Chapter 7 Vector Analysis

7-1 Vector Functions

One-variable vector function: $\vec{R}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$

Multi-variable vector function: $\vec{F}(x,y,z) = F_x(x,y,z)\,\hat{x} + F_y(x,y,z)\hat{y} + F_z(x,y,z)\hat{z}$

The *derivatives* of vector functions: $\dfrac{d\vec{R}(t)}{df(t)} = \dfrac{d\vec{R}(t)}{dt}\cdot\dfrac{dt}{df(t)} = \dfrac{d\vec{R}(t)}{dt} \Big/ \dfrac{df(t)}{dt}$

$$\frac{\partial \vec{F}(x,y,z)}{\partial g(x,y,z)} = \frac{\partial \vec{F}(x,y,z)}{\partial x}\cdot\frac{\partial x}{\partial g(x,y,z)} + \frac{\partial \vec{F}(x,y,z)}{\partial y}\cdot\frac{\partial y}{\partial g(x,y,z)} + \frac{\partial \vec{F}(x,y,z)}{\partial z}\cdot\frac{\partial z}{\partial g(x,y,z)}$$

Eg. For $\vec{R}(t) = 2t\hat{x} - \cos(3t)\hat{y} + t^3\hat{z}$ **,** $0 \le t \le 1$**, find** $\vec{R}'(t) = d\vec{R}(t)/dt$ **. Let *s*(*t*)=** $\int_0^1 \sqrt{4 + 9\sin^2(3t) + 9t^4}\cdot dt$**, then find** $d\vec{R}(t)/ds(t)$**.**

(Sol) $d\vec{R}(t)/dt = 2\hat{x} + 3\sin(3t)\hat{y} + 3t^2\hat{z}$

$$\therefore\ \frac{d\vec{R}(t)}{ds(t)} = \frac{d\vec{R}(t)}{dt}\cdot\frac{dt}{ds(t)} = \frac{\left[\dfrac{d\vec{R}(t)}{dt}\right]}{\left[\dfrac{ds(t)}{dt}\right]} = \frac{2\hat{x} + 3\sin(3t)\hat{y} + 3t^2\hat{z}}{\sqrt{4 + 9\sin^2(3t) + 9t^4}}$$

Some theorems of derivatives of vector functions:

1. $(\vec{F}\cdot\vec{G})' = \vec{F}'\cdot\vec{G} + \vec{F}\cdot\vec{G}'$

2. $(\vec{F}\times\vec{G})' = \vec{F}'\times\vec{G} + \vec{F}\times\vec{G}'$

3. $(\vec{F}\times\vec{F}')' = \vec{F}\times\vec{F}''$

(Proof) $(\vec{F}\times\vec{F}')' = \vec{F}'\times\vec{F}' + \vec{F}\times\vec{F}'' = \vec{F}\times\vec{F}''$

4. For $\vec{R}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$ **, if** $\vec{R}(t)$ **does not change direction, then** $\vec{R}(t)\times\vec{R}'(t) = 0$**, and vice versa.**

5. Let $\vec{R}(t)$ **denote the position of a particle at time *t*. If the particle moves so that equal areas are swept out in equal times, then we have** $\vec{R}(t)\times\vec{R}''(t) = 0$**, and vice versa. (Kepler's law)**

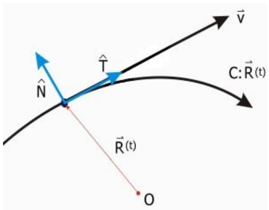
(Proof) 2 area $= \vec{R}(t)\times[\vec{R}(t+\Delta t) - \vec{R}(t)] \approx \vec{R}(t)\times\vec{R}'(t)\cdot\Delta t$

If area $= 0 \Leftrightarrow \vec{R}(t)\times\vec{R}'(t)\cdot\Delta t = 0 \Leftrightarrow \vec{R}(t)\times\vec{R}'(t) = 0$

Equal area in equal time

$\Leftrightarrow \vec{R}(t)\times\vec{R}'(t) = \text{constant} \Leftrightarrow [\vec{R}(t)\times\vec{R}'(t)]' = 0 \Leftrightarrow \vec{R}(t)\times\vec{R}''(t) = 0$

7-2 Differential Geometry



Position vector: $\vec{R}(t)=x(t)\,\hat{x}+y(t)\,\hat{y}+z(t)\,\hat{z}$

Velocity: $\vec{v}(t)=\vec{R}'(t)=\dfrac{dx(t)}{dt}\,\hat{x}+\dfrac{dy(t)}{dt}\,\hat{y}+\dfrac{dz(t)}{dt}\,\hat{z}$

Arc length: $s(t)=\int_{t_1}^{t_2}|\vec{R}'(t)|\,dt,\ \dfrac{ds}{dt}=|\vec{R}'(t)|=|\vec{v}(t)|$

Acceleration: $\vec{a}(t)=\vec{v}'(t)=\dfrac{d^2x(t)}{dt^2}\,\hat{x}+\dfrac{d^2y(t)}{dt^2}\,\hat{y}+\dfrac{d^2z(t)}{dt^2}\,\hat{z}$

Curvature: $\kappa=\left|\dfrac{d\hat{T}}{ds}\right|=\left|\dfrac{d\hat{T}}{dt}\cdot\dfrac{dt}{ds}\right|=\left|\dfrac{d\hat{T}}{dt}\right|\cdot\dfrac{1}{|\vec{v}|}$, where $\hat{T}=\dfrac{\vec{R}'(t)}{|\vec{R}'(t)|}=\dfrac{\vec{v}(t)}{|\vec{v}(t)|}=\dfrac{d\vec{R}(t)}{ds(t)}$

Eg. C: $\vec{R}(t)=t\hat{x}+(t-2)\hat{y}+(3t-1)\hat{z}$ **is a straight line.**

$$\hat{T}=\frac{\vec{v}}{|\vec{v}|}=\frac{\hat{x}+\hat{y}+3\hat{z}}{\sqrt{11}},\quad \kappa=\left|\frac{d\hat{T}}{ds}\right|=\left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right|=0.$$

Eg. C: $\vec{R}(t)=2\cos(t)\,\hat{x}+2\sin(t)\hat{y}+4\hat{z}$ **is a circle of radius 2 at z=4.**

$$\hat{T}=\frac{-2\sin(t)\,\hat{x}+2\cos(t)\hat{y}}{2},\quad \kappa=\left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right|=\frac{1}{2}=\frac{1}{r}.$$

Eg. Consider the curve: $\vec{r}=a\cos(t)\hat{x}+a\sin(t)\hat{y}+bt\hat{z}$ **,** $0\le t\le 2\pi$**. What is the equation of tangential vector at $t=\pi/2$. [1991 中山機研]**

(Sol.) $\vec{v}=\dfrac{d\vec{r}}{dt}=-a\sin(t)\,\hat{x}+a\cos(t)\hat{y}+b\hat{z}$. At $t=\dfrac{\pi}{2}\Rightarrow\vec{v}=-a\hat{x}+b\hat{z}$

$$\hat{T}=\frac{\vec{v}}{|\vec{v}|}=\frac{-a\hat{x}+b\hat{z}}{\sqrt{a^2+b^2}}$$

Theorem $\vec{a}=\dfrac{d|\vec{v}|}{dt}\hat{T}+\dfrac{|\vec{v}|^2}{\rho}\hat{N}=$ **tangential acceleration+centripetal acceleration**

