

Chapter 7 Vector Analysis

7-1 Vector Functions

One-variable vector function: $\vec{R}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$

Multi-variable vector function: $\vec{F}(x, y, z) = F_x(x, y, z)\hat{x} + F_y(x, y, z)\hat{y} + F_z(x, y, z)\hat{z}$

The derivatives of vector functions: $\frac{d\vec{R}(t)}{df(t)} = \frac{d\vec{R}(t)}{dt} \cdot \frac{dt}{df(t)} = \frac{d\vec{R}(t)}{dt} / \frac{df(t)}{dt}$

$$\frac{\partial \vec{F}(x, y, z)}{\partial g(x, y, z)} = \frac{\partial \vec{F}(x, y, z)}{\partial x} \cdot \frac{\partial x}{\partial g(x, y, z)} + \frac{\partial \vec{F}(x, y, z)}{\partial y} \cdot \frac{\partial y}{\partial g(x, y, z)} + \frac{\partial \vec{F}(x, y, z)}{\partial z} \cdot \frac{\partial z}{\partial g(x, y, z)}$$

Eg. For $\vec{R}(t) = 2t\hat{x} - \cos(3t)\hat{y} + t^3\hat{z}$, $0 \leq t \leq 1$, **find** $\vec{R}'(t) = d\vec{R}(t)/dt$. **Let** $s(t) = \int_0^1 \sqrt{4 + 9\sin^2(3t) + 9t^4} \cdot dt$, **then find** $d\vec{R}(t)/ds(t)$.

(Sol) $d\vec{R}(t)/dt = 2\hat{x} + 3\sin(3t)\hat{y} + 3t^2\hat{z}$

$$\therefore \frac{d\vec{R}(t)}{ds(t)} = \frac{d\vec{R}(t)}{dt} \cdot \frac{dt}{ds(t)} = \frac{\left[\frac{d\vec{R}(t)}{dt} \right]}{\left[\frac{ds(t)}{dt} \right]} = \frac{2\hat{x} + 3\sin(3t)\hat{y} + 3t^2\hat{z}}{\sqrt{4 + 9\sin^2(3t) + 9t^4}}$$

Some theorems of derivatives of vector functions:

1. $(\vec{F} \cdot \vec{G})' = \vec{F}' \cdot \vec{G} + \vec{F} \cdot \vec{G}'$

2. $(\vec{F} \times \vec{G})' = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$

3. $(\vec{F} \times \vec{F}')' = \vec{F} \times \vec{F}''$

(Proof) $(\vec{F} \times \vec{F}')' = \vec{F}' \times \vec{F}' + \vec{F} \times \vec{F}'' = \vec{F} \times \vec{F}''$

4. **For** $\vec{R}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$, **if** $\vec{R}(t)$ **does not change direction, then** $\vec{R}(t) \times \vec{R}'(t) = 0$, **and vice versa.**

5. **Let** $\vec{R}(t)$ **denote the position of a particle at time** t . **If the particle moves so that equal areas are swept out in equal times, then we have** $\vec{R}(t) \times \vec{R}''(t) = 0$, **and vice versa. (Kepler's law)**

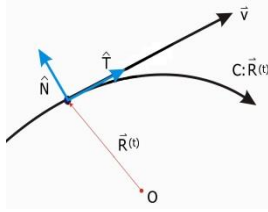
(Proof) $2 \text{ area} = \vec{R}(t) \times [\vec{R}(t + \Delta t) - \vec{R}(t)] \approx \vec{R}(t) \times \vec{R}'(t) \cdot \Delta t$

If $\text{area} = 0 \Leftrightarrow \vec{R}(t) \times \vec{R}'(t) \cdot \Delta t = 0 \Leftrightarrow \vec{R}(t) \times \vec{R}'(t) = 0$

Equal area in equal time

$$\Leftrightarrow \vec{R}(t) \times \vec{R}'(t) = \text{constant} \Leftrightarrow [\vec{R}(t) \times \vec{R}'(t)]' = 0 \Leftrightarrow \vec{R}(t) \times \vec{R}''(t) = 0$$

7-2 Differential Geometry



Position vector: $\vec{R}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$

Velocity: $\vec{v}(t) = \vec{R}'(t) = \frac{dx(t)}{dt}\hat{x} + \frac{dy(t)}{dt}\hat{y} + \frac{dz(t)}{dt}\hat{z}$

