Chapter 6 Partial Differential Equations (PDE)

6-1 Classification of Partial Differential Equations

The first-order linear PDE: $a(x,y)\dfrac{\partial u}{\partial x} + b(x,y)\dfrac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0$

The second-order linear PDE: $a(x,y)\dfrac{\partial^2 u}{\partial x^2} + b(x,y)\dfrac{\partial^2 u}{\partial x \partial y} + c(x,y)\dfrac{\partial^2 u}{\partial y^2}$

$$+ d(x,y)\dfrac{\partial u}{\partial x} + e(x,y)\dfrac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0$$

$$\begin{cases} hyperbolic\ at\ (x_0, y_0): \ \Delta(x_0, y_0) = b(x_0, y_0)^2 - 4a(x_0, y_0)c(x_0, y_0) > 0 \\ elliptic\ at\ (x_0, y_0): \ \Delta(x_0, y_0) < 0 \\ parabolic\ at\ (x_0, y_0): \ \Delta(x_0, y_0) = 0 \end{cases}$$

Eg. Which is the type of partial differential equation for $4\dfrac{\partial^2 u}{\partial x^2} = 3\dfrac{\partial^2 u}{\partial y^2}$ **? [2018 台大電研]**

(Sol.) $4\dfrac{\partial^2 u}{\partial x^2} - 3\dfrac{\partial^2 u}{\partial y^2} = 0$, $\because$ Δ=0-4·4·(-3)=48>0, $\therefore$ it is hyperbolic.

Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) + b\dfrac{\partial u}{\partial t} + F$

Heat equation or Diffusion equation: $\dfrac{\partial u}{\partial t} = a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = a^2 \nabla^2 u$

Laplace's and Poisson's equations: $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \begin{cases} 0 \\ \rho \end{cases}$

Schrodinger's equation: $i\hbar\dfrac{\partial \psi}{\partial t} = \dfrac{-\hbar^2}{2m}\nabla^2 \psi + V\psi$ in quantum mechanics.

6-2 Separation-of-Variable Method

Eg. Solve $\dfrac{\partial^2\theta}{\partial x^2}=\dfrac{\partial^2\theta}{\partial t^2}$ **,** $\theta(x,0)=x$**,** $\theta(0,t)=0$**,** $\dfrac{\partial\theta}{\partial t}\Big|_{t=0}=0$ **, and** $\dfrac{\partial\theta}{\partial x}\Big|_{x=1}=0$ **.**

(Sol.) Let $\theta(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{T(t)}=\lambda$

$$X(0)=0,\ X'(1)=0\Rightarrow-\lambda=\left[\dfrac{(2n-1)\pi}{2}\right]^2,\quad X_n=C_n\sin\left[\dfrac{(2n-1)\pi}{2}x\right]$$

$$\dfrac{T''(t)}{T(t)}=-\left[\dfrac{(2n-1)\pi}{2}\right]^2,\ T(0)=\text{constant},\ T'(0)=0\Rightarrow T_n=d_n\cos\left[\dfrac{(2n-1)\pi}{2}t\right],$$

$$\therefore\ \theta(x,t)=\sum_{n=1}^{\infty}A_n\cos\left[\dfrac{(2n-1)\pi}{2}t\right]\cdot\sin\left[\dfrac{(2n-1)\pi}{2}x\right]$$

$$\theta(x,0)=x=\sum_{n=1}^{\infty}A_n\sin\left[\dfrac{(2n-1)\pi}{2}x\right]\Rightarrow A_n=\dfrac{\displaystyle\int_{-1}^{1}x\sin\left[\dfrac{(2n-1)\pi}{2}x\right]dx}{\displaystyle\int_{-1}^{1}\sin^2\left[\dfrac{(2n-1)\pi}{2}x\right]dx}=\dfrac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$ **,** $u(0,t)=u(1,t)=u(x,0)=0$**, and** $\dfrac{\partial u}{\partial t}\Big|_{t=0}=\sin(\pi x)$ **.**

(Sol.) $u(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T''(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$,

$\dfrac{T''(t)}{T(t)}=-(n\pi)^2$ and $T(0)=0$, $T'(0)=$constant$\Rightarrow T(t)=d_n\sin(n\pi t)$,

$$\therefore\ u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot\sin(n\pi t),\quad \dfrac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}n\pi A_n\sin(n\pi x)\cdot\cos(n\pi t)$$

