

Chapter 6 Partial Differential Equations (PDE)

6-1 Classification of Partial Differential Equations

The first-order linear PDE: $a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} + f(x, y)u + g(x, y) = 0$

The second-order linear PDE: $a(x, y) \frac{\partial^2 u}{\partial x^2} + b(x, y) \frac{\partial^2 u}{\partial x \partial y} + c(x, y) \frac{\partial^2 u}{\partial y^2} + d(x, y) \frac{\partial u}{\partial x} + e(x, y) \frac{\partial u}{\partial y} + f(x, y)u + g(x, y) = 0$

$\left\{ \begin{array}{l} \text{hyperbolic at } (x_0, y_0): \Delta(x_0, y_0) = b(x_0, y_0)^2 - 4a(x_0, y_0)c(x_0, y_0) > 0 \\ \text{elliptic at } (x_0, y_0): \Delta(x_0, y_0) < 0 \\ \text{parabolic at } (x_0, y_0): \Delta(x_0, y_0) = 0 \end{array} \right.$

Eg. Which is the type of partial differential equation for $4 \frac{\partial^2 u}{\partial x^2} = 3 \frac{\partial^2 u}{\partial y^2}$? [2018 台大電研]

(Sol.) $4 \frac{\partial^2 u}{\partial x^2} - 3 \frac{\partial^2 u}{\partial y^2} = 0$, $\therefore \Delta = 0 - 4 \cdot 4 \cdot (-3) = 48 > 0$, \therefore it is hyperbolic.

Wave equation: $\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + b \frac{\partial u}{\partial t} + F$

Heat equation or Diffusion equation: $\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = a^2 \nabla^2 u$

Laplace's and Poisson's equations: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \begin{cases} 0 \\ \rho \end{cases}$

Schrodinger's equation: $i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi$ in quantum mechanics.



6-2 Separation-of-Variable Method

Eg. Solve $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$, $\theta(x,0)=x$, $\theta(0,t)=0$, $\frac{\partial \theta}{\partial t}|_{t=0} = 0$, and $\frac{\partial \theta}{\partial x}|_{x=1} = 0$.

(Sol.) Let $\theta(x,t) = X(x)T(t)$, $X''(x)T(t) = X(x)T''(t)$, $\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$

$$X(0)=0, X'(1)=0 \Rightarrow -\lambda = \left[\frac{(2n-1)\pi}{2} \right]^2, \quad X_n = C_n \sin \left[\frac{(2n-1)\pi}{2} x \right]$$

$$\frac{T''(t)}{T(t)} = - \left[\frac{(2n-1)\pi}{2} \right]^2, \quad T(0)=\text{constant}, T'(0)=0 \Rightarrow T_n = d_n \cos \left[\frac{(2n-1)\pi}{2} t \right],$$

$$\therefore \theta(x,t) = \sum_{n=1}^{\infty} A_n \cos \left[\frac{(2n-1)\pi}{2} t \right] \cdot \sin \left[\frac{(2n-1)\pi}{2} x \right]$$

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin \left[\frac{(2n-1)\pi}{2} x \right] \Rightarrow A_n = \frac{\int_{-1}^1 x \sin \left[\frac{(2n-1)\pi}{2} x \right] dx}{\int_{-1}^1 \sin^2 \left[\frac{(2n-1)\pi}{2} x \right] dx} = \frac{8(-1)^{n+1}}{(2n-1)^2 \pi^2}$$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t)=u(1,t)=u(x,0)=0$, and $\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x)$.

(Sol.) $u(x,t) = X(x)T(t)$, $X''(x)T(t) = X(x)T''(t)$, $\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$,

$$X(0)=0=X(1) \Rightarrow \lambda = -(n\pi)^2 \text{ and } X(x) = C_n \sin(n\pi x),$$

$$\frac{T''(t)}{T(t)} = -(n\pi)^2 \text{ and } T(0)=0, T'(0)=\text{constant} \Rightarrow T(t) = d_n \sin(n\pi t),$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cdot \sin(n\pi t), \quad \frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty} n\pi A_n \sin(n\pi x) \cdot \cos(n\pi t)$$

$$\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x) \Rightarrow \pi A_1 = 1 \Rightarrow A_1 = 1/\pi \text{ but } A_n = 0 \text{ for } n \neq 1 \Rightarrow u(x,t) = \frac{1}{\pi} \sin(\pi x) \cdot \sin(\pi t)$$

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0 < x < 1$, $t \geq 0$.