Arc length: $s(t) = \int_{t_1}^{t_2} |\vec{R}'(t)| dt$, $\frac{ds}{dt} = |\vec{R}'(t)| = |\vec{v}(t)|$

Acceleration: $\vec{a}(t) = \vec{v}'(t) = \frac{d^2x(t)}{dt^2}\hat{x} + \frac{d^2y(t)}{dt^2}\hat{y} + \frac{d^2z(t)}{dt^2}\hat{z}$

Curvature: $\kappa = \left| \frac{d\hat{T}}{ds} \right| = \left| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right| = \left| \frac{d\hat{T}}{dt} \right| \cdot \frac{1}{|\vec{v}|}$, where $\hat{T} = \frac{\vec{R}'(t)}{|\vec{R}'(t)|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{R}(t)}{ds(t)}$

Ex. C: $\vec{R}(t) = t\hat{x} + (t-2)\hat{y} + (3t-1)\hat{z}$ is a straight line.

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{x} + \hat{y} + 3\hat{z}}{\sqrt{11}}, \quad \kappa = \left| \frac{d\hat{T}}{ds} \right| = \left| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right| = 0.$$

Ex. C: $\vec{R}(t) = 2\cos(t)\hat{x} + 2\sin(t)\hat{y} + 4\hat{z}$ is a circle of radius 2 at $z=4$.

$$\hat{T} = \frac{-2\sin(t)\hat{x} + 2\cos(t)\hat{y}}{2}, \quad \kappa = \left| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right| = \frac{1}{2} = \frac{1}{r}.$$

Ex. Consider the curve: $\vec{r} = a\cos(t)\hat{x} + a\sin(t)\hat{y} + bt\hat{z}$, $0 \leq t \leq 2\pi$. What is the equation of tangential vector at $t=\pi/2$. [1991 中山機研]

(Sol.) $\vec{v} = \frac{d\vec{r}}{dt} = -a\sin(t)\hat{x} + a\cos(t)\hat{y} + b\hat{z}$. At $t = \frac{\pi}{2} \Rightarrow \vec{v} = -a\hat{x} + b\hat{z}$

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-a\hat{x} + b\hat{z}}{\sqrt{a^2 + b^2}}$$

Theorem $\vec{a} = \frac{d|\vec{v}|}{dt}\hat{T} + \frac{|\vec{v}|^2}{\rho}\hat{N}$ = tangential acceleration + centripetal acceleration

(Proof) Define $\hat{N} = \rho \frac{d\hat{T}}{ds}$. It is easily shown that $\hat{N} \perp \hat{T}$ because we have the

following proof: $\hat{T} \cdot \hat{T} = 1$, $\frac{d(\hat{T} \cdot \hat{T})}{ds} = 0$, $\hat{T} \cdot \frac{d\hat{T}}{ds} + \frac{d\hat{T}}{ds} \cdot \hat{T} = 2\hat{T} \cdot \frac{d\hat{T}}{ds} = 0$, $\hat{T} \cdot \rho \frac{d\hat{T}}{ds} = 0 \Rightarrow \hat{N} \perp \hat{T}$

$$\begin{aligned} \vec{a}(t) &= \frac{d\vec{v}(t)}{dt} = \frac{d}{dt} [|\vec{v}(t)| \hat{T}] = \frac{d|\vec{v}(t)|}{dt} \cdot \hat{T} + |\vec{v}(t)| \cdot \frac{d\hat{T}}{dt} \\ &= \frac{d|\vec{v}(t)|}{dt} \cdot \hat{T} + |\vec{v}(t)| \cdot \left(\frac{ds}{dt} \cdot \frac{d\hat{T}}{ds} \right) = \frac{d|\vec{v}(t)|}{dt} \cdot \hat{T} + |\vec{v}(t)|^2 \frac{d\hat{T}}{ds} = \frac{d|\vec{v}(t)|}{dt} \hat{T} + \frac{|\vec{v}(t)|^2}{\rho} \hat{N}, \end{aligned}$$