$\dfrac{\partial u}{\partial t}\Big|_{t=0}=\sin(\pi x)\Rightarrow\pi A_1=1\Rightarrow A_1=1/\pi$ but $A_n=0$ for $n\ne1\Rightarrow u(x,t)=\dfrac{1}{\pi}\sin(\pi x)\cdot\sin(\pi t)$

Eg. Solve $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$ **,** $u(x,0)=3\sin(2\pi x)$**,** $u(0,t)=u(1,t)=0$**,** $0<x<1$**,** $t\ge0$**.**

(Sol.) $u(x,t)=X(x)T(t)$, $X(x)T'(t)=X''(x)T(t)$, $\dfrac{X''(x)}{X(x)}=\dfrac{T'(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$, $\dfrac{T'(t)}{T(t)}=-(n\pi)^2$ and $T(0)=$constant

$\Rightarrow T(t)=d_n e^{-n^2\pi^2 t}$, $\therefore\ u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot e^{-n^2\pi^2 t}$ and $u(x,0)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)$

$u(x,0)=3\sin(2\pi x)\Rightarrow A_2=3$ but $A_n=0$ for $n\ne2\Rightarrow u(x,t)=3e^{-4\pi^2 t}\cdot\sin(2\pi x)$

Eg. Solve $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$, **$y(x,0)$=$f(x)$=** $0.1\left(1 - \left|\dfrac{2x}{\pi} - 1\right|\right) = \begin{cases} \dfrac{0.2x}{\pi} & , 0 < x < \dfrac{\pi}{2} \\ \left(1 - \dfrac{x}{\pi}\right)0.2 & , \dfrac{\pi}{2} < x < \pi \end{cases}$,

and $y(x,t)$= $\displaystyle\sum_{n=1}^{\infty} b_n(t)\sin(nx)$. (a) Find the Fourier sine series for $f(x)$ on $[0,\pi]$. (b) Find the ordinary differential equation and the initial condition for $b_n(t)$. (c) Find $y(x,t)$. [1990 中央電研]

(Sol.) (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} a_n \sin\dfrac{n\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin(nx)$, $2L = 2\pi$, $L = \pi$, $\dfrac{n\pi x}{L} = nx$

$$a_n = \dfrac{1}{L}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \dfrac{2}{L}\int_{0}^{\pi} f(x)\sin(nx)dx$$

$$= \dfrac{2}{\pi}\left[\int_0^{\pi/2}\dfrac{0.2x}{\pi}\cdot\sin(nx)dx + \int_{\pi/2}^{\pi}0.2\left(1 - \dfrac{x}{\pi}\right)\sin(nx)dx\right] = \dfrac{0.8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty}\dfrac{0.8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)\cdot\sin(nx)$$

(b) $y(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n(t)\sin(nx)$, $\quad y(x,0) = \displaystyle\sum_{n=1}^{\infty} b_n(0)\sin(nx) = f(x)$

$$\Rightarrow \sum_{n=1}^{\infty} b_n''(t)\sin(nx) = \sum_{n=1}^{\infty} b_n(t)\cdot n^2\cdot(-\sin(nx))$$

$$\Rightarrow b_n''(t) + n^2 b_n(t) = 0 \quad \text{is the ordinary differential equation.}$$

$$\Rightarrow b_n(t) = \alpha_n\cos(nt) + \beta_n\sin(nt)$$

$$y(x,0) = f(x) = \sum_{n=1}^{\infty}\dfrac{0.8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)\cdot\sin(nx) = \sum_{n=1}^{\infty} b_n(0)\sin(nx)$$

$$\Rightarrow b_n(0) = \dfrac{0.8}{n^2\pi^2}\cdot\sin\left(\dfrac{n\pi}{2}\right) = \alpha_n \quad \text{(the initial condition)}, \quad \beta_n = 0$$

(c) $y(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n(t)\sin(nx) = \sum_{n=1}^{\infty}\alpha_n\cos(nt)\cdot\sin(nx)$

$$= \sum_{n=1}^{\infty}\dfrac{0.8}{n^2\pi^2}\sin\left(\dfrac{n\pi}{2}\right)\cos(nt)\cdot\sin(nx).$$