(Sol.) $u(x,t) = X(x)T(t)$, $X(x)T'(t) = X''(x)T(t)$, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda$,

$$X(0)=0=X(1) \Rightarrow \lambda = -(n\pi)^2 \text{ and } X(x) = C_n \sin(n\pi x), \quad \frac{T'(t)}{T(t)} = -(n\pi)^2 \text{ and } T(0)=\text{constant}$$

$$\Rightarrow T(t) = d_n e^{-n^2 \pi^2 t}, \quad \therefore u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cdot e^{-n^2 \pi^2 t} \text{ and } u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$$u(x,0) = 3\sin(2\pi x) \Rightarrow A_2 = 3 \text{ but } A_n = 0 \text{ for } n \neq 2 \Rightarrow u(x,t) = 3e^{-4\pi^2 t} \cdot \sin(2\pi x)$$

Eg. Solve $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$, $y(x,0)=f(x)=0.1\left(1-\left|\frac{2x}{\pi}-1\right|\right) = \begin{cases} \frac{0.2x}{\pi}, & 0 < x < \frac{\pi}{2} \\ \left(1-\frac{x}{\pi}\right)0.2, & \frac{\pi}{2} < x < \pi \end{cases}$, and

$y(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$. (a) Find the Fourier sine series for $f(x)$ on $[0,\pi]$. (b) Find the ordinary differential equation and the initial condition for $b_n(t)$. (c) Find $y(x,t)$. [1990 中央電研]

(Sol.) (a) $f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin(nx)$, $2L = 2\pi$, $L = \pi$, $\frac{n\pi x}{L} = nx$

$$a_n = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{2}{L} \int_0^{\pi} f(x) \sin(nx) dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{0.2x}{\pi} \cdot \sin(nx) dx + \int_{\pi/2}^{\pi} 0.2 \left(1 - \frac{x}{\pi}\right) \sin(nx) dx \right] = \frac{0.8}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{0.8}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \cdot \sin(nx)$$

(b) $y(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx)$, $y(x,0) = \sum_{n=1}^{\infty} b_n(0) \sin(nx) = f(x)$

$$\Rightarrow \sum_{n=1}^{\infty} b_n''(t) \sin(nx) = \sum_{n=1}^{\infty} b_n(t) \cdot n^2 \cdot (-\sin(nx))$$

$\Rightarrow b_n''(t) + n^2 b_n(t) = 0$ is the ordinary differential equation.

$$\Rightarrow b_n(t) = \alpha_n \cos(nt) + \beta_n \sin(nt)$$

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} \frac{0.8}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \cdot \sin(nx) = \sum_{n=1}^{\infty} b_n(0) \sin(nx)$$

$$\Rightarrow b_n(0) = \frac{0.8}{n^2 \pi^2} \cdot \sin\left(\frac{n\pi}{2}\right) = \alpha_n \quad (\text{the initial condition}), \quad \beta_n = 0$$

(c) $y(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(nx) = \sum_{n=1}^{\infty} \alpha_n \cos(nt) \cdot \sin(nx)$

$$= \sum_{n=1}^{\infty} \frac{0.8}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \cos(nt) \cdot \sin(nx).$$

6-3 Laplace Transform Solutions of Partial Differential Equations

Eg. Solve $\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} = y$, $\theta(x,0)=0$ for $x \geq 0$, and $\theta(0,y)=y$ for $y \geq 0$. [1990 台大化工研]