(Proof) Define $\hat{N}=\rho\dfrac{d\hat{T}}{ds}$. It is easily shown that $\hat{N}\perp\hat{T}$ because we have the following proof: $\hat{T}\cdot\hat{T}=1,\ \dfrac{d(\hat{T}\cdot\hat{T})}{ds}=0,\ \hat{T}\cdot\dfrac{d\hat{T}}{ds}+\dfrac{d\hat{T}}{ds}\cdot\hat{T}=2\hat{T}\cdot\dfrac{d\hat{T}}{ds}=0,\ \hat{T}\cdot\dfrac{\rho d\hat{T}}{ds}=0\Rightarrow\hat{N}\perp\hat{T}$

$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=\frac{d}{dt}[|\vec{v}(t)|\hat{T}]=\frac{d|\vec{v}(t)|}{dt}\cdot\hat{T}+|\vec{v}(t)|\cdot\frac{d\hat{T}}{dt}$$

$$=\frac{d|\vec{v}(t)|}{dt}\cdot\hat{T}+|\vec{v}(t)|\cdot\left(\frac{ds}{dt}\cdot\frac{d\hat{T}}{ds}\right)=\frac{d|\vec{v}(t)|}{dt}\cdot\hat{T}+|\vec{v}(t)|^2\frac{d\hat{T}}{ds}=\frac{d|\vec{v}(t)|}{dt}\hat{T}+\frac{|\vec{v}(t)|^2}{\rho}\hat{N},$$

Eg. For $\vec{R}(t)=[\cos(t)+t\sin(t)]\hat{x}+[\sin(t)-t\cos(t)]\hat{y}+t^2\hat{z}$ **,** t**>0, we have**

$\vec{v}(t)=t\cos(t)\hat{x}+t\sin(t)\hat{y}+2t\hat{z}$ **,** $\vec{a}(t)=[\cos(t)-t\sin(t)]\hat{x}+[\sin(t)+t\cos(t)]\hat{y}+2\hat{z}$

$$\hat{T}=\frac{\vec{v}(t)}{|\vec{v}(t)|}=\frac{1}{\sqrt{5}}\cos(t)\hat{x}+\frac{1}{\sqrt{5}}\sin(t)\hat{y}+\frac{2}{\sqrt{5}}\hat{z},\quad \rho=\frac{1}{\kappa}=\frac{1}{\left|\frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}\right|}=\frac{1}{\left|\frac{d\hat{T}}{dt}\cdot\frac{1}{|\vec{v}(t)|}\right|}=5t$$

$$\hat{N}=\rho\frac{d\hat{T}}{ds}=\rho\frac{dt}{ds}\cdot\frac{d\hat{T}}{dt}=\frac{\rho}{|\vec{v}(t)|}\cdot\frac{d\hat{T}}{dt}=-\sin(t)\hat{x}+\cos(t)\hat{y}$$

Binormal vector: $\hat{B}=\hat{T}\times\hat{N}$

Fernet formulae:
$$\begin{cases}\dfrac{d\hat{T}}{ds}=\kappa\hat{N}=\dfrac{\hat{N}}{\rho} & (1)\\[2ex] \dfrac{d\hat{N}}{ds}=-\kappa\hat{T}+\tau\hat{B} & (2)\\[2ex] \dfrac{d\hat{B}}{ds}=-\tau\hat{N} & (3)\end{cases}$$

Torsion **of a curve: τ**

(Proof of (3)) $\dfrac{d\hat{B}}{ds}=\dfrac{d}{ds}(\hat{T}\times\hat{N})=\dfrac{d\hat{T}}{ds}\times\hat{N}+\hat{T}\times\dfrac{d\hat{N}}{ds}=\hat{T}\times(-\kappa\hat{T})+\hat{T}\times(\tau\hat{B})=-\tau\hat{N}$

Note: $\hat{T}$, $\hat{N}$, *and* $\hat{B}$ are unit vectors.

Basic theorems of curvature and torsion:

$\kappa=\dfrac{|\vec{R}'\times\vec{R}''|}{|\vec{R}'|^3}$, $\tau=[\hat{T},\hat{N},\hat{N}']=\dfrac{1}{\kappa^2}[\vec{R}',\vec{R}'',\vec{R}''']$, **where** $[\vec{A},\vec{B},\vec{C}]\equiv\vec{A}\cdot(\vec{B}\times\vec{C})$

Eg. For $C:\vec{R}(t)=3\cos(t)\hat{x}+3\sin(t)\hat{y}+4t\hat{z}$ **, find** $\hat{T}$, $\hat{N}$, $\hat{B}$, ***κ*, *τ*, and *ρ*.**

(Sol.) $\hat{T}=\dfrac{\vec{v}}{|\vec{v}|}=-\dfrac{3}{5}\sin(t)\hat{x}+\dfrac{3}{5}\cos(t)\hat{y}+\dfrac{4}{5}\hat{z}$, $\kappa=\dfrac{|\vec{R}'\times\vec{R}''|}{|R'|^3}=\dfrac{3}{25}$, $\rho=\dfrac{1}{\kappa}=\dfrac{25}{3}$

$\hat{N}=\rho\dfrac{d\hat{T}}{ds}=\dfrac{\rho}{|\vec{v}(t)|}\cdot\dfrac{d\hat{T}}{dt}=-\cos(t)\hat{x}-\sin(t)\hat{y}$, $\hat{B}=\hat{T}\times\hat{N}=\dfrac{4}{5}\sin(t)\hat{x}-\dfrac{4}{5}\cos(t)\hat{y}+\dfrac{3}{5}\hat{z}$

$\dfrac{d\hat{B}}{ds}=\dfrac{1}{|\vec{v}(t)|}\dfrac{d\hat{B}}{dt}=\dfrac{4}{25}\cos(t)\hat{x}+\dfrac{4}{25}\sin(t)\hat{y}=-\tau\hat{N}=-\tau(-\cos(t)\hat{x}-\sin(t)\hat{y})$

$\Rightarrow \tau=\dfrac{4}{25}$

諾貝爾物理獎得主楊振寧作詩「贊陳氏級」稱讚數學大師/美國加州柏克萊大學教授陳省身對微分幾何學的貢獻：「天衣豈無縫，匠心剪接成，渾然歸一體，廣邃妙絕倫。造化愛幾何，四力纖維能，千古寸心事，歐高黎卡陳。」

7-3 Gradient, Divergence, and Curl in the Rectangular Coordinate System

Gradient $\nabla\phi(x,y,z)$ **:** $\nabla\phi(x,y,z)=\frac{\partial\phi(x,y,z)}{\partial x}\hat{x}+\frac{\partial\phi(x,y,z)}{\partial y}\hat{y}+\frac{\partial\phi(x,y,z)}{\partial z}\hat{z}$ is a vector function if $\phi(x,y,z)$ is a scalar function.

