}{a^2+t^2}\right]=\int_{-\infty}^{\infty}\dfrac{1}{a^2+t^2}e^{-i\omega t}dt=\dfrac{\pi}{a}\cdot e^{-a|\omega|}$ and $\Im\left[t^n f(t)\right]=(i)^n\dfrac{d^n}{d\omega^n}F(\omega)$

$$\therefore \Im[\frac{1}{a+it}]=\Im[\frac{a}{a^2+t^2}]-\Im[\frac{it}{a^2+t^2}]=a\cdot\frac{\pi}{a}e^{-a|\omega|}-i\cdot i\frac{d}{d\omega}[\frac{\pi}{a}e^{-a|\omega|}]=\pi e^{-a|\omega|}\cdot[1-\mathrm{sgn}(\omega)]$$

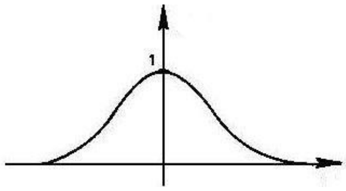
$$=\begin{cases}0, & \omega\geq 0\\ 2\pi e^{a\omega}, & \omega<0\end{cases}=2\pi e^{a\omega}\cdot u(-\omega)$$

Eg. Find $\Im(e^{-a^2x^2})$.

(Sol.) $\int_{-\infty}^{\infty} e^{-a^2x^2}\cdot e^{-i\omega x}dx=\int_{-\infty}^{\infty} e^{-a^2\left(x^2+i\frac{\omega}{a^2}x\right)}dx=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty} e^{-a^2\left[x^2+i\frac{\omega}{a^2}x-\frac{\omega^2}{4a^4}\right]}\cdot dx$

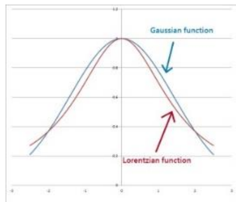
$=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty} e^{-a^2\left[x+i\frac{\omega}{2a^2}\right]^2}\cdot dx=e^{\frac{-\omega^2}{4a^2}}\cdot\int_{-\infty}^{\infty} e^{-a^2u^2}\cdot du \leftarrow \left(u=x+i\frac{\omega}{2a^2}\right)$

$=\frac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}} \quad \left(Note:\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a^2(u^2+v^2)}dudv=\int_{o}^{\infty}\int_{o}^{2\pi} e^{-a^2r^2}rdrd\theta=\frac{\pi}{a^2}\right)$



***f*(*x*)=**$e^{-a^2x^2}$ ***F*(ω)=**$\frac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}$

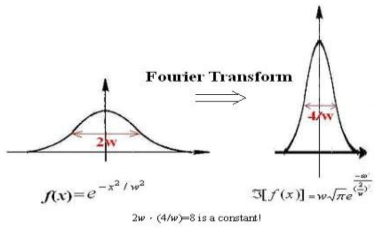
Note: $f(x)=\frac{1}{a^2+x^2}$ and $g(x)=e^{-a^2x^2}$ are similar to each other. But their respective Fourier transforms look quite different!



Eg. Determine $\Im[xe^{-x^2}]$. **[2022台大電研工數C]**

(Sol.) a=1, ∴ $\Im[e^{-x^2}]=\sqrt{\pi}e^{\frac{-\omega^2}{4}}$

$\Im[xe^{-x^2}]=i\frac{d}{d\omega}(\sqrt{\pi}e^{\frac{-\omega^2}{4}})=\frac{-i\omega\sqrt{\pi}e^{\frac{-\omega^2}{4}}}{2}$



Eg. Determine $\Im^{-1}[e^{-b^2\omega^2}]$.

(Sol.) $\Im(e^{-a^2x^2})=\frac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}$, $\int_{-\infty}^{\infty} e^{-a^2x^2}\cdot e^{-i\omega x}dx=\int_{-\infty}^{\infty} e^{-a^2x^2}\cdot e^{i(-\omega)x}dx$

$=\int_{-\infty}^{\infty} e^{-a^2u^2}\cdot e^{i(-\omega)u}du=\frac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}$, $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a^2u^2}\cdot e^{i(-\omega)u}du=\frac{1}{2a\sqrt{\pi}}e^{\frac{-\omega^2}{4a^2}}$

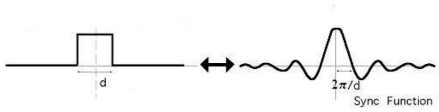
-$\omega\rightarrow x$ and $a\rightarrow b$ (simultaneously), and then $u\rightarrow\omega$

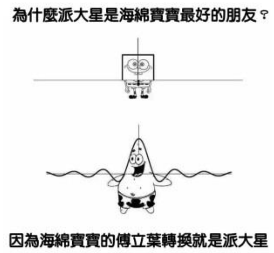
$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-b^2\omega^2}\cdot e^{ix\omega}d\omega=\frac{1}{2b\sqrt{\pi}}e^{\frac{-x^2}{4b^2}}$, ∴ $\Im^{-1}[e^{-b^2\omega^2}]=\frac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}}$

Eg. For two rectangular functions: $f(x)=\begin{cases}1, |x|\le 1\\ 0, |x|\rangle 1\end{cases}$, $g(x)=\begin{cases}1, |x|\le 2\\ 0, |x|\rangle 2\end{cases}$, find

(a) $\Im[f(x)]$, (b) $\Im[g(x)]$, (c) $\int_{-\infty}^{\infty}\frac{\sin(x)}{x}dx$ [文化電機轉學考], and $\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega$ [2008 成大電研].