(Sol.) **Method 1: By Laplace transform**

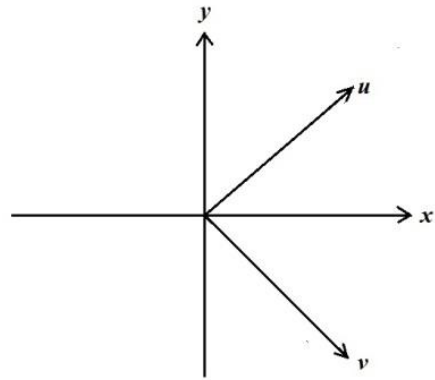
$$L[\theta(x, y)] = \int_0^{\infty} \theta(x, y)e^{-sy} dy = \Theta(x, s), \quad L[y] = \frac{1}{s^2} = L[\theta(0, y)] = \Theta(0, s)$$

$$\Rightarrow \frac{d\Theta(x, s)}{dx} + s\Theta(x, s) - \theta(x, 0) = \frac{1}{s^2} \Rightarrow \Theta(x, s) = A(s) \cdot e^{-sx} + \frac{1}{s^3}$$

$$\Theta(0, s) = A(s) + \frac{1}{s^3} = \frac{1}{s^2} \Rightarrow A(s) = \frac{1}{s^2} - \frac{1}{s^3} \Rightarrow \Theta(x, s) = \left(\frac{1}{s^2} - \frac{1}{s^3} \right) e^{-sx} + \frac{1}{s^3}$$

$$\Rightarrow \theta(x, y) = \left[y - x - \frac{1}{2}(y-x)^2 \right] \cdot u(y-x) + \frac{y^2}{2}$$

$$= \begin{cases} \frac{y^2}{2}, & y \leq x \\ y - x - \frac{1}{2}(y-x)^2 + \frac{y^2}{2}, & y \geq x \end{cases}$$



Method 2: Define $u=x+y$ and $v=x-y$, $\therefore x = \frac{u+v}{2}$,

$$y = \frac{u-v}{2},$$

we have $\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial u} + \frac{\partial \theta}{\partial v}$

and $\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial u} - \frac{\partial \theta}{\partial v}$

$$\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} = 2 \frac{\partial \theta}{\partial u} = y = \frac{u-v}{2}, \quad \frac{\partial \theta}{\partial u} = \frac{u-v}{4}, \quad \therefore \theta(u, v) = \frac{1}{4} \left(\frac{u^2}{2} - uv \right) + C(v)$$

1° $\theta(x,0)=0$ for $x \geq 0 \Rightarrow \theta\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = 0$, $u=v$ and we have

$$\theta = \frac{1}{4} \left(\frac{v^2}{2} - v^2 \right) + C(v) = 0,$$

$$\therefore C(v) = \frac{v^2}{8} \text{ for } v \geq 0 \text{ or } x \geq y \Rightarrow \theta = \frac{1}{4} \left(\frac{u^2}{2} - uv \right) + \frac{v^2}{8} = \frac{1}{8} (u-v)^2 = \frac{y^2}{2} \text{ for } x \geq y$$

2° $\theta(0,y)=y$ for $y \geq 0 \Rightarrow \theta\left(\frac{u+v}{2}=0, \frac{u-v}{2}\right) = \frac{u-v}{2}$, $u=-v$ and we have

$$\theta = \frac{1}{4} \left(\frac{v^2}{2} + v^2 \right) + C(v) = -v, \quad \therefore C(v) = -v - \frac{3v^2}{8} \text{ for } v \leq 0 \text{ or } x \leq y$$

$$\Rightarrow \theta = \frac{1}{4} \left(\frac{u^2}{2} - uv \right) - v - \frac{3v^2}{8} = \frac{1}{8} (u-v)^2 - v - \frac{v^2}{2} = \frac{y^2}{2} + (y-x) - \frac{(x-y)^2}{2} \text{ for } x \leq y$$

Note: This partial differential equation cannot be solved by separation of variables.

Eg. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t)=u(1,t)=u(x,0)=0$, and $\frac{\partial u}{\partial t}|_{t=0} = \sin(\pi x)$.

(Sol.) $L[u(x,t)] = \int_0^\infty u(x,t)e^{-st} dt = U(x,s)$

$$\frac{d^2 U(x,s)}{dx^2} = s^2 U(x,s) - su(x,0) - \frac{\partial u}{\partial t}|_{t=0} = s^2 U(x,s) - \sin(\pi x)$$

$$\Rightarrow \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -\sin(\pi x) \Rightarrow U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{\sin \pi x}{s^2 + \pi^2}$$

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \Rightarrow \begin{cases} U(0,s) = 0 \\ U(1,s) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow u(x,t) = \frac{1}{\pi} \sin(\pi x) \cdot \sin(\pi t)$$

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0 < x < 1$, $t \geq 0$.