Eg. For $\phi(x,y,z)=e^{xyz}$**,** $\nabla\varphi(x,y,z)=yze^{xyz}\hat{x}+xze^{xyz}\hat{y}+xye^{xyz}\hat{z}$**.**

Divergence $\nabla\cdot\vec{F}(x,y,z)$ **:** $\nabla\cdot\vec{F}(x,y,z)=\frac{\partial F_x(x,y,z)}{\partial x}+\frac{\partial F_y(x,y,z)}{\partial y}+\frac{\partial F_z(x,y,z)}{\partial z}$ is a scalar function if $\vec{F}(x,y,z)=F_x(x,y,z)\hat{x}+F_y(x,y,z)\hat{y}+F_z(x,y,z)\hat{z}$ is a vector function.

Curl $\nabla\times\vec{F}(x,y,z)$**:**

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_x&F_y&F_z\end{vmatrix}=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{x}+\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{y}+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{z}$$

is a vector function if $\vec{F}(x,y,z)=F_x(x,y,z)\hat{x}+F_y(x,y,z)\hat{y}+F_z(x,y,z)\hat{z}$ is a vector function.

Eg. $\vec{F}=x\hat{x}+y\hat{y}+z\hat{z}$**, compute** $\nabla\cdot\vec{F}$ **and** $\nabla\times\vec{F}$**.**

(Sol.) $\nabla\cdot\vec{F}=1+1+1=3$, $\nabla\times\vec{F}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&y&z\end{vmatrix}=0\hat{x}+0\hat{y}+0\hat{z}=0$

Eg. $\vec{F}(x,y,z)=x^2\hat{x}-2x^2y\hat{y}+2yz^4\hat{z}$**, find** $\nabla\times\vec{F}$ **and** $\nabla\cdot\vec{F}$ **at (1,-1,1). [1991 中山電研]**

(Sol.) $\nabla\cdot\vec{F}=2x-2x^2+8yz^3,\ at\ (1,-1,1)\Rightarrow\nabla\cdot\vec{F}=-8$

$\nabla\times\vec{F}=2z^4\hat{x}-4xy\hat{z},\ at\ (1,-1,1)\Rightarrow\nabla\times\vec{F}=2\hat{x}+4\hat{z}$

Laplacian operator: $\nabla^2\phi=\nabla\cdot\nabla\phi=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})\phi$

Eg. For $\phi(x,y,z)=e^{xyz}$**,** $\nabla^2\phi(x,y,z)=y^2z^2e^{xyz}+x^2z^2e^{xyz}+x^2y^2e^{xyz}$**.**

Theorems ***(a)*** $\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$

(b) $\nabla\cdot(\nabla\times\vec{F})=0,\ \forall\vec{F}\in C^2$ ***(c)*** $\nabla\times(\nabla\phi)=0,\ \forall\phi\in C^2$

(d) $\nabla\times(\vec{F}\times\vec{G})=(\vec{G}\cdot\nabla)\vec{F}-(\vec{F}\cdot\nabla)\vec{G}+(\nabla\cdot\vec{G})\vec{F}-(\nabla\cdot\vec{F})\vec{G}$

(e) $\nabla(\vec{F}\cdot\vec{G})=(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$

(f) $\nabla\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G})$

(g) $\nabla\cdot(\phi\vec{F})=(\nabla\phi)\cdot\vec{F}+\phi(\nabla\cdot\vec{F})$

Theorem $\nabla\phi(x,y,z)\perp$ **the surface of** $\phi(x,y,z)$**=constant.**

(Proof) $\because$ $\phi(x,y,z)$=constant

$$\therefore\ d\phi(x,y,z)=0=\frac{\partial\phi(x,y,z)}{\partial x}dx+\frac{\partial\phi(x,y,z)}{\partial y}dy+\frac{\partial\phi(x,y,z)}{\partial z}dz$$

$$=\left(\frac{\partial\varphi}{\partial x}\hat{x}+\frac{\partial\varphi}{\partial y}\hat{y}+\frac{\partial\varphi}{\partial z}\hat{z}\right)\cdot(dx\hat{x}+dy\hat{y}+dz\hat{z})=\nabla\phi\cdot d\vec{R}$$

$\therefore$ $\nabla\phi(x,y,z)\perp d\vec{R}$, and $d\vec{R}$ is the tangential increment on the surface $\phi(x,y,z)$

Eg. Find the tangential plane and normal line to $z=x^2+y^2$ at (2,-2,8).

(Sol.) Let $\phi(x,y,z)=z-x^2-y^2$, $\nabla\varphi=-2x\hat{x}-2y\hat{y}+\hat{z}$, and (2,-2,8) is on the surface.

For $z-x^2-y^2=0$, the normal vector at (2,-2,8) is $-4\hat{x}+4\hat{y}+\hat{z}$.

The tangential plane at (2,-2,8) is $-4(x-2)+4(y+2)+(z-8)=0\Rightarrow -4x+4y+z=-8$

The normal line is $\dfrac{x-2}{-4}=\dfrac{y+2}{4}=\dfrac{z-8}{1}$.

Eg. Find a unit normal vector of $z^2=4(x^2+y^2)$ at (1,0,2). [1993 中山電研、2007 師大光電所]

Theorem $\nabla\cdot\vec{F}(x,y,z)$ **is the outward flux per unit volume of the flow at point (x,y,z) and time t.**

(Proof) $\vec{F}\ at\ P=F_x\hat{x}+F_y\hat{y}+F_z\hat{z}$

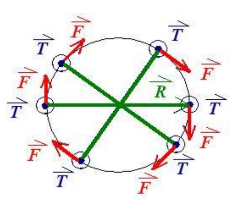
The x-direction flux $=\left(F_x+\dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\,\Delta z-\left(F_x-\dfrac{1}{2}\dfrac{\partial F_x}{\partial x}\Delta x\right)\Delta y\,\Delta z=\dfrac{\partial F_x}{\partial x}\Delta x\,\Delta y\,\Delta z$

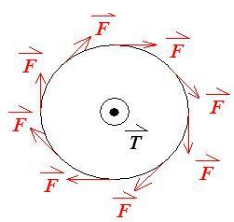
The y-direction flux $=\dfrac{\partial F_y}{\partial y}\Delta x\,\Delta y\,\Delta z$, and the z- direction flux $=\dfrac{\partial F_z}{\partial z}\Delta x\,\Delta y\,\Delta z$

$$\therefore\ \text{Total flux}=\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)\cdot\Delta x\,\Delta y\,\Delta z/\Delta x\,\Delta y\,\Delta z=\nabla\cdot\vec{F}$$

Theorem If $\vec{T}=\vec{F}\times\vec{R}$ **and** $\vec{R}=x\hat{x}+y\hat{y}+z\hat{z}$**, then** $\vec{F}=\dfrac{1}{2}\nabla\times\vec{T}$**.**

(Proof) $\nabla\times\vec{T}=\nabla\times(\vec{F}\times\vec{R})=\nabla\times\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\F_x&F_y&F_z\\x&y&z\end{vmatrix}$