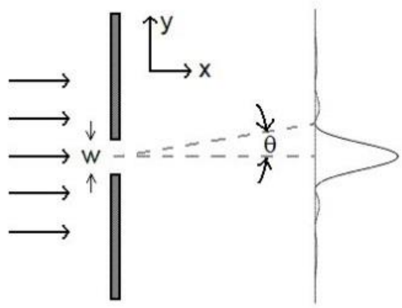
(Sol.) (a) $\Im[f(x)]=\int_{-1}^{1}e^{-i\omega x}dx=\left.\frac{e^{-i\omega x}}{-i\omega}\right|_{-1}^{1}=\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}=\frac{2\left(e^{i\omega}-e^{-i\omega}\right)}{2i\omega}=\frac{2\sin(\omega)}{\omega}$





(b) $\because\ \Im[f(ax)]=\frac{1}{a}F\left(\frac{\omega}{a}\right)$, $a>0$,

$$\Im[g(x)]=\Im\left[f\left(\frac{x}{2}\right)\right]=2\cdot\frac{2\sin(2\omega)}{2\omega}=\frac{2\sin(2\omega)}{\omega}$$

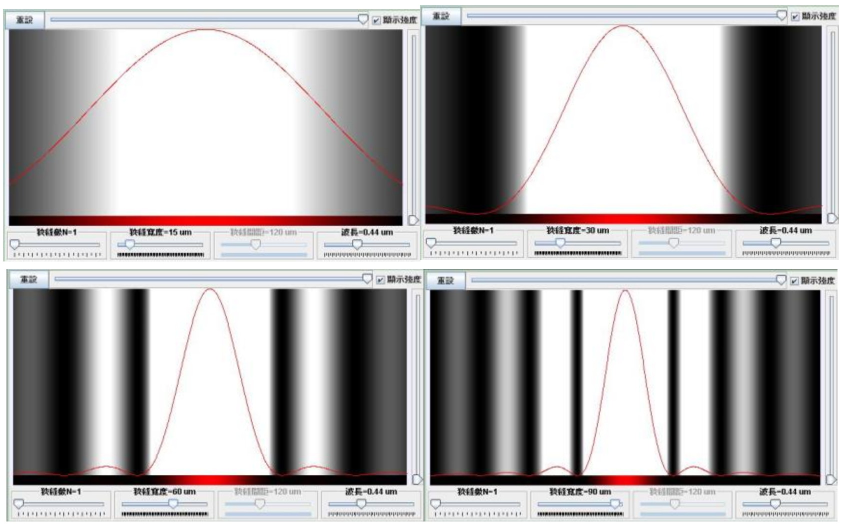


(c) $f(x)=\Im^{-1}\left\{\frac{2\sin\omega}{\omega}\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin\omega}{\omega}e^{i\omega x}d\omega$,

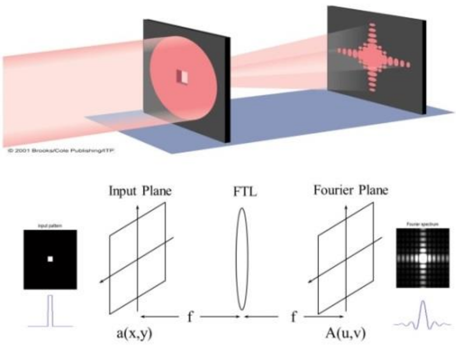
$$f(0)=1\Rightarrow\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\pi$$

(d) According to $\int_{-\infty}^{\infty}|f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2d\omega$,

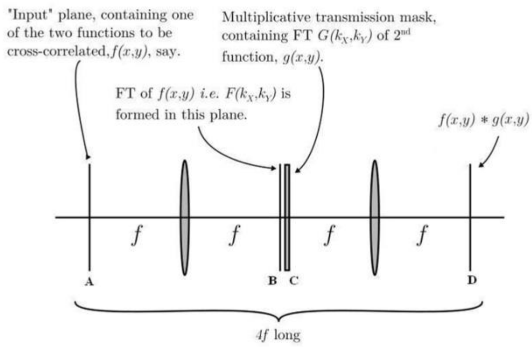
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin(\omega)}{\omega}\right)^2d\omega=\int_{-1}^{1}1^2dx=2\Rightarrow\int_{-\infty}^{\infty}\frac{\sin^2\omega}{\omega^2}d\omega=\pi$$



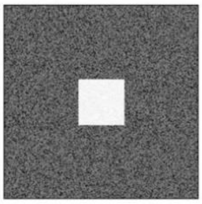
Optical Fourier Transform



Optical Signal Processing



Examples of Optical Signal Processing



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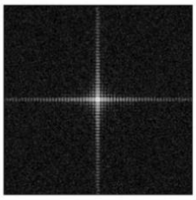
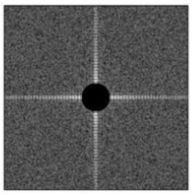
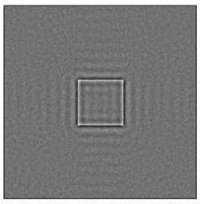


Fourier transform at B

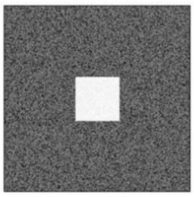


Applying a high-pass filter to the Fourier transform at C

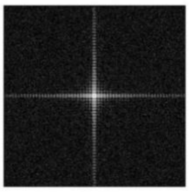


The resulting edge-enhanced image at D

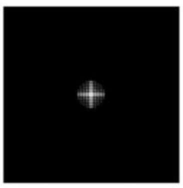
Examples of Optical Signal Processing (Cont')