Eg. For $\vec{R}(t) = [\cos(t) + t \sin(t)]\hat{x} + [\sin(t) - t \cos(t)]\hat{y} + t^2\hat{z}$, $t > 0$, **we have**

$$\vec{v}(t) = t \cos(t)\hat{x} + t \sin(t)\hat{y} + 2t\hat{z}, \quad \vec{a}(t) = [\cos(t) - t \sin(t)]\hat{x} + [\sin(t) + t \cos(t)]\hat{y} + 2\hat{z}$$

$$\hat{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{\sqrt{5}}\cos(t)\hat{x} + \frac{1}{\sqrt{5}}\sin(t)\hat{y} + \frac{2}{\sqrt{5}}\hat{z}, \quad \rho = \frac{1}{\kappa} = \frac{1}{\left| \frac{d\hat{T}}{dt} \cdot \frac{dt}{ds} \right|} = \frac{1}{\left| \frac{d\hat{T}}{dt} \cdot \frac{1}{|\vec{v}(t)|} \right|} = 5t$$

$$\hat{N} = \rho \frac{d\hat{T}}{ds} = \rho \frac{dt}{ds} \cdot \frac{d\hat{T}}{dt} = \frac{\rho}{|\vec{v}(t)|} \cdot \frac{d\hat{T}}{dt} = -\sin(t)\hat{x} + \cos(t)\hat{y}$$

Binormal vector: $\hat{B} = \hat{T} \times \hat{N}$

$$\text{Fernet formulae: } \begin{cases} \frac{d\hat{T}}{ds} = \kappa\hat{N} = \frac{\hat{N}}{\rho} & (1) \\ \frac{d\hat{N}}{ds} = -\kappa\hat{T} + \tau\hat{B} & (2) \\ \frac{d\hat{B}}{ds} = -\tau\hat{N} & (3) \end{cases}$$

Torsion of a curve: τ

$$\text{(Proof of (3)) } \frac{d\hat{B}}{ds} = \frac{d}{ds}(\hat{T} \times \hat{N}) = \frac{d\hat{T}}{ds} \times \hat{N} + \hat{T} \times \frac{d\hat{N}}{ds} = \hat{T} \times (-\kappa\hat{T}) + \hat{T} \times (\tau\hat{B}) = -\tau\hat{N}$$

Note: \hat{T} , \hat{N} , and \hat{B} are unit vectors.

Basic theorems of curvature and torsion:

$$\kappa = \frac{|\vec{R}' \times \vec{R}''|}{|\vec{R}'|^3}, \quad \tau = [\hat{T}, \hat{N}, \hat{N}'] = \frac{1}{\kappa^2} [\vec{R}', \vec{R}'', \vec{R}'''], \quad \text{where } [\vec{A}, \vec{B}, \vec{C}] \equiv \vec{A} \cdot (\vec{B} \times \vec{C})$$

Eg. For $C: \vec{R}(t) = 3\cos(t)\hat{x} + 3\sin(t)\hat{y} + 4t\hat{z}$, **find** \hat{T} , \hat{N} , \hat{B} , κ , τ , and ρ .

$$\text{(Sol.) } \hat{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{3}{5}\sin(t)\hat{x} + \frac{3}{5}\cos(t)\hat{y} + \frac{4}{5}\hat{z}, \quad \kappa = \frac{|\vec{R}' \times \vec{R}''|}{|\vec{R}'|^3} = \frac{3}{25}, \quad \rho = \frac{1}{\kappa} = \frac{25}{3}$$

$$\hat{N} = \rho \frac{d\hat{T}}{ds} = \frac{\rho}{|\vec{v}(t)|} \cdot \frac{d\hat{T}}{dt} = -\cos(t)\hat{x} - \sin(t)\hat{y}, \quad \hat{B} = \hat{T} \times \hat{N} = \frac{4}{5}\sin(t)\hat{x} - \frac{4}{5}\cos(t)\hat{y} + \frac{3}{5}\hat{z}$$

$$\frac{d\hat{B}}{ds} = \frac{1}{|\vec{v}(t)|} \frac{d\hat{B}}{dt} = \frac{4}{25}\cos(t)\hat{x} + \frac{4}{25}\sin(t)\hat{y} = -\tau\hat{N} = -\tau(-\cos(t)\hat{x} - \sin(t)\hat{y})$$

$$\Rightarrow \tau = \frac{4}{25}$$

諾貝爾物理獎得主楊振寧作詩「贊陳氏級」稱讚數學大師/美國加州柏克萊大學教授陳省身對微分幾何學的貢獻：「天衣豈無縫，匠心剪接成，渾然歸一體，廣邃妙絕倫。造化愛幾何，四力纖維能，千古寸心事，歐高黎卡陳。」

7-3 Gradient, Divergence, and Curl in the Rectangular Coordinate System

Gradient $\nabla\phi(x, y, z)$: $\nabla\phi(x, y, z) = \frac{\partial\phi(x, y, z)}{\partial x}\hat{x} + \frac{\partial\phi(x, y, z)}{\partial y}\hat{y} + \frac{\partial\phi(x, y, z)}{\partial z}\hat{z}$ is a vector function if $\phi(x, y, z)$ is a scalar function.