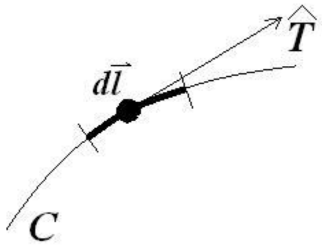
$$=\nabla\times[(F_yz-F_zy)\hat{x}+(F_zx-F_xz)\hat{y}+(F_xy-F_yx)\hat{z}]$$

$$=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\F_yz-F_zy&F_zx-F_xz&F_xy-F_yx\end{vmatrix}=2(F_x\hat{x}+F_y\hat{y}+F_z\hat{z})=2\vec{F}\Rightarrow\vec{F}=\frac{1}{2}\nabla\times\vec{T}$$

7-4 Line Integrals & Surface Integrals

Line integral: $\exists$ Curve C: $\vec{R}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$, $a \le t \le b$

$\exists$ Vector $\vec{F}(x,y,z) = F_x(x,y,z)\,\hat{x} + F_y(x,y,z)\hat{y} + F_z(x,y,z)\hat{z}$



(a) $\int_C \vec{F}\cdot d\vec{l} = \int_a^b \vec{F}[x(t),y(t),z(t)]\cdot\vec{R}'(t)dt$,

(b) $\int_C \vec{F}\times d\vec{l} = \int_a^b \vec{F}[x(t),y(t),z(t)]\times\vec{R}'(t)dt$,

(c) $\int_C f(x,y,z)dl = \int_a^b f[x(t),y(t),z(t)]\cdot|\vec{R}'(t)|\,dt$

Eg. For $C:\vec{R}(t) = x(t)\,\hat{x} + y(t)\hat{y} + z(t)\hat{z} = \cos(t)\hat{x} + \sin(t)\hat{y} + \hat{z}$, $0 \le t \le 2\pi$, **and** $\vec{F}(x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}$, **determine** $\int_C \vec{F}\cdot d\vec{l}$ **and** $\int_C \vec{F}\times d\vec{l}$ **.**

(Sol.) $C:\vec{R}(t) = \cos(t)\,\hat{x} + \sin(t)\,\hat{y} + \hat{z}$, $\vec{R}'(t) = -\sin(t)\,\hat{x} + \cos(t)\,\hat{y}$

$\vec{F}(x(t),y(t),z(t)) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z} = \cos(t)\,\hat{x} + \sin(t)\,\hat{y} + \hat{z}$

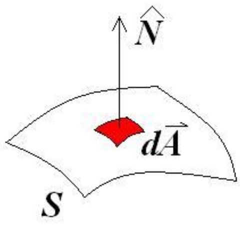
$\vec{F}\cdot d\vec{l} = \vec{F}(x(t),y(t),z(t))\cdot\vec{R}'(t)dt = (\cos(t)\,\hat{x} + \sin(t)\,\hat{y} + \hat{z})\cdot(-\sin(t)\,\hat{x} + \cos(t)\,\hat{y})dt$
$= [-\cos(t)\sin(t) + \sin(t)\cos(t)]dt = 0$

$$\vec{F}\times d\vec{l} = \vec{F}(x(t),y(t),z(t))\times\vec{R}'(t)dt = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos(t) & \sin(t) & 1 \\ -\sin(t) & \cos(t) & 0 \end{vmatrix} dt = [-\cos(t)\hat{x} - \sin(t)\hat{y} + \hat{z}]dt$$

$\therefore$ $\int_C \vec{F}\cdot d\vec{l} = 0$, $\int_C \vec{F}\times d\vec{l} = \int_0^{2\pi}[-\cos(t)\hat{x} - \sin(t)\hat{y} + \hat{z}]dt = 2\pi\hat{z}$

Surface integral: $\exists$ Surface S that has a projection on the xy-plane: $\phi(x,y,z) = z - \varphi(x,y) = 0$. $\vec{N} = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z} = -\frac{\partial\varphi}{\partial x}\hat{x} - \frac{\partial\varphi}{\partial y}\hat{y} + \hat{z}$ is a normal vector on S, and $\hat{N} = \vec{N}/|\vec{N}|$.

$\exists$ Vector $\vec{F}(x,y,z) = F_x(x,y,z)\,\hat{x} + F_y(x,y,z)\hat{y} + F_z(x,y,z)\hat{z}$



(a) $\iint_S \vec{F}\cdot d\vec{A} = \iint_D \vec{F}(x,y,z)\cdot\hat{N}\sqrt{1 + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

(b) $\iint_S \vec{F}\times d\vec{A} = \iint_D \vec{F}(x,y,z)\times\hat{N}\sqrt{1 + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

(c) $\iint_S f(x,y,z)dA = \iint_D f(x,y,z)\sqrt{1 + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}\,dxdy$,

Note: $d\vec{l} = \hat{T}dl$ is parallel to the tangential direction of the curve, but $d\vec{A} = \hat{N}dA$ is normal to the surface.

Eg. For $S: x^2+y^2+z^2=a^2, z\ge 0$ **, and** $\vec{F}(x,y,z)=x\hat{x}+y\hat{y}+z\hat{z}$ **, determine** $\iint_S \vec{F}\cdot d\vec{A}$ **and** $\iint_S \vec{F}\times d\vec{A}$**.**

(Sol.) $S:\ x^2+y^2+z^2=a^2 \wedge z\ge 0 \Rightarrow z=\sqrt{a^2-x^2-y^2}$, $\dfrac{\partial\varphi}{\partial x}=\dfrac{-x}{\sqrt{a^2-x^2-y^2}}=-\dfrac{x}{z}$

$\dfrac{\partial\varphi}{\partial y}=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}=-\dfrac{y}{z}\Rightarrow \vec{N}=-\dfrac{\partial\varphi}{\partial x}\hat{x}-\dfrac{\partial\varphi}{\partial y}\hat{y}+\hat{z}=\dfrac{x}{z}\hat{x}+\dfrac{y}{z}\hat{y}+\hat{z}$, $|\vec{N}|=\dfrac{a}{z}$

$\hat{N}=\dfrac{\vec{N}}{|\vec{N}|}=\dfrac{x\hat{x}+y\hat{y}+z\hat{z}}{a}$, $\vec{F}(x,y,z)=x\hat{x}+y\hat{y}+z\hat{z}$, and $\sqrt{1+\left(\dfrac{\partial\varphi}{\partial x}\right)^2+\left(\dfrac{\partial\varphi}{\partial y}\right)^2}=\dfrac{a}{z}$

$$\vec{F}\cdot d\vec{A}=\vec{F}\cdot\hat{N}\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}dxdy=(x\hat{x}+y\hat{y}+z\hat{z})\cdot(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{a})\frac{adxdy}{z}$$

$$=\frac{x^2+y^2+z^2}{z}dxdy=\frac{a^2dxdy}{\sqrt{a^2-x^2-y^2}}\Rightarrow$$

$$\iint_S \vec{F}\cdot d\vec{A}=\iint_{x^2+y^2\le a^2}\frac{a^2dxdy}{\sqrt{a^2-x^2-y^2}}=\int_0^{2\pi}\int_0^a\frac{a^2rdrd\theta}{\sqrt{a^2-r^2}}=2\pi a^2\cdot\int_0^a\frac{rdr}{\sqrt{a^2-r^2}}=\frac{2\pi a^2}{2}\cdot\int_0^1\frac{dr^2}{\sqrt{a^2-r^2}}$$

$$\underset{u=r^2}{\overset{\uparrow}{=}}\pi a^2\cdot\int_0^{a^2}\frac{du}{\sqrt{a^2-u}}=-2\pi a^2\cdot\sqrt{a^2-u}\,\Big|_{u=0}^{u=a^2}=2\pi a^3$$

$$\vec{F}\times d\vec{A}=\vec{F}\times\hat{N}\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}dxdy=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\ x&y&z\\ \frac{x}{a}&\frac{y}{a}&\frac{z}{a}\end{vmatrix}\frac{adxdy}{z}=0\ \Rightarrow\ \iint_S\vec{F}\times d\vec{A}=0$$