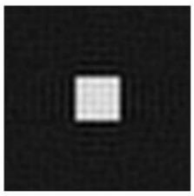
Input Image



Fourier transform at B

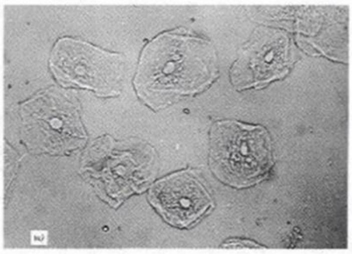


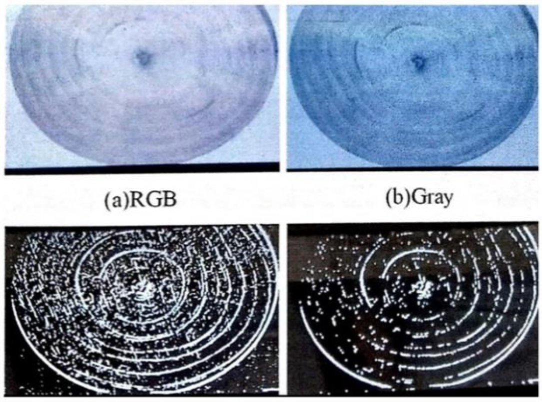
Applying a low-pass filter to the Fourier transform at C

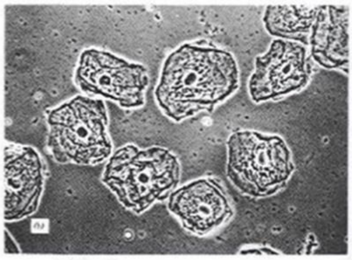


The resulting image at D

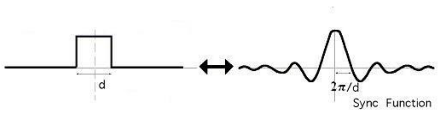
Comparison between Phase Contrast Microscopy by Optical Signal Processing (Left) and Conventional Image Processing of Sharpening Edges (Right)







Summary

$f(x)$	$F(\omega)=\Im[f(x)]$	Examples
$f(x)=\begin{cases}1, \lvert x\rvert \le a\\ 0, \lvert x\rvert > a\end{cases}$	$\dfrac{2\sin(a\omega)}{\omega}$	
$e^{-a\lvert x\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$\Im\left[e^{-\frac{1}{2}\lvert x\rvert}\right]=\dfrac{1}{\omega^2+\frac{1}{4}}=\dfrac{4}{4\omega^2+1}$, $\Im\left[e^{-\lvert x\rvert}\right]=\dfrac{2}{\omega^2+1}$, $\Im\left[e^{-2\lvert x\rvert}\right]=\dfrac{4}{\omega^2+4}$, $\Im\left[e^{-3\lvert x\rvert}\right]=\dfrac{6}{\omega^2+9}$, etc.
$\dfrac{1}{a^2+x^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$	$\Im\left[\dfrac{1}{1+x^2}\right]=\pi e^{-\lvert\omega\rvert}$, $\Im\left[\dfrac{1}{4+x^2}\right]=\dfrac{\pi}{2}e^{-2\lvert\omega\rvert}$, $\Im\left[\dfrac{1}{9+x^2}\right]=\dfrac{\pi}{3}e^{-3\lvert\omega\rvert}$, etc.
$e^{-a^2x^2}$	$\dfrac{\sqrt{\pi}}{a}e^{\frac{-\omega^2}{4a^2}}$	$\Im(e^{\frac{-x^2}{4}})=2\sqrt{\pi}e^{-\omega^2}$, $\Im(e^{-x^2})=\sqrt{\pi}e^{\frac{-\omega^2}{4}}$, $\Im(e^{-4x^2})=\dfrac{\sqrt{\pi}}{2}e^{\frac{-\omega^2}{16}}$, $\Im(e^{-9x^2})=\dfrac{\sqrt{\pi}}{3}e^{\frac{-\omega^2}{36}}$, etc.

5-2 Fourier Series of a Periodical Function

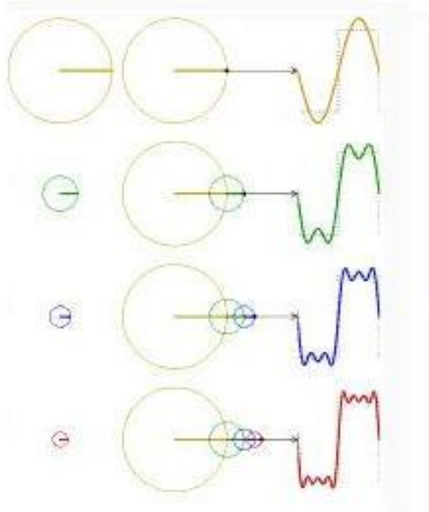
Video Example of Fourier Series of Periodical Rectangular Function:

$$\frac{4\sin\theta}{\pi}$$

$$\frac{4\sin\theta}{\pi}+\frac{4\sin 3\theta}{3\pi}$$

$$\frac{4\sin\theta}{\pi}+\frac{4\sin 3\theta}{3\pi}+\frac{4\sin 5\theta}{5\pi}$$

$$\frac{4\sin\theta}{\pi}+\frac{4\sin 3\theta}{3\pi}+\frac{4\sin 5\theta}{5\pi}+\frac{4\sin 7\theta}{7\pi}$$



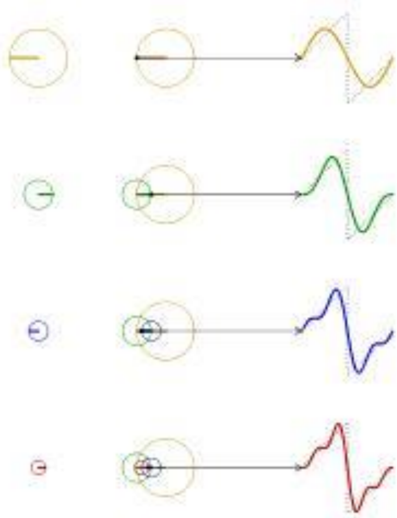
Video Example of Fourier Series of Periodical Sawtooth Function:

$$\frac{2\sin\theta}{-\pi}$$

$$\frac{2\sin\theta}{-\pi}+\frac{2\sin 2\theta}{2\pi}$$

$$\frac{2\sin\theta}{-\pi}+\frac{2\sin 2\theta}{2\pi}+\frac{2\sin 3\theta}{-3\pi}$$

$$\frac{2\sin\theta}{-\pi}+\frac{2\sin 2\theta}{2\pi}+\frac{2\sin 3\theta}{-3\pi}+\frac{2\sin 4\theta}{4\pi}$$



$$\begin{aligned}\pi &= 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots\right)\\ &= \sqrt{6\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)}\\ &= \sqrt{\sqrt{90\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)}}\\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2\end{aligned}$$