Eg. For $\phi(x, y, z) = e^{xyz}$, $\nabla\phi(x, y, z) = yze^{xyz}\hat{x} + xze^{xyz}\hat{y} + xye^{xyz}\hat{z}$.

Divergence $\nabla \cdot \vec{F}(x, y, z)$: $\nabla \cdot \vec{F}(x, y, z) = \frac{\partial F_x(x, y, z)}{\partial x} + \frac{\partial F_y(x, y, z)}{\partial y} + \frac{\partial F_z(x, y, z)}{\partial z}$

is a scalar function if $\vec{F}(x, y, z) = F_x(x, y, z)\hat{x} + F_y(x, y, z)\hat{y} + F_z(x, y, z)\hat{z}$ is a vector function.

Curl $\nabla \times \vec{F}(x, y, z)$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$
 is a vector

function if $\vec{F}(x, y, z) = F_x(x, y, z)\hat{x} + F_y(x, y, z)\hat{y} + F_z(x, y, z)\hat{z}$ is a vector function.

Eg. $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$, compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

$$\text{(Sol.) } \nabla \cdot \vec{F} = 1+1+1=3, \quad \nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{x} + 0\hat{y} + 0\hat{z} = 0$$

Eg. $\vec{F}(x, y, z) = x^2\hat{x} - 2x^2y\hat{y} + 2yz^4\hat{z}$, find $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$ at $(1, -1, 1)$. [1991 中山電研]

$$\text{(Sol.) } \nabla \cdot \vec{F} = 2x - 2x^2 + 8yz^3, \text{ at } (1, -1, 1) \Rightarrow \nabla \cdot \vec{F} = -8$$

$$\nabla \times \vec{F} = 2z^4\hat{x} - 4xy\hat{z}, \text{ at } (1, -1, 1) \Rightarrow \nabla \times \vec{F} = 2\hat{x} + 4\hat{z}$$

Laplacian operator: $\nabla^2\phi = \nabla \cdot \nabla\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$

Eg. For $\phi(x, y, z) = e^{xyz}$, $\nabla^2\phi(x, y, z) = y^2z^2e^{xyz} + x^2z^2e^{xyz} + x^2y^2e^{xyz}$.

Theorems (a) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(b) $\nabla \cdot (\nabla \times \vec{F}) = 0, \forall \vec{F} \in C^2$ **(c)** $\nabla \times (\nabla\phi) = 0, \forall \phi \in C^2$

(d) $\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G} + (\nabla \cdot \vec{G})\vec{F} - (\nabla \cdot \vec{F})\vec{G}$

(e) $\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F})$

(f) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$

(g) $\nabla \cdot (\phi\vec{F}) = (\nabla\phi) \cdot \vec{F} + \phi(\nabla \cdot \vec{F})$

Theorem $\nabla\phi(x, y, z) \perp$ the surface of $\phi(x, y, z)=\text{constant}$.

(Proof) $\because \phi(x, y, z)=\text{constant}$

$$\therefore d\phi(x, y, z) = 0 = \frac{\partial\phi(x, y, z)}{\partial x} dx + \frac{\partial\phi(x, y, z)}{\partial y} dy + \frac{\partial\phi(x, y, z)}{\partial z} dz$$

$$= \left(\frac{\partial\phi}{\partial x} \hat{x} + \frac{\partial\phi}{\partial y} \hat{y} + \frac{\partial\phi}{\partial z} \hat{z} \right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z}) = \nabla\phi \cdot d\vec{R}$$

$\therefore \nabla\phi(x, y, z) \perp d\vec{R}$, and $d\vec{R}$ is the tangential increment on the surface $\phi(x, y, z)$

Eg. Find the tangential plane and normal line to $z=x^2+y^2$ at $(2, -2, 8)$.

(Sol.) Let $\phi(x, y, z)=z-x^2-y^2$, $\nabla\phi = -2x\hat{x} - 2y\hat{y} + \hat{z}$, and $(2, -2, 8)$ is on the surface.

For $z-x^2-y^2=0$, the normal vector at $(2, -2, 8)$ is $-4\hat{x} + 4\hat{y} + \hat{z}$.