Eg. Compute the surface integral $\iint_S (x+y+z)dA$ **, where** $S: z=x+y, 0\le y\le x, 0\le x\le 1$**. [1990 台大農工研]**

(Sol.) $\sqrt{1+\left(\dfrac{\partial\varphi}{\partial x}\right)^2+\left(\dfrac{\partial\varphi}{\partial y}\right)^2}=\sqrt{3}$, $\iint_S(x+y+z)dA=\int_0^1\int_0^x 2(x+y)\cdot\sqrt{3}dydx=\sqrt{3}$

Eg. $f=\frac{1}{r}$, $r=\sqrt{x^2+y^2+z^2}$ **, find** $\iint_S \nabla f\cdot\hat{n}dA$ **for** S**:** $x^2+y^2+z^2=a^2$**. [1990 中山電研]**

(Sol.) $S:\ x^2+y^2+z^2=a^2 \Rightarrow z=\pm\sqrt{a^2-x^2-y^2}$

For $z=\sqrt{a^2-x^2-y^2} \Rightarrow \vec{N}=-\frac{\partial\varphi}{\partial x}\hat{x}-\frac{\partial\varphi}{\partial y}\hat{y}+\hat{z}=\frac{x}{z}\hat{x}+\frac{y}{z}\hat{y}+\hat{z}$,

$$\left|\vec{N}\right|=\frac{a}{z},\quad \hat{N}=\frac{\vec{N}}{|\vec{N}|}=\frac{1}{a}(x\hat{x}+y\hat{y}+z\hat{z}),\ \sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}=\frac{a}{z},$$

$$\nabla f=\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=\frac{-(x\hat{x}+y\hat{y}+z\hat{z})}{(\sqrt{x^2+y^2+z^2})^3}=\frac{-(x\hat{x}+y\hat{y}+z\hat{z})}{a^3}$$

$$\Rightarrow \nabla(f)\cdot\hat{n}dA=\nabla(f)\cdot\hat{N}\sqrt{1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2}dxdy=-\frac{x^2+y^2+z^2}{a^3z}dxdy=\frac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}}$$

$$\iint_{z=\sqrt{a^2-x^2-y^2}}\nabla\ \ f\cdot\hat{n}dA=\iint_{x^2+y^2\le a^2}\frac{-dxdy}{a\cdot\sqrt{a^2-x^2-y^2}}=\frac{-1}{a}\int_0^{2\pi}\int_0^a\frac{rdrd\theta}{\sqrt{a^2-r^2}}=\frac{-2\pi}{a}\int_0^a\frac{rdr}{\sqrt{a^2-r^2}}$$

$=-2\pi$. Similarly, $\iint_{z=-\sqrt{a^2-x^2-y^2}}\nabla\ \ f\cdot\hat{n}dA=-2\pi$ for $z=-\sqrt{a^2-x^2-y^2}$

$\therefore\ \iint_S\nabla f\cdot\hat{n}dA=(-2\pi)+(-2\pi)=-4\pi$

Green's theorem Let *C* be a regular, closed, positively-oriented curve enclosing a region *D*, $\vec{F}(x,y)=F_x(x,y)\hat{x}+F_y(x,y)\hat{y}$ **.**

$$\oint_C\vec{F}\cdot d\vec{R}=\oint_C F_x(x,y)dx+F_y(x,y)dy=\iint_D\left[\frac{\partial F_y(x,y)}{\partial x}-\frac{\partial F_x(x,y)}{\partial y}\right]dxdy$$

Eg. $\hat{F}=\frac{-y\hat{x}+x\hat{y}}{x^2+y^2}$ **, find** $\oint_C\vec{F}\cdot d\vec{R}$**, where *C* is any closed curve.**

(Sol.) $\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}=0,\ \forall(x,y)\neq 0$, $\oint_C\vec{F}\cdot d\vec{R}=0$ if C does not enclose 0.

Else, $\oint\vec{F}\cdot d\vec{r}=2\pi$

Stokes's theorem Let *S* be a regular surface with coherently oriented boundary *C*, $\iint_S(\nabla\times\vec{F})\cdot d\vec{A}=\oint_C\vec{F}\cdot d\vec{\ell}$ **.**

Divergence theorem **Let S be a regular, positive-oriented closed surface, enclosing a region V,** $\oiint_S \vec{F}\cdot d\vec{A} = \iiint_V (\nabla\cdot\vec{F})dV$ **.**

Eg. Compute $\oiint_S \vec{F}\cdot d\vec{A}$ **, where** $\vec{F} = (y^2+z^2)^{\frac{2}{3}}\hat{x} + \sin(x^2+z)\hat{y} + e^{x^2-y^2}\hat{z}$ **,**

$S:\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$**. [1990 成大工程科學所]**

(Sol.) $\nabla\cdot\vec{F}=0 \Rightarrow \oiint_S \vec{F}\cdot d\vec{A} = \iiint_V (\nabla\cdot\vec{F})dxdydz = 0$

Eg. $f=\dfrac{1}{r}$, $r=\sqrt{x^2+y^2+z^2}$ **, find** $\iint_S \nabla f\cdot\hat{n}dA$ **for S: $x^2+y^2+z^2=a^2$. [1990 中山電研]**

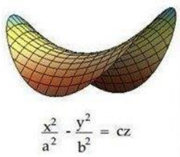
(Sol.) $\vec{F}=\nabla f=\nabla(\dfrac{1}{\sqrt{x^2+y^2+z^2}})=\dfrac{-1}{(\sqrt{x^2+y^2+z^2})^3}(x\hat{x}+y\hat{y}+z\hat{z})=\dfrac{-1}{a^3}(x\hat{x}+y\hat{y}+z\hat{z})$,

$\nabla\cdot\vec{F}=\dfrac{-3}{a^3}$, $\iint_S \nabla f\cdot\hat{n}dA = \oiint_S \vec{F}\cdot d\vec{A} = \iiint_V (\nabla\cdot\vec{F})dxdydz = \dfrac{-3}{a^3}\cdot\dfrac{4a^3\pi}{3} = -4\pi$

Green's identities

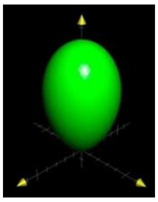
$\phi,\ \psi$ are scalars, $\begin{cases}(a)\iiint_V [\phi\nabla^2\psi+\nabla\phi\cdot\nabla\psi]dxdydz = \iint_S (\phi\nabla\psi)\cdot d\vec{A} \\ (b)\iiint_V [\phi\nabla^2\psi-\psi\nabla^2\phi]dxdydz = \iint_S [\phi\nabla\psi-\psi\nabla\phi]\cdot d\vec{A}\end{cases}$.