Fourier series: $f(x)$ is a periodical function with period=$2L$ and defined on an interval: $-L \leqq x \leqq L$. $f(x+2L)=f(x)$, and then $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right)+b_n \sin\left(\frac{n\pi x}{L}\right)\right]$, where

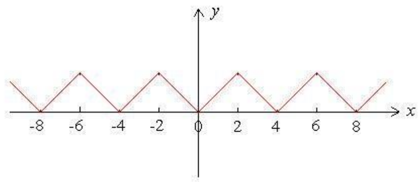
$$\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ \text{and}\ \ b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

In case $f(x)$ is $\begin{matrix} odd \\ even \end{matrix} \Rightarrow \begin{matrix} a_n=0 \\ b_n=0 \end{matrix}$

Parseval's Identity for Fourier series: $\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$

Orthogonalities:

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=\begin{cases}0, n\neq m\\ L, n=m\end{cases}\quad \text{and}\quad \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\begin{cases}0, n\neq m\\ L, n=m\end{cases}$$



Eg. Expand $f(x)=\begin{cases} x, 0\le x\le 2 \\ -x, -2\le x\le 0\end{cases}$, $f(x+4)=f(x)$ into Fourier series and $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

(Sol.) $f(x)=\begin{cases} x, 0\le x\le 2 \\ -x, -2\le x\le 0\end{cases}$, $f(x+4)=f(x)$, $2L=4$, $L=2$,

$\because$ Even function, $\therefore$ $b_n=0$, $\frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L} f(x)dx=\frac{1}{4}\left[\int_{-2}^{0}-x\,dx+\int_{0}^{2}x\,dx\right]=1$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}-x\cos\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{2}{2}\cdot\int_{0}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx=\left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)-\frac{4}{n^2\pi^2}\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\right]\Bigg|_0^2$$

$$=\frac{4}{n^2\pi^2}\left[\cos(n\pi)-1\right]=\begin{cases}\frac{-8}{n^2\pi^2}, n: odd\\ 0, n: even\end{cases}=\frac{-8}{(2m-1)^2\pi^2}, m=1,2,\cdots\cdots$$

$$\therefore\ f(x)=1+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos\left(\frac{n\pi x}{2}\right)=1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\ldots\ldots\right)$$

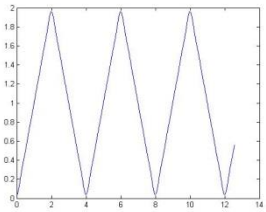
$$f(0)=0=1-\frac{8}{\pi^2}\left(\cos 0+\frac{1}{3^2}\cos 0+\frac{1}{5^2}\cos 0+\ldots\ldots\right)$$

$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots=\frac{\pi^2}{8}$$

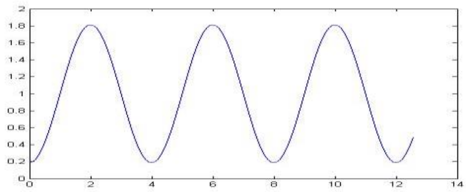
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$$1+\sum_{n=1}^{10}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos(\frac{n\pi x}{2})=1-\sum_{i=1}^{5}\frac{8}{(2i-1)^2\pi^2}\cos(\frac{(2i-1)\pi x}{2}).$$

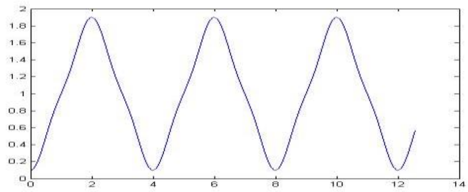
```
>>x = 0:0.001:4*pi; y=1;
>>for i=1:5
y=y-8*cos((2*i-1)*pi*x/2)/(2*i-1)^2/pi^2
end
>>plot (x,y)
```



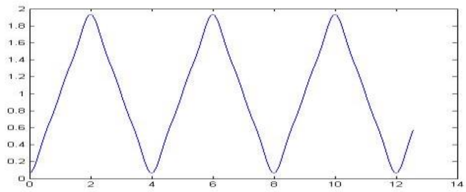
$$1-\frac{8}{\pi^2}\cos(\frac{\pi x}{2})$$



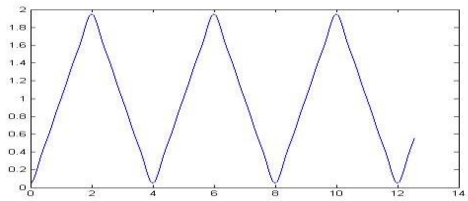
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}\right)$$



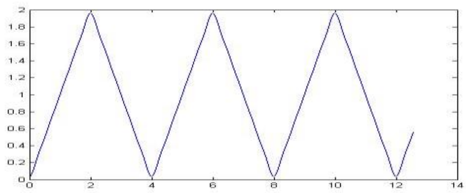
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}\right)$$



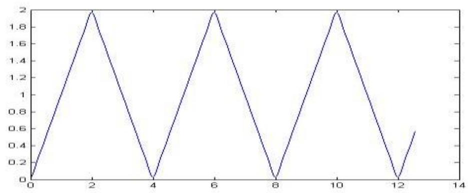
$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\frac{1}{7^2}\cos\frac{7\pi x}{2}\right)$$