(Sol.) $L[u(x,t)] = \int_0^\infty u(x,t)e^{-st} dt = U(x,s) \Rightarrow sU(x,s) - u(x,0) = \frac{d^2}{dx^2} U(x,s)$

$$\frac{d^2}{dx^2} U(x,s) - sU(x,s) = -3\sin(2\pi x) \Rightarrow U(x,s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x} + \frac{3}{s + 4\pi^2} \cdot \sin(2\pi x)$$

$$L[u(0,t)] = U(0,s) = 0, \quad L[u(1,t)] = U(1,s) = 0 \Rightarrow c_1 = 0, \quad c_2 = 0$$

$$U(x,s) = \frac{3}{s + 4\pi^2} \cdot \sin(2\pi x) \Rightarrow u(x,t) = L^{-1} \left[\frac{3}{s + 4\pi^2} \cdot \sin(2\pi x) \right] = 3e^{-4\pi^2 t} \cdot \sin(2\pi x)$$



6-4 Fourier Transform Solutions of Partial Differential Equations

Eg. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = e^{-x^2}$, $-\infty < x < \infty$, $t > 0$. [2011 成大電研、2001 台大電研類
似題]

$$\text{(Sol.) } \mathfrak{F}[u(x,t)] = U(\omega,t), \quad \mathfrak{F}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = -\omega^2 U(\omega,t)$$

$$\frac{d}{dt}U(\omega,t) = -\omega^2 U(\omega,t) \Rightarrow U(\omega,t) = Ae^{-\omega^2 t}$$

$$\text{According to } \mathfrak{F}[e^{-a^2 x^2}] = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}},$$

$$U(\omega,0) = A = \mathfrak{F}[u(x,0)] = \mathfrak{F}[e^{-x^2}] = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} \Rightarrow A = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}}$$

$$\text{Let } b^2 = t + 1/4 \text{ and according to } \mathfrak{F}^{-1}[e^{-b^2 \omega^2}] = \frac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}},$$

$$\Rightarrow u(x,t) = \mathfrak{F}^{-1}[U(\omega,t)] = \mathfrak{F}^{-1}\left[\sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} e^{-\omega^2 t}\right] = \sqrt{\pi} \cdot \mathfrak{F}^{-1}\left[e^{-(t+\frac{1}{4})\omega^2}\right] = \frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

Eg. Solve $\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = 4e^{-5|x|}$, $\frac{\partial u(x,0)}{\partial t} = 0$, $-\infty < x < \infty$, $t > 0$.

$$\text{(Sol.) } \mathfrak{F}[u(x,t)] = U(\omega,t), \quad \mathfrak{F}\left[9 \frac{\partial^2 u(x,t)}{\partial x^2}\right] = -9\omega^2 U(\omega,t)$$

$$\frac{d^2}{dt^2}U(\omega,t) = -9\omega^2 U(\omega,t) \Rightarrow U(\omega,t) = A \cos(3\omega t) + B \sin(3\omega t)$$

$$\frac{d}{dt}U(\omega,t) = -3\omega A \sin(3\omega t) + 3\omega B \cos(3\omega t)$$

$$\text{According to } \mathfrak{F}[e^{-a|x|}] = \frac{2a}{a^2 + \omega^2}, \quad U(\omega,0) = A = \mathfrak{F}[u(x,0)] = \mathfrak{F}[4e^{-5|x|}] = \frac{40}{25 + \omega^2}$$

$$\mathfrak{F}\left[\frac{\partial u(x,0)}{\partial t}\right] = \frac{d}{dt}U(\omega,0) = 0 = 3\omega B \Rightarrow B = 0, \quad \therefore U(\omega,t) = \frac{40}{\omega^2 + 25} \cdot \cos(3\omega t)$$

$$\text{By } \mathfrak{F}^{-1}[e^{ja\omega}F(\omega)] = f(x+a) \text{ and } \mathfrak{F}^{-1}[e^{-ja\omega}F(\omega)] = f(x-a),$$