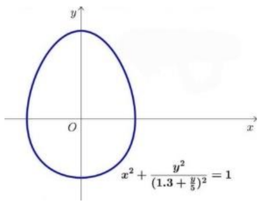
Hyperbolic Paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=cz$



Egg Equations

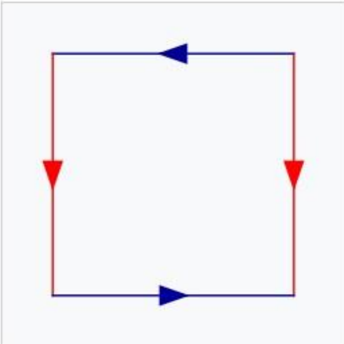
$x^2+y^2+z^2=z^{1.5}$



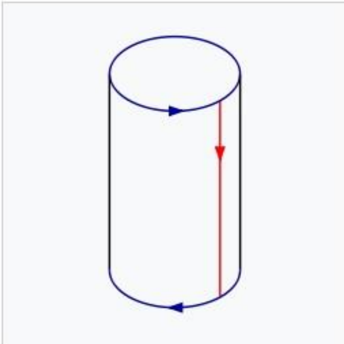


Mobius Belt/Strip

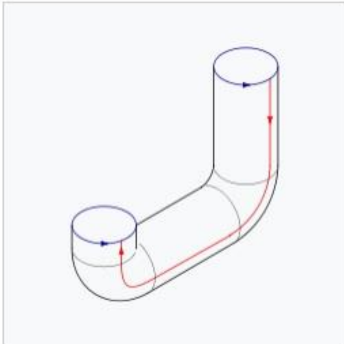
Klein Bottle



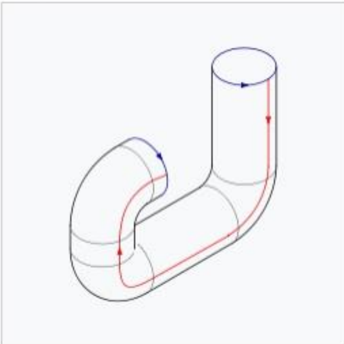
Schritt 1



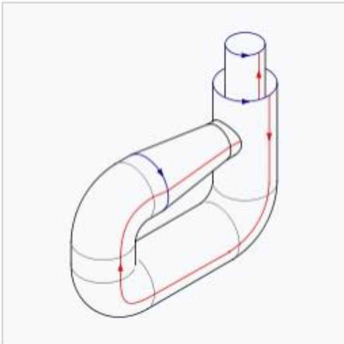
Schritt 2



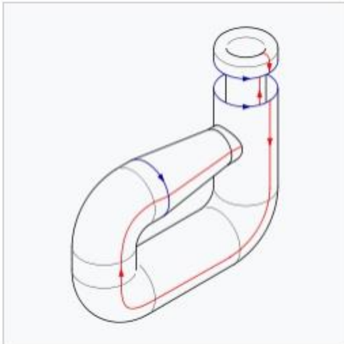
Schritt 3



Schritt 4



Schritt 5



Schritt 6

7-5 Potential Theory

Potential: ϕ is called a potential for the vector field $\vec{F}$ if $\vec{F}=\nabla\phi$ or $\vec{F}=-\nabla\phi$

Test for a potential: If $\vec{F}$ and $\nabla\cdot\vec{F}$ are continuous in a simply-connected domain Ω, then $\vec{F}$ has a potential function. $\Leftrightarrow \nabla\times\vec{F}=0$

(Proof) 〝$\Rightarrow$〞: $\vec{F}=\pm\nabla\phi \Rightarrow \nabla\times\vec{F}=0$.

〝$\Leftarrow$〞: $\nabla\times\vec{F}=0$, $\oint_C \vec{F}\cdot d\vec{\ell}=\iint_S(\nabla\times\vec{F})\cdot d\vec{A}=0$

Let C be the boundary of ϕ (x,y,z)=constant, then we choose $\vec{F}=\pm\nabla\phi$, then $\int_C \vec{F}\cdot d\vec{\ell}=0$ is always valid. $\Rightarrow \vec{F}$ has a potential function.

Eg. Check (a) $\vec{F}=2xy\hat{x}+z^2\hat{y}+(x-y+z)\hat{z}$,

and (b) $\vec{F}=(yze^{xyz}-4x)\hat{x}+(xze^{xyz}+z)\hat{y}+(xye^{xyz}+y)\hat{z}$, **which does have a potential? [1991 成大機研]**

(Sol.)

(a) $$\nabla\times\vec{F}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy & z^2 & x-y+z\end{vmatrix}=(-2z-1)\hat{x}-\hat{y}-2x\hat{z}\neq 0, \quad \therefore \text{ no potential.}$$

(b)

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yze^{xyz}-4x & xze^{xyz}+z & xye^{xyz}+y\end{vmatrix}=(1+xe^{xyz}+x^2yze^{xyz}-1-xe^{xyz}-x^2yze^{xyz})\hat{x}$$

$$+(ye^{xyz}+xy^2ze^{xyz}-ye^{xyz}-xy^2ze^{xyz})\hat{y}+(ze^{xyz}+xyz^2-ze^{xyz}-xyz^2)\hat{z}=0,$$

$\therefore$ there exists a potential.

$$\vec{F}=\nabla\phi=\frac{\partial\phi}{\partial x}\hat{x}+\frac{\partial\phi}{\partial y}\hat{y}+\frac{\partial\phi}{\partial z}\hat{z}=(yze^{xyz}-4x)\hat{x}+(xze^{xyz}+z)\hat{y}+(xye^{xyz}+y)\hat{z}$$

$$\frac{\partial\phi}{\partial x}=yze^{xyz}-4x\Rightarrow\phi=\int(yze^{xyz}-4x)dx=e^{xyz}-2x^2+I(y,z)$$

$$\frac{\partial\phi}{\partial y}=xze^{xyz}+z\Rightarrow\phi=\int(xze^{xyz}+z)dy=e^{xyz}+yz+J(x,z)$$

$$\frac{\partial\phi}{\partial z}=xye^{xyz}+y\Rightarrow\phi=\int(xye^{xyz}+y)dz=e^{xyz}+yz+K(x,y),\ \therefore\phi=e^{xyz}-2x^2+yz+C$$

Eg. $\vec{F}=(x^2+y^2+z^2)^n(x\hat{x}+y\hat{y}+z\hat{z})$ **, find a scalar potential ϕ (*x*,*y*,*z*) so that** $\vec{F}=-\nabla\phi$. **[1990 台大材研]** (Ans.) $\phi(x,y,z)=\dfrac{-(x^2+y^2+z^2)^{n+1}}{2(n+1)}+C$

Theorem If $\vec{F}$ has a potential, then the line integral of $\vec{F}$ is independent of path in Ω. That is, $\int_C \vec{F}\cdot d\vec{\ell}=\int_K \vec{F}\cdot d\vec{\ell}$**, whenever *C* and *K* are regular curves in Ω with the same initial point and the same terminal point.**

(Proof) $\int_C \vec{F}\cdot d\vec{\ell}-\int_K \vec{F}\cdot d\vec{\ell}=\oint_{C'} \vec{F}\cdot d\vec{\ell}$

If $\vec{F}$ has a potential, then $\oint_{C'} \vec{F}\cdot d\vec{\ell}=0$, $\therefore$ $\int_C \vec{F}\cdot d\vec{\ell}=\int_K \vec{F}\cdot d\vec{\ell}$.