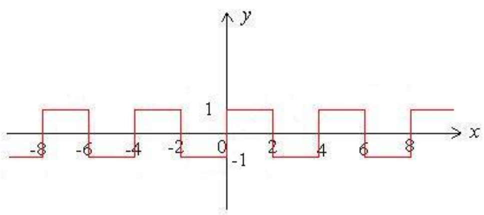


$$1-\frac{8}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\frac{1}{7^2}\cos\frac{7\pi x}{2}+\frac{1}{9^2}\cos\frac{9\pi x}{2}\right)$$



$$1+\sum_{n=1}^{22}\frac{4}{n^2\pi^2}(\cos n\pi-1)\cos(\frac{n\pi x}{2})$$





Eg. Expand $f(x)=\begin{cases}1\,,\,0\le x\le 2\\ -1\,,\,-2\le x\le 0\end{cases}$ **and** $f(x+4)=f(x)$ **into Fourier series. Find (a)** $\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}$ **and (b)** $\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}$ **. [文化電機轉學考]**

(Sol.) $2L=4\,,\,L=2\,,\ f(x)=\begin{cases}1\,,\,0\le x\le 2\\ -1\,,\,-2\le x\le 0\end{cases}$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$. $\because$ Odd function, $\therefore\ a_n=0,\ \forall n$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}-\sin\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{2}{n\pi}-\frac{2\cos(n\pi)}{n\pi}=\frac{2}{n\pi}[1-\cos(n\pi)]\Rightarrow f(x)=\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$

(a) Set $x=1$,

$$f(1)=1=\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi}{2}\right)\right\}=\frac{4}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\ldots\right\}$$

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\ldots=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}=\frac{\pi}{4}$$

(b) $\frac{1}{L}\int_{-L}^{L}|f(x)|^2\,dx=\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}\left[a_n^{\,2}+b_n^{\,2}\right]$,

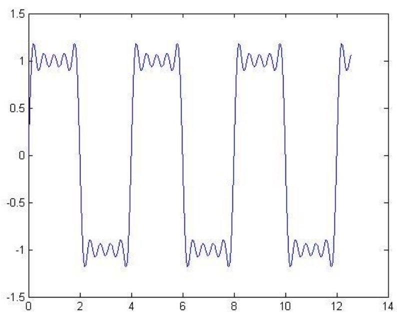
$$\frac{1}{2}\int_{-2}^{2}1^2dx=2=\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[1-\cos(n\pi)]^2=\frac{16}{\pi^2}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots\right]$$

$$\Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}$$

$$\therefore\quad \pi=4\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\right)=\sqrt{8\cdot\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\ldots\right)}$$

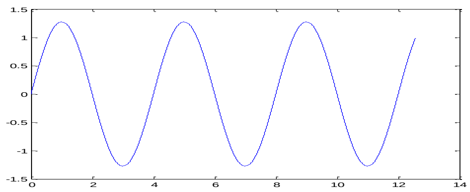
In **Matlab** language, we can use the following instructions to obtain the finite sum of $\sum_{n=1}^{10}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$ $=$ $\sum_{i=1}^{5}\left\{\frac{4}{(2i-1)\pi}\cdot\sin\left(\frac{(2i-1)\pi x}{2}\right)\right\}$.

```
>>x = 0:0.001:4*pi;
>>y=0;
>>for i=1:5
y=y+4*sin((2*i-1)*pi*x/2)/(2*i-1)/pi
end
```

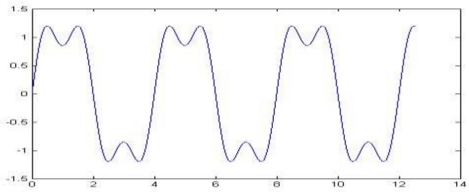


>>plot (x,y)

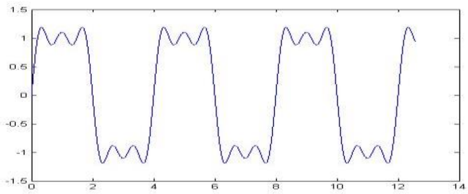
$$\frac{4}{\pi}\sin(\frac{\pi x}{2})$$



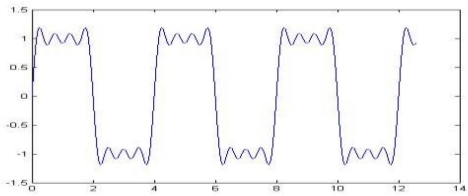
$$\sum_{n=1}^{4}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



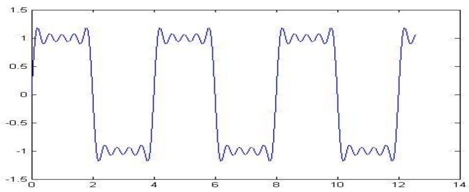
$$\sum_{n=1}^{6}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



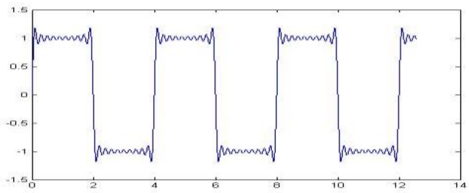
$$\sum_{n=1}^{8}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



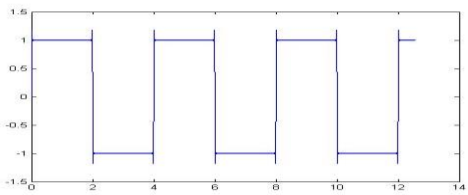
$$\sum_{n=1}^{10}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