The tangential plane at $(2, -2, 8)$ is $-4(x-2) + 4(y+2) + (z-8) = 0 \Rightarrow -4x + 4y + z = -8$

The normal line is $\frac{x-2}{-4} = \frac{y+2}{4} = \frac{z-8}{1}$.

Eg. Find a unit normal vector of $z^2=4(x^2+y^2)$ at $(1, 0, 2)$. [1993 中山電研、2007 師大光電所]

Theorem $\nabla \cdot \vec{F}(x, y, z)$ is the outward flux per unit volume of the flow at point (x, y, z) and time t .

(Proof) \vec{F} at $P = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$

$$\text{The } x\text{-direction flux} = \left(F_x + \frac{1}{2} \frac{\partial F_x}{\partial x} \Delta x \right) \Delta y \Delta z - \left(F_x - \frac{1}{2} \frac{\partial F_x}{\partial x} \Delta x \right) \Delta y \Delta z = \frac{\partial F_x}{\partial x} \Delta x \Delta y \Delta z$$

$$\text{The } y\text{-direction flux} = \frac{\partial F_y}{\partial y} \Delta x \Delta y \Delta z, \text{ and the } z\text{-direction flux} = \frac{\partial F_z}{\partial z} \Delta x \Delta y \Delta z$$

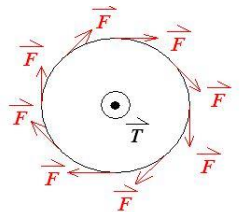
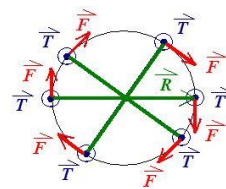
$$\therefore \text{Total flux} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \cdot \Delta x \Delta y \Delta z / \Delta x \Delta y \Delta z = \nabla \cdot \vec{F}$$

Theorem If $\vec{T} = \vec{F} \times \vec{R}$ and $\vec{R} = x\hat{x} + y\hat{y} + z\hat{z}$, then $\vec{F} = \frac{1}{2} \nabla \times \vec{T}$.

$$\text{(Proof)} \quad \nabla \times \vec{T} = \nabla \times (\vec{F} \times \vec{R}) = \nabla \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ F_x & F_y & F_z \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(F_y z - F_z y)\hat{x} + (F_z x - F_x z)\hat{y} + (F_x y - F_y x)\hat{z}]$$

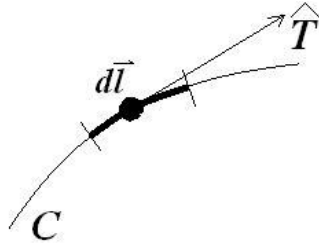
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_y z - F_z y & F_z x - F_x z & F_x y - F_y x \end{vmatrix} = 2(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}) = 2\vec{F} \Rightarrow \vec{F} = \frac{1}{2} \nabla \times \vec{T}$$



7-4 Line Integrals & Surface Integrals

Line integral: \exists Curve $C: \vec{R}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$, $a \leq t \leq b$

\exists Vector $\vec{F}(x, y, z) = F_x(x, y, z)\hat{x} + F_y(x, y, z)\hat{y} + F_z(x, y, z)\hat{z}$



$$(a) \int_C \vec{F} \cdot d\vec{l} = \int_a^b \vec{F}[x(t), y(t), z(t)] \cdot \vec{R}'(t) dt,$$

$$(b) \int_C \vec{F} \times d\vec{l} = \int_a^b \vec{F}[x(t), y(t), z(t)] \times \vec{R}'(t) dt,$$

$$(c) \int_C f(x, y, z) dl = \int_a^b f[x(t), y(t), z(t)] \cdot |\vec{R}'(t)| dt$$

Ex. For $C: \vec{R}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} = \cos(t)\hat{x} + \sin(t)\hat{y} + \hat{z}$, $0 \leq t \leq 2\pi$, and $\vec{F}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$, **determine** $\int_C \vec{F} \cdot d\vec{l}$ and $\int_C \vec{F} \times d\vec{l}$.