6-3 Laplace Transform Solutions of Partial Differential Equations

Eg. Solve $\dfrac{\partial \theta}{\partial x} + \dfrac{\partial \theta}{\partial y} = y$, $\theta(x,0)=0$ **for** $x \geq 0$, **and** $\theta(0,y)=y$ **for** $y \geq 0$. **[1990 台大化工研]**
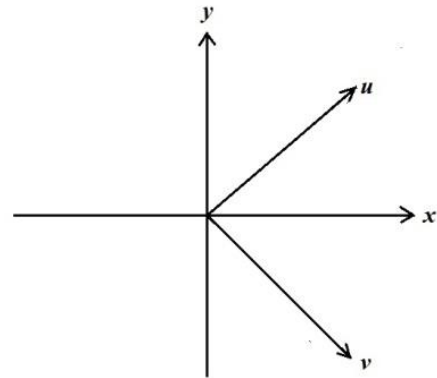
(Sol.)　**Method 1: By Laplace transform**

$$L[\theta(x,y)] = \int_0^\infty \theta(x,y)e^{-sy}dy = \Theta(x,s) , \quad L[y] = \frac{1}{s^2} = L[\theta(0,y)] = \Theta(0,s)$$

$$\Rightarrow \frac{d\Theta(x,s)}{dx} + s\Theta(x,s) - \theta(x,0) = \frac{1}{s^2} \Rightarrow \Theta(x,s) = A(s)\cdot e^{-sx} + \frac{1}{s^3}$$

$$\Theta(0,s) = A(s) + \frac{1}{s^3} = \frac{1}{s^2} \Rightarrow A(s) = \frac{1}{s^2} - \frac{1}{s^3} \Rightarrow \Theta(x,s) = \left(\frac{1}{s^2} - \frac{1}{s^3}\right)e^{-sx} + \frac{1}{s^3}$$

$$\Rightarrow \theta(x,y) = \left[y - x - \frac{1}{2}(y-x)^2\right]\cdot u(y-x) + \frac{y^2}{2}$$

$$= \begin{cases} \dfrac{y^2}{2}, & y \leq x \\[2mm] y - x - \dfrac{1}{2}(y-x)^2 + \dfrac{y^2}{2}, & y \geq x \end{cases}$$

Method 2: Define $u=x+y$ and $v=x-y$, $\therefore x = \dfrac{u+v}{2}$,

$y = \dfrac{u-v}{2}$,

we have $\dfrac{\partial \theta}{\partial x} = \dfrac{\partial \theta}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial \theta}{\partial v}\cdot\dfrac{\partial v}{\partial x} = \dfrac{\partial \theta}{\partial u} + \dfrac{\partial \theta}{\partial v}$

and $\dfrac{\partial \theta}{\partial y} = \dfrac{\partial \theta}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial \theta}{\partial v}\cdot\dfrac{\partial v}{\partial y} = \dfrac{\partial \theta}{\partial u} - \dfrac{\partial \theta}{\partial v}$

$\dfrac{\partial \theta}{\partial x} + \dfrac{\partial \theta}{\partial y} = 2\dfrac{\partial \theta}{\partial u} = y = \dfrac{u-v}{2}$, $\dfrac{\partial \theta}{\partial u} = \dfrac{u-v}{4}$, $\therefore \theta(u,v) = \dfrac{1}{4}\left(\dfrac{u^2}{2} - uv\right) + C(v)$

$1°$　$\theta(x,0)=0$　for　$x \geq 0$ $\Rightarrow$　$\theta(\dfrac{u+v}{2}$, $\dfrac{u-v}{2} = 0) = 0$,　$u=v$　and　we　have

$\theta = \dfrac{1}{4}\left(\dfrac{v^2}{2} - v^2\right) + C(v) = 0$,

$\therefore C(v) = \dfrac{v^2}{8}$　for $v \geq 0$ or $x \geq y$ $\Rightarrow \theta = \dfrac{1}{4}\left(\dfrac{u^2}{2} - uv\right) + \dfrac{v^2}{8} = \dfrac{1}{8}(u-v)^2 = \dfrac{y^2}{2}$　for $x \geq y$

$2°$ $\theta(0,y)=y$ for $y \geq 0 \Rightarrow$　$\theta(\dfrac{u+v}{2} = 0, \dfrac{u-v}{2}) = \dfrac{u-v}{2}$, $u=-v$ and we have

$\theta = \dfrac{1}{4}\left(\dfrac{v^2}{2} + v^2\right) + C(v) = -v$, $\therefore C(v) = -v - \dfrac{3v^2}{8}$　for $v \leq 0$ or $x \leq y$

$\Rightarrow \theta = \dfrac{1}{4}\left(\dfrac{u^2}{2} - uv\right) - v - \dfrac{3v^2}{8} = \dfrac{1}{8}(u-v)^2 - v - \dfrac{v^2}{2} = \dfrac{y^2}{2} + (y-x) - \dfrac{(x-y)^2}{2}$　for $x \leq y$

Note: This partial differential equation cannot be solved by separation of variables.