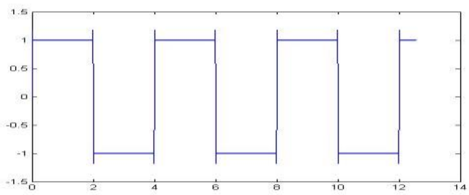
$$\sum_{n=1}^{20}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



$$\sum_{n=1}^{1000}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$

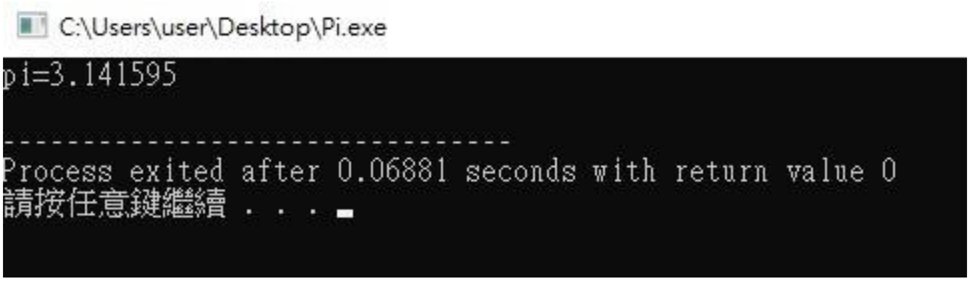


$$\sum_{n=1}^{2000}\left\{\frac{2}{n\pi}[1-\cos(n\pi)]\cdot\sin\left(\frac{n\pi x}{2}\right)\right\}$$



```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>

main()
{
float x,y,z,pi,sum=0.0;
  for(int i=1; i<=10000000;i++)
  {
  x=pow(-1.0,i+1);
  y=1.0/(2*i-1);
  z=x*y;
  sum+=z;
  }
  pi=sum*4.0;
  printf("pi=%.6f \n",pi);
}
```



Eg. (a) Expand $f(x)=\begin{cases} k, 0\le x\le \pi \\ -k, -\pi\le x\le 0 \end{cases}$ and $f(x+2\pi)=f(x)$ into Fourier series. (b) Find $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\ldots$. [2018 台大電研、2015 師大電研與 2017 台聯大電研類似題 for k=1]

(Sol.) (a) $2L=2\pi$, $L=\pi$, $f(x)=\begin{cases} k, 0\le x\le \pi \\ -k, -\pi\le x\le 0 \end{cases}$

$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$. ∵ Odd function, ∴ a_n=0, $\forall n$

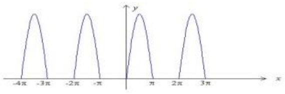
$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{k}{\pi}\left[\int_{-\pi}^{0}-\sin\left(\frac{n\pi x}{\pi}\right)dx+\int_{0}^{\pi}\sin\left(\frac{n\pi x}{\pi}\right)dx\right]$$

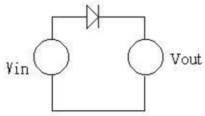
$$=\frac{2k}{n\pi}-\frac{2k\cos(n\pi)}{n\pi}=\frac{2k}{n\pi}\left[1-\cos(n\pi)\right]\Rightarrow f(x)=\sum_{n=1}^{\infty}\left\{\frac{2k}{n\pi}\left[1-\cos(n\pi)\right]\cdot\sin(nx)\right\}$$

(b) Set $x=\pi/2$,

$$f\left(\frac{\pi}{2}\right)=k=\sum_{n=1}^{\infty}\left\{\frac{2k}{n\pi}\left[1-\cos(n\pi)\right]\cdot\sin\left(\frac{n\pi}{2}\right)\right\}=\frac{4k}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\ldots\right\}$$

$$\Rightarrow 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-+\ldots\ldots=\frac{\pi}{4}$$





Eg. Find the Fourier series of $f(x)=\begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$ and use the results to show that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\times 3}-\frac{1}{3\times 5}+\frac{1}{5\times 7}-\frac{1}{7\times 9}+\ldots\ldots$ **. [2004 台大電研]**

(Sol.)

$(a)\ \ 2L=2\pi,\ \ L=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\frac{a_0}{2}=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}\sin xdx\right]=\frac{1}{2\pi}[1-\cos\pi]=\frac{1}{\pi}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi x}{L})dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot\cos(nx)dx+\int_{0}^{\pi}\sin(x)\cdot\cos(nx)dx\right]=\frac{1}{\pi}\int_{0}^{\pi}\frac{1}{2}[\sin(x+nx)+\sin(x-nx)]dx$$

$$=\frac{1}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n}+\frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{1+n}+\frac{1}{1-n}\right)-\frac{\cos(1+n)\pi}{1+n}-\frac{\cos(1-n)\pi}{1-n}\right]=\frac{1}{2\pi}\left[\frac{2}{1-n^2}+\frac{\cos n\pi}{1+n}+\frac{\cos n\pi}{1-n}\right]$$

$$=\frac{1}{2\pi}\left[\frac{2}{1-n^2}+\frac{2\cos n\pi}{1-n^2}\right]=\frac{1}{2\pi}\cdot\frac{2}{1-n^2}(1+\cos n\pi)$$

$$=\frac{1+\cos n\pi}{\pi(1-n^2)}(n\neq 1)=\begin{cases} 0,\forall n=3,5,7,\cdots \\ \frac{2}{\pi(1-n^2)},\forall n=2,4,6,\cdots \end{cases}$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot\sin(nx)dx+\int_{0}^{\pi}\sin(x)\cdot\sin(nx)dx\right]$$

$$=\frac{1}{\pi}\left\{\int_{0}^{\pi}\frac{1}{2}[\cos(x-nx)-\cos(x+nx)]dx\right\}=\frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{1-n}-\frac{\sin(1+n)\pi}{1+n}\right]=\begin{cases} 1/2, & n=1 \\ 0, & n>1 \end{cases}$$

$$\therefore\ \ f(x)=\frac{1}{\pi}+\frac{2}{\pi}\left[\frac{\cos(2x)}{-3}+\frac{\cos(4x)}{-15}+\frac{\cos(6x)}{-35}+\frac{\cos(8x)}{-63}+\ldots\right]+\frac{1}{2}\sin(x)$$