Theorem Let $\vec{F}$ be a 2-D vector field of a simply-connected domain Ω. Then $\vec{F}$ has a potential on Ω. $\Leftrightarrow \dfrac{\partial F_y}{\partial x}=\dfrac{\partial F_x}{\partial y}$. (In this case, $\vec{F}=F_x(x,y)\hat{x}+F_y(x,y)\hat{y}$)

(Proof) If $\vec{F}=\nabla\phi$, then $F_x=\dfrac{\partial\phi(x,y)}{\partial x}$, $F_y=\dfrac{\partial\phi(x,y)}{\partial y}$, $\dfrac{\partial F_x}{\partial y}=\dfrac{\partial^2\phi}{\partial y\partial x}=\dfrac{\partial^2\phi}{\partial x\partial y}=\dfrac{\partial F_y}{\partial x}$

By Green's theorem, $\oint_C F_x dx+F_y dy=\iint_D\left(\dfrac{\partial F_y}{\partial x}-\dfrac{\partial F_x}{\partial y}\right)dxdy=0$, $\therefore$ $\vec{F}$ is conservative on Ω.

Eg. Evaluate $\int_\Gamma \vec{v}\cdot d\vec{R}$ **, where** $\vec{v}=2x^2\hat{x}-2yz\hat{y}-(y^2+3)\hat{z}$ **and Γ is some complicated path from (0,0,0) to (0,0,4). [2016 成大電研]**

(Sol.) $\nabla\times\vec{v}=0$, $\therefore$ $\vec{v}$ is conservative. , $\therefore \exists\phi(x,y,z)$ such that $\vec{v}=\nabla\phi$

$$\frac{\partial\phi}{\partial x}=2x^2\Rightarrow\phi=\int 2x^2dx=\frac{2x^3}{3}+I(y,z)$$

$$\frac{\partial\phi}{\partial y}=-2yz\Rightarrow\phi=\int -2yzdy=-y^2z+J(x,z)$$

$$\frac{\partial\phi}{\partial z}=-y^2-3\Rightarrow\phi=\int(-y^2-3)dz=-y^2z-3z+K(x,y)$$

$$\therefore\ \phi=-y^2z-3z+\frac{2x^3}{3}+C\Rightarrow\int\vec{v}\cdot d\vec{r}=\phi(x,y,z)\Big|_{(0,0,0)}^{(0,0,4)}=-12$$

Eg. (a) Is $\vec{F}(x,y,z)=-\frac{y}{z}\hat{x}-\frac{x}{z}\hat{y}+\frac{xy}{z^2}\hat{z}$ conservative in the region *z*>0? (b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=-\frac{y}{z}\hat{x}-\frac{x}{z}\hat{y}+\frac{xy}{z^2}\hat{z}$, *C* is a piecewisely smooth curve from (1,1,1) to (2,-1,3) and not crossing the *xy*-plane. [1990 成大化工所]

(Sol.) (a) $\nabla\times\vec{F}=0$ in the region *z*>0, $\therefore$ $\vec{F}$ is conservative.

(b) $\nabla\times\vec{F}=0$, $\therefore \exists\phi(x,y,z)$ such that $\vec{F}=\nabla\phi$

$$\frac{\partial\phi}{\partial x}=-\frac{y}{z}\Rightarrow\phi=\int-\frac{y}{z}dx=\frac{-xy}{z}+I(y,z)$$

$$\frac{\partial\phi}{\partial y}=-\frac{x}{z}\Rightarrow\phi=\int-\frac{x}{z}dy=\frac{-xy}{z}+J(x,z)$$

$$\frac{\partial\phi}{\partial z}=\frac{xy}{z^2}\Rightarrow\phi=\int\frac{xy}{z^2}dz=\frac{-xy}{z}+K(x,y)$$

$$\Rightarrow\phi(x,y,z)=-\frac{xy}{z}+C\Rightarrow\int\vec{F}\cdot d\vec{r}=\phi(x,y,z)\Big|_{(1,1,1)}^{(2,-1,3)}=\frac{5}{3}$$

7-6 Curvilinear Coordinates

Coordinate transformation: $\begin{cases} x = x(q_1, q_2, q_3) \\ y = y(q_1, q_2, q_3) \\ z = z(q_1, q_2, q_3) \end{cases} \Leftrightarrow \begin{cases} q_1 = q_1(x, y, z) \\ q_2 = q_2(x, y, z) \\ q_3 = q_3(x, y, z) \end{cases}$

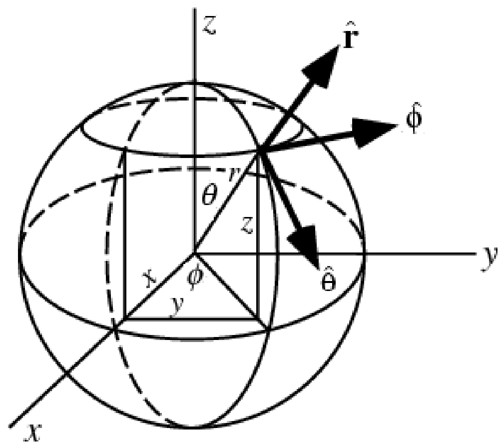
Scalar factors: If (q_1, q_2, q_3) are orthogonal system, then

$$h_{ij} = \begin{cases} 0, i \neq j \\ \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right]^{1/2}, i = j \end{cases}$$

Eg. Rectangular coordinate
$\Leftrightarrow$ Spherical coordinate

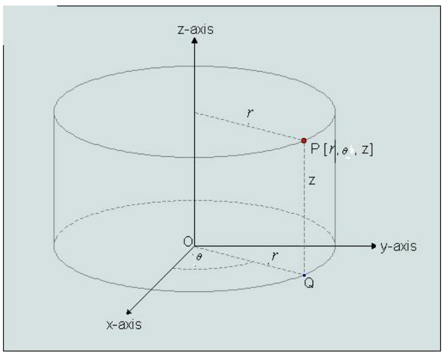
$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \sin^{-1}\left(\sqrt{x^2 + y^2} \Big/ \sqrt{x^2 + y^2 + z^2}\right) \\ \phi = \cos^{-1}\left(x \Big/ \sqrt{x^2 + y^2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\ h_\phi = h_{33} = \left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2\right]^{1/2} = r\sin\theta \end{cases}$$



Differential length vector in the spherical coordinate:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r\sin\theta d\phi$$



Eg. Rectangular coordinate $\Leftrightarrow$ Cylindrical coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