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, **u(0,t)=u(1,t)=u(x,0)=0, and** $\dfrac{\partial u}{\partial t}_{|t=0} = \sin(\pi x)$ **.**

(Sol.) $L[u(x,t)] = \displaystyle\int_0^\infty u(x,t)e^{-st}dt = U(x,s)$

$$\frac{d^2 U(x,s)}{dx^2} = s^2 U(x,s) - su(x,0) - \frac{\partial u}{\partial t}_{|t=0} = s^2 U(x,s) - \sin(\pi x)$$

$$\Rightarrow \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -\sin(\pi x) \Rightarrow U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{\sin \pi x}{s^2 + \pi^2}$$

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \Rightarrow \begin{cases} U(0,s) = 0 \\ U(1,s) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow u(x,t) = \frac{1}{\pi}\sin(\pi x)\cdot \sin(\pi t)$$

Eg. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, **u(x,0)=3sin(2πx), u(0,t)=u(1,t)=0, 0<x<1, t ≥ 0.**

(Sol.) $L[u(x,t)] = \displaystyle\int_0^\infty u(x,t)e^{-st}dt = U(x,s) \Rightarrow sU(x,s) - u(x,0) = \dfrac{d^2}{dx^2}U(x,s)$

$$\frac{d^2}{dx^2}U(x,s) - sU(x,s) = -3\sin(2\pi x) \Rightarrow U(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x} + \frac{3}{s + 4\pi^2}\cdot \sin(2\pi x)$$

$$L[u(0,t)] = U(0,s) = 0,\quad L[u(1,t)] = U(1,s) = 0 \Rightarrow c_1 = 0,\ c_2 = 0$$

$$U(x,s) = \frac{3}{s + 4\pi^2}\cdot \sin(2\pi x) \Rightarrow u(x,t) = L^{-1}\left[\frac{3}{s + 4\pi^2}\cdot \sin(2\pi x)\right] = 3e^{-4\pi^2 t}\cdot \sin(2\pi x)$$

6-4 Fourier Transform Solutions of Partial Differential Equations

Eg. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $u(x,0)=e^{-x^2}$, $-\infty < x < \infty$, $t>0$. **[2011 成大電研、2001 台大電研類似題]**

(Sol.) $\Im[u(x,t)]=U(\omega,t)$, $\Im\left[\dfrac{\partial^2 u(x,t)}{\partial x^2}\right]=-\omega^2 U(\omega,t)$

$$\frac{d}{dt}U(\omega,t)=-\omega^2 U(\omega,t) \Rightarrow U(\omega,t)=Ae^{-\omega^2 t}$$

According to $\Im[e^{-a^2 x^2}]=\dfrac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$,

$$U(\omega,0)=A=\Im[u(x,0)]=\Im[e^{-x^2}]=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}} \Rightarrow A=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}$$

Let $b^2=t+1/4$ and according to $\Im^{-1}[e^{-b^2\omega^2}]=\dfrac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}}$,

$$\Rightarrow u(x,t)=\Im^{-1}[U(\omega,t)]=\Im^{-1}[\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}e^{-\omega^2 t}]=\sqrt{\pi}\cdot \Im^{-1}[e^{-(t+\frac{1}{4})\omega^2}]=\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

Eg. Solve $\dfrac{\partial^2 u}{\partial t^2} = 9\dfrac{\partial^2 u}{\partial x^2}$, $u(x,0)=4e^{-5|x|}$, $\dfrac{\partial u(x,0)}{\partial t}=0$, $-\infty < x < \infty$, $t>0$.

(Sol.) $\Im[u(x,t)]=U(\omega,t)$, $\Im\left[9\dfrac{\partial^2 u(x,t)}{\partial x^2}\right]=-9\omega^2 U(\omega,t)$

$$\frac{d^2}{dt^2}U(\omega,t)=-9\omega^2 U(\omega,t) \Rightarrow U(\omega,t)=A\cos(3\omega t)+B\sin(3\omega t)$$

$$\frac{d}{dt}U(\omega,t)=-3\omega A\sin(3\omega t)+3\omega B\cos(3\omega t)$$

According to $\Im\left[e^{-a|x|}\right]=\dfrac{2a}{a^2+\omega^2}$, $U(\omega,0)=A=\Im[u(x,0)]=\Im[4e^{-5|x|}]=\dfrac{40}{25+\omega^2}$