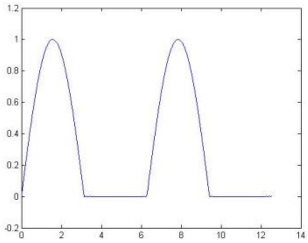
$$(b)\ \ f(-\frac{\pi}{2})=0=\frac{1}{\pi}+\frac{2}{\pi}\left[\frac{\cos(-\pi)}{-3}+\frac{\cos(-2\pi)}{-15}+\frac{\cos(-3\pi)}{-35}+\frac{\cos(-4\pi)}{-63}+\ldots\right]-\frac{1}{2}$$

$$\frac{1}{2}=\frac{1}{\pi}+\left[\frac{2}{3\pi}-\frac{2}{15\pi}+\frac{2}{35\pi}-\frac{2}{63\pi}+-\ldots\right]=\frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\frac{1}{63}+-\ldots\right)$$

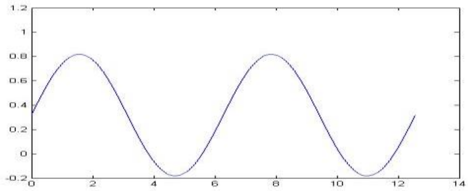
$$\therefore\ \ \frac{\pi}{4}=\frac{1}{2}+\frac{1}{1\times 3}-\frac{1}{3\times 5}+\frac{1}{5\times 7}-\frac{1}{7\times 9}+-\ldots\ldots$$

In **Matlab** language, we can use the following instructions to obtain the finite sum of $\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}+\frac{\cos(6x)}{35}+\frac{\cos(8x)}{63}+\ldots+\frac{\cos(40x)}{1599}\right]+\frac{1}{2}\sin(x)$.

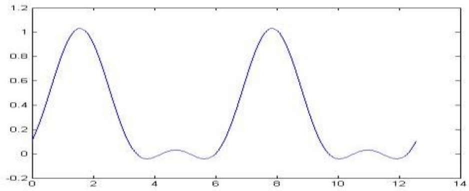
```
>>x = 0:0.001:4*pi; y=1/pi+sin(x)/2;
>>for n=1:20
y=y-2*cos(2*n*x)/pi/(4*n^2-1)
end
>>plot (x,y)
```



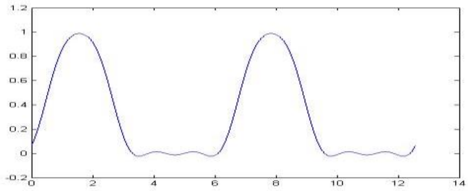
$\frac{1}{\pi}+\frac{1}{2}\sin(x)$



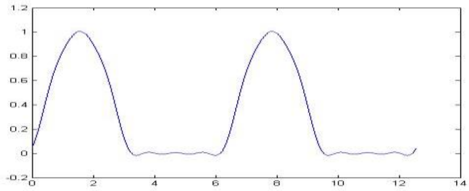
$\frac{1}{\pi}-\frac{2}{\pi}\cdot\frac{\cos(2x)}{3}+\frac{1}{2}\sin(x)$



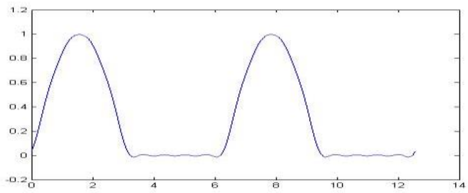
$\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}\right]+\frac{1}{2}\sin(x)$



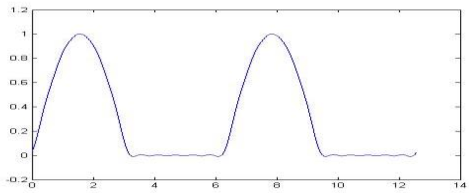
$\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{\cos(2x)}{3}+\frac{\cos(4x)}{15}+\frac{\cos(6x)}{35}\right]+\frac{1}{2}\sin(x)$



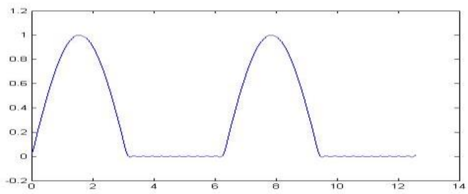
$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{4}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$



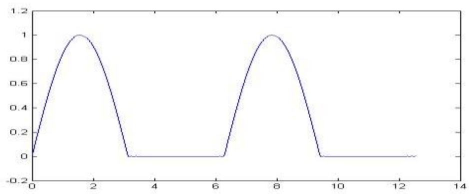
$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{5}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$

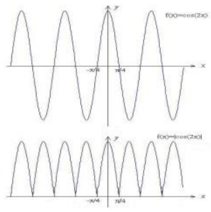


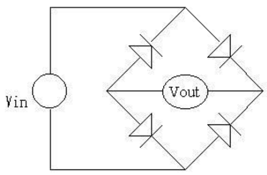
$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{10}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$



$\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{20}\frac{\cos(2nx)}{4n^2-1}+\frac{1}{2}\sin(x)$







Eg. Find the Fourier series of $|\cos(2x)|$ and calculate $\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$ [1990 交大材研、成大電研]