(Sol.) $C: \vec{R}(t) = \cos(t)\hat{x} + \sin(t)\hat{y} + \hat{z}$, $\vec{R}'(t) = -\sin(t)\hat{x} + \cos(t)\hat{y}$

$\vec{F}(x(t), y(t), z(t)) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} = \cos(t)\hat{x} + \sin(t)\hat{y} + \hat{z}$

$\vec{F} \cdot d\vec{l} = \vec{F}(x(t), y(t), z(t)) \cdot \vec{R}'(t) dt = (\cos(t)\hat{x} + \sin(t)\hat{y} + \hat{z}) \cdot (-\sin(t)\hat{x} + \cos(t)\hat{y}) dt$
 $= [-\cos(t)\sin(t) + \sin(t)\cos(t)] dt = 0$

$$\vec{F} \times d\vec{l} = \vec{F}(x(t), y(t), z(t)) \times \vec{R}'(t) dt = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos(t) & \sin(t) & 1 \\ -\sin(t) & \cos(t) & 0 \end{vmatrix} dt = [-\cos(t)\hat{x} - \sin(t)\hat{y} + \hat{z}] dt$$

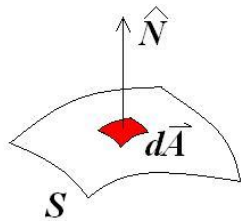
$$\therefore \int_C \vec{F} \cdot d\vec{l} = 0, \quad \int_C \vec{F} \times d\vec{l} = \int_0^{2\pi} [-\cos(t)\hat{x} - \sin(t)\hat{y} + \hat{z}] dt = 2\pi\hat{z}$$

Surface integral: \exists Surface S that has a projection on the xy -plane:

$\phi(x, y, z) = z - \phi(x, y) = 0$. $\vec{N} = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} = -\frac{\partial \phi}{\partial x}\hat{x} - \frac{\partial \phi}{\partial y}\hat{y} + \hat{z}$ is a normal

vector on S , and $\hat{N} = \vec{N} / |\vec{N}|$.

\exists Vector $\vec{F}(x, y, z) = F_x(x, y, z)\hat{x} + F_y(x, y, z)\hat{y} + F_z(x, y, z)\hat{z}$



$$(a) \iint_S \vec{F} \cdot d\vec{A} = \iint_D \vec{F}(x, y, z) \cdot \hat{N} \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dx dy,$$

$$(b) \iint_S \vec{F} \times d\vec{A} = \iint_D \vec{F}(x, y, z) \times \hat{N} \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dx dy,$$

$$(c) \iint_S f(x, y, z) dA = \iint_D f(x, y, z) \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dx dy,$$

Note: $d\vec{l} = \hat{T} dl$ is parallel to the tangential direction of the curve, but $d\vec{A} = \hat{N} dA$ is normal to the surface.

Ex. For $S: x^2 + y^2 + z^2 = a^2, z \geq 0$, and $\vec{F}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$, determine $\iint_S \vec{F} \cdot d\vec{A}$ and $\iint_S \vec{F} \times d\vec{A}$.

(Sol.) $S: x^2 + y^2 + z^2 = a^2 \wedge z \geq 0 \Rightarrow z = \sqrt{a^2 - x^2 - y^2}$, $\frac{\partial \phi}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{z}$

$\frac{\partial \phi}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}} = -\frac{y}{z} \Rightarrow \vec{N} = -\frac{\partial \phi}{\partial x} \hat{x} - \frac{\partial \phi}{\partial y} \hat{y} + \hat{z} = \frac{x}{z} \hat{x} + \frac{y}{z} \hat{y} + \hat{z}$, $|\vec{N}| = \frac{a}{z}$

$\hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{a}$, $\vec{F}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$, and $\sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} = \frac{a}{z}$

$\vec{F} \cdot d\vec{A} = \vec{F} \cdot \hat{N} \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dxdy = (x\hat{x} + y\hat{y} + z\hat{z}) \cdot \left(\frac{x\hat{x} + y\hat{y} + z\hat{z}}{a}\right) \frac{adxdy}{z}$
 $= \frac{x^2 + y^2 + z^2}{z} dxdy = \frac{a^2 dxdy}{\sqrt{a^2 - x^2 - y^2}} \Rightarrow$

$\iint_S \vec{F} \cdot d\vec{A} = \iint_{x^2 + y^2 \leq a^2} \frac{a^2 dxdy}{\sqrt{a^2 - x^2 - y^2}} = \int_0^{2\pi} \int_0^a \frac{a^2 r dr d\theta}{\sqrt{a^2 - r^2}} = 2\pi a^2 \cdot \int_0^a \frac{r dr}{\sqrt{a^2 - r^2}} = \frac{2\pi a^2}{2} \cdot \int_0^1 \frac{dr^2}{\sqrt{a^2 - r^2}}$
 $\stackrel{u=r^2}{=} \pi a^2 \cdot \int_0^{a^2} \frac{du}