$\Im[\dfrac{\partial u(x,0)}{\partial t}]=\dfrac{d}{dt}U(\omega,0)=0=3\omega B \Rightarrow$ B=0, $\therefore$ $U(\omega,t)=\dfrac{40}{\omega^2+25}\cdot\cos(3\omega t)$

By $\Im^{-1}[e^{ja\omega}F(\omega)]=f(x+a)$ and $\Im^{-1}[e^{-ja\omega}F(\omega)]=f(x-a)$,

$u(x,t)=\Im^{-1}[U(\omega,t)]=\Im^{-1}[\dfrac{40}{\omega^2+25}\cdot\cos(3\omega t)]=\Im^{-1}[\dfrac{40}{\omega^2+25}\cdot\dfrac{e^{i3\omega t}+e^{-i3\omega t}}{2}]$

$=\Im^{-1}[\dfrac{20}{\omega^2+25}\cdot(e^{i3\omega t}+e^{-i3\omega t})]=2\Im^{-1}[\dfrac{2\cdot 5}{\omega^2+25}\cdot(e^{i3\omega t}+e^{-i3\omega t})]=2e^{-5|x-3t|}+2e^{-5|x+3t|}$

6-5 Miscellaneous Methods of Solving Partial Differential Equations

Error function: $erf(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-u^2} du$

Complementary error function: $erfc(x) = 1 - erf(x)$

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$**,** $u(0,y)=0$ **and** $u(x,1)=x^2$ **.**

(Sol.) Set $u(x,y)=f(y+mx)$ and $\varepsilon \equiv y+mx$ $\Rightarrow$ $m^2 f''(\varepsilon) + 2mf''(\varepsilon) + f''(\varepsilon)=0$, $m^2+2m+1=0$, m=-1, -1

$\therefore$ $u(x,y)=f(y-x)+xg(y-x)$. $u(0,y)=0=f(y-0)+0\cdot g(y-0)=f(y)$ $\Rightarrow$ $f(1-x)=0$ and $f(y-x)=0$

$u(x,1)=x^2=f(1-x)+xg(1-x)=xg(1-x)$ $\Rightarrow$ $g(1-x)=x$, $g(x)=1-x$ $\Rightarrow$ $g(y-x)=1-y+x$

$$\therefore u(x,y)=x\cdot(1-y+x)$$

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} + u$ **.** **[2015 台大電子所工數 K]**

(Sol.) $u(x,y) = X(x)Y(y)$, $[X''(x)+2X'(x)]\cdot Y(y) = X(x)\cdot[Y'(y)+Y(y)]$,

$\dfrac{X''(x)+2X'(x)}{X(x)} = \dfrac{Y'(y)+Y(y)}{Y(y)} = \lambda \Rightarrow X''(x)+2X'(x)-\lambda X(x)=0$ and $Y'(y)+(1-\lambda)Y(y)=0$

$X''(x)+2X'(x)-\lambda X(x)=0$, $r^2+2r-\lambda=0$, $r=-1\pm\sqrt{1+\lambda}$

1˙ If $1+\lambda>0$, $\sqrt{1+\lambda}=k \in R$, $\lambda=k^2-1$ and $r=-1\pm k \Rightarrow X(x)=Ae^{(-1+k)x}+Be^{(-1-k)x}$, $1-\lambda=2-k^2$,

$Y'(y)+(1-\lambda)Y(y)=0 \Rightarrow Y(y)=C'e^{-(1-\lambda)y}=C'e^{(k^2-2)y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot[\int_0^\infty C(k)e^{-kx+k^2 y}dk + \int_0^\infty D(k)e^{kx+k^2 y}dk]$

2˙ If $1+\lambda=0$, $\lambda=-1$ and $r=-1$, $-1 \Rightarrow X(x)=Ae^{-x}+Bxe^{-x}$, $Y'(y)+2Y(y)=0 \Rightarrow Y(y)=C'e^{-2y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot(C+Dx)$

3˙ If $1+\lambda<0$, $\sqrt{1+\lambda}=\pm i\omega$, $r=-1\pm i\omega \Rightarrow X(x)=Ae^{-x}\cos(\omega x)+Be^{-x}\sin(\omega x)$,

$\lambda=-\omega^2-1$, $1-\lambda=2+\omega^2$, $Y'(y)+(2+\omega^2)Y(y)=0$, $Y(y)=C'e^{(-\omega^2-2)y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot[\int_0^\infty C(\omega)e^{-\omega^2 y}\cos(\omega x)d\omega + \int_0^\infty D(\omega)e^{-\omega^2 y}\sin(\omega x)d\omega]$