(Sol.) (a) $f(x)=f\left(x+\frac{\pi}{2}\right)$, $f(x)=\cos(2x)$, $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$

$$2L=\frac{\pi}{2},\ L=\frac{\pi}{4},\ \frac{n\pi x}{L}=4nx,\quad \frac{a_0}{2}=\frac{1}{2L}\int_{-L}^{L}\cos(2x)dx=\frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2x dx=\frac{2}{\pi}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}\cos(2x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx=\frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos(2x)\cos(4nx)dx$$

$$=\frac{4}{\pi}\cdot\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}\left[\cos((2+4n)x)+\cos((2-4n)x)\right]dx$$

$$=\frac{2}{\pi}\cdot\left[\frac{\sin(2+4n)x}{2+4n}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}+\frac{\sin(2-4n)x}{2-4n}\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right]$$

$$=\frac{2}{\pi}\cdot\left[\frac{2\sin\frac{(2n+1)}{2}\pi}{2(2n+1)}+\frac{2\sin\frac{(2n-1)}{2}\pi}{2(2n-1)}\right]=\frac{2}{\pi}\cdot\left[\frac{(-1)^n}{2n+1}+\frac{(-1)^{n+1}}{2n-1}\right]$$

$$=\frac{2}{\pi}\cdot\left[\frac{(-1)^n}{2n+1}-\frac{(-1)^n}{2n-1}\right]=-\frac{4}{\pi}\cdot\frac{(-1)^n}{4n^2-1}=\frac{4}{\pi}\cdot\frac{(-1)^{n+1}}{4n^2-1}$$

$\because$ Even function, $\therefore\ b_n=0$ $\quad\Rightarrow f(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)=|\cos(2x)|$

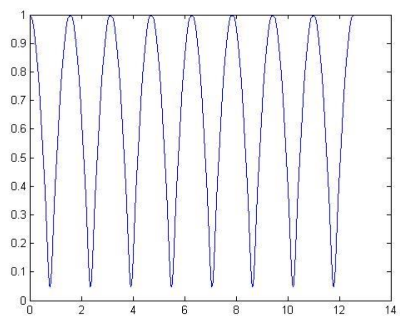
(b) $x=0,\ f(x)=1=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi}{4}\left(1-\frac{2}{\pi}\right)\ \Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}=\frac{\pi}{4}\left(\frac{2}{\pi}-1\right)$$

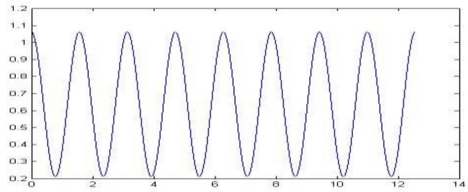
In **Matlab** language, we can use the following instructions to obtain the finite sum of

$\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{6}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$.

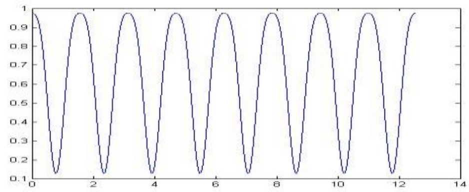
```
>>x = 0:0.001:4*pi; y=2/pi;
>>for n=1:6
y=y+4*(-1)^(n+1)*cos(4*n*x)/(4*n^2-1)/pi
end
>>plot (x,y)
```



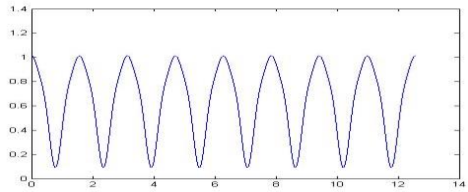
$$\frac{2}{\pi} + \frac{4}{3\pi}\cos(4x)$$



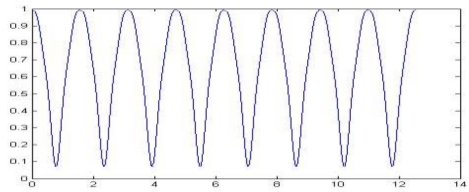
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{2}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



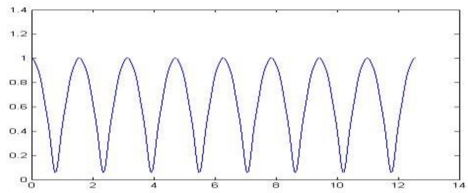
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{3}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



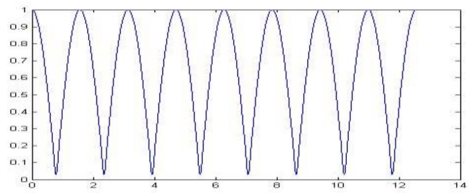
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{4}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



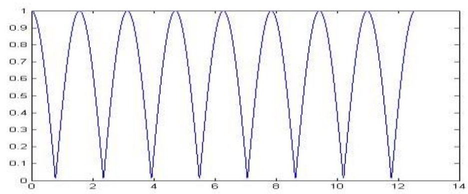
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{5}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



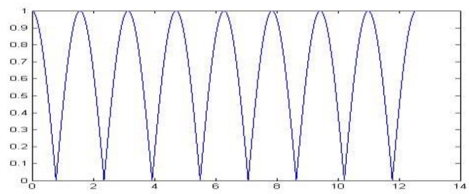
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{6}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



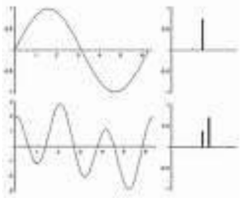
$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{10}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{100}\frac{(-1)^{n+1}}{4n^2-1}\cdot\cos(4nx)$$



Discrete spectrum of *f*(*t*): $\because\ \cos(\frac{n\pi t}{L}) = \frac{e^{i\frac{n\pi t}{L}} + e^{-i\frac{n\pi t}{L}}}{2}$, $\sin(\frac{n\pi t}{L}) = \frac{e^{i\frac{n\pi t}{L}} - e^{-i\frac{n\pi t}{L}}}{2i}$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right] = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi t}{L}} = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t},$$

where $\omega_n = \frac{n\pi}{L}$.