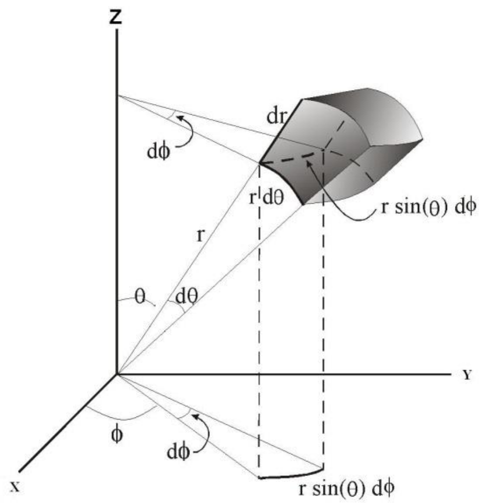
$$\Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\dfrac{\partial x}{\partial r}\right)^2 + \left(\dfrac{\partial y}{\partial r}\right)^2 + \left(\dfrac{\partial z}{\partial r}\right)^2\right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\dfrac{\partial x}{\partial \theta}\right)^2 + \left(\dfrac{\partial y}{\partial \theta}\right)^2 + \left(\dfrac{\partial z}{\partial \theta}\right)^2\right]^{1/2} = r \\ h_z = h_{33} = \left[\left(\dfrac{\partial x}{\partial z}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + \left(\dfrac{\partial z}{\partial z}\right)^2\right]^{1/2} = 1 \end{cases}$$

Differential length vector in the cylindrical coordinate: $d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_z dz$

Differential arc length: $ds = [(h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2]^{1/2}$

Differential elements of area on the $q_i q_j$-plane: $dA_{ij} = ds_i\ ds_j = h_i h_j dq_i dq_j$

Differential element of volume: $dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$



Eg. Spherical coordinate:

$$\begin{cases} ds = \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]^{1/2} \\ dA_{\theta,\phi} = r^2 \sin\theta d\theta d\phi, dA_{r,\theta} = r dr d\theta, etc. \\ dV = r^2 \sin\theta dr d\theta d\phi \end{cases}$$

Eg. Cylindrical coordinate:
$$\begin{cases} ds = \left[dr^2 + r^2 d\theta^2 + dz^2\right]^{1/2} \\ dA_{\theta,z} = r d\theta dz, dA_{r,\theta} = r dr d\theta, dA_{r,z} = dr dz. \\ dV = r dr d\theta dz \end{cases}$$

Jacobian determinants: $(p_1,p_2,p_3) \Leftrightarrow (q_1,q_2,q_3)$

$$dp_i dp_j = \left| \det \begin{bmatrix} \partial p_i / \partial q_i & \partial p_j / \partial q_i \\ \partial p_i / \partial q_j & \partial p_j / \partial q_j \end{bmatrix} \right| \cdot dq_i dq_j$$

and $dp_1dp_2dp_3 = \left|\det\begin{bmatrix} \partial p_1/\partial q_1 & \partial p_2/\partial q_1 & \partial p_3/\partial q_1 \\ \partial p_1/\partial q_2 & \partial p_2/\partial q_2 & \partial p_3/\partial q_2 \\ \partial p_1/\partial q_3 & \partial p_2/\partial q_3 & \partial p_3/\partial q_2 \end{bmatrix}\right| \cdot dq_1dq_2dq_3$

Eg. Let *I*= $\iiint_V f(x,y,z)dxdydz$ **. Transform the integral from (*x*,*y*,*z*) into (*r*,*θ*,*φ*).**

$$|J| = \left|\det\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{bmatrix}\right| = \left|r^2\sin\theta\cos^2\theta\cos^2\phi\right.$$

$$\left.+r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi + r^2\sin^3\theta\cos^2\phi\right| = \left|r^2\sin\theta\cos^2\theta + r^2\sin^3\theta\right| = r^2\sin\theta = h_1h_2h_3$$

$$\therefore\ I = \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2\sin\theta dr d\theta d\phi$$

Del operators in (q_1,q_2,q_3) coordinate system:

$$\nabla\psi(q_1,q_2,q_3) = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial q_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\hat{u}_3$$

$$\nabla\cdot\vec{F}(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}(F_1h_2h_3) + \frac{\partial}{\partial q_2}(F_2h_1h_3) + \frac{\partial}{\partial q_3}(F_3h_1h_2)\right)$$

$$\nabla\times\vec{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix}$$

$$\nabla^2\psi(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right],$$

where $\vec{F} = F_1\hat{u}_1 + F_2\hat{u}_2 + F_3\hat{u}_3$

Eg. Spherical coordinate system: h_r=1, h_θ=*r*, h_φ=*r*sin*θ*

$$\vec{F} = F_r\hat{a}_r + F_\theta\hat{a}_\theta + F_\phi\hat{a}_\phi$$

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\sin\theta\cdot F_\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(F_\phi)$$

$$\nabla\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ F_r & rF_\theta & r\sin\theta\cdot F_\phi \end{vmatrix},\quad \nabla\psi = \frac{\partial\psi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{a}_\phi$$

$$\nabla^2\psi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2}$$

Eg. Cylindrical coordinate system: h_r=1, h_θ=r, h_z=1

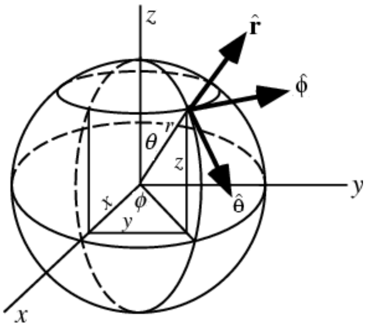
$$\vec{F} = F_r\hat{a}_r + F_\theta\hat{a}_\theta + F_z\hat{a}_z$$

$$\nabla\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{a}_r + \left(\frac{\partial F_\theta}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{a}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right)\hat{a}_z$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{a}_\theta + \frac{\partial\psi}{\partial z}\hat{a}_z,\quad \nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Unit vector conversions between distinct coordinates:



Eg. Rectangular coordinate $\Leftrightarrow$ Spherical coordinate

(Proof) $\hat{x}\cdot\hat{r} = \sin\theta\cos\phi$, $\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$,

$\hat{z}\cdot\hat{r} = \cos\theta$, $\therefore$ $\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

Similarly, $\hat{x}\cdot\hat{\theta} = \cos\theta\cos\phi$, $\hat{y}\cdot\hat{\theta} = \cos\theta\sin\phi$,

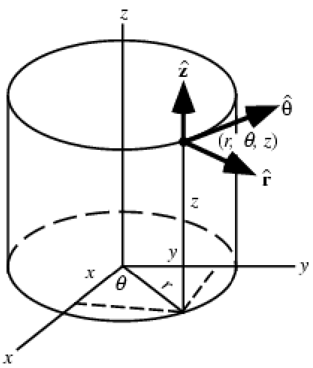
$\hat{z}\cdot\hat{\theta} = -\sin\theta$, $\therefore$ $\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$,

And $\hat{x}\cdot\hat{\phi} = -\sin\phi$, $\hat{y}\cdot\hat{\phi} = \cos\phi$, $\hat{z}\cdot\hat{\phi} = 0$,

$\therefore$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\cdot\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix},$$

or $$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\cdot\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix}$$



Eg. Rectangular coordinate $\Leftrightarrow$ Cylindrical coordinate

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$$