$$u(x,t) = \mathfrak{F}^{-1}[U(\omega,t)] = \mathfrak{F}^{-1}\left[\frac{40}{\omega^2 + 25} \cdot \cos(3\omega t)\right] = \mathfrak{F}^{-1}\left[\frac{40}{\omega^2 + 25} \cdot \frac{e^{i3\omega t} + e^{-i3\omega t}}{2}\right]$$

$$= \mathfrak{F}^{-1}\left[\frac{20}{\omega^2 + 25} \cdot (e^{i3\omega t} + e^{-i3\omega t})\right] = 2\mathfrak{F}^{-1}\left[\frac{2 \cdot 5}{\omega^2 + 25} \cdot (e^{i3\omega t} + e^{-i3\omega t})\right] = 2e^{-5|x-3t|} + 2e^{-5|x+3t|}$$

6-5 Miscellaneous Methods of Solving Partial Differential Equations

Error function: $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

Complementary error function: $erfc(x) = 1 - erf(x)$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0,y)=0$ and $u(x,1)=x^2$.

(Sol.) Set $u(x,y)=f(y+mx)$ and $\varepsilon \equiv y+mx \Rightarrow m^2 f''(\varepsilon) + 2mf''(\varepsilon) + f''(\varepsilon) = 0$, $m^2 + 2m + 1 = 0$,
 $m = -1, -1$

$\therefore u(x,y) = f(y-x) + xg(y-x)$. $u(0,y) = 0 = f(y-0) + 0 \cdot g(y-0) = f(y) \Rightarrow f(1-x) = 0$ and $f(y-x) = 0$

$u(x,1) = x^2 = f(1-x) + xg(1-x) = xg(1-x) \Rightarrow g(1-x) = x$, $g(x) = 1-x \Rightarrow g(y-x) = 1-y+x$

$\therefore u(x,y) = x \cdot (1-y+x)$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + u$. [2015 台大電子所工數 K]

(Sol.) $u(x,y) = X(x)Y(y)$, $[X''(x) + 2X'(x)] \cdot Y(y) = X(x) \cdot [Y'(y) + Y(y)]$,

$\frac{X''(x) + 2X'(x)}{X(x)} = \frac{Y'(y) + Y(y)}{Y(y)} = \lambda \Rightarrow X''(x) + 2X'(x) - \lambda X(x) = 0$ and $Y'(y) + (1-\lambda)Y(y) = 0$

$X''(x) + 2X'(x) - \lambda X(x) = 0$, $r^2 + 2r - \lambda = 0$, $r = -1 \pm \sqrt{1+\lambda}$

1. If $1+\lambda > 0$, $\sqrt{1+\lambda} = k \in R$, $\lambda = k^2 - 1$ and $r = -1 \pm k \Rightarrow X(x) = Ae^{(-1+k)x} + Be^{(-1-k)x}$, $1-\lambda = 2-k^2$,

$Y'(y) + (1-\lambda)Y(y) = 0 \Rightarrow Y(y) = C'e^{-(1-\lambda)y} = C'e^{(k^2-2)y}$

$\Rightarrow u(x,y) = e^{-x-2y} \cdot \left[\int_0^\infty C(k)e^{-kx+k^2y} dk + \int_0^\infty D(k)e^{kx+k^2y} dk \right]$

2. If $1+\lambda = 0$, $\lambda = -1$ and $r = -1, -1 \Rightarrow X(x) = Ae^{-x} + Bxe^{-x}$, $Y'(y) + 2Y(y) = 0 \Rightarrow Y(y) = C'e^{-2y}$

$\Rightarrow u(x,y) = e^{-x-2y} \cdot (C + Dx)$

3. If $1+\lambda < 0$, $\sqrt{1+\lambda} = \pm i\omega$, $r = -1 \pm i\omega \Rightarrow X(x) = Ae^{-x}\cos(\omega x) + Be^{-x}\sin(\omega x)$,

$\lambda = -\omega^2 - 1$, $1-\lambda = 2+\omega^2$, $Y'(y) + (2+\omega^2)Y(y) = 0$, $Y(y) = C'e^{(-\omega^2-2)y}$

$\Rightarrow u(x,y) = e^{-x-2y} \cdot \left[\int_0^\infty C(\omega)e^{-\omega^2y} \cos(\omega x) d\omega + \int_0^\infty D(\omega)e^{-\omega^2y} \sin(\omega x) d\omega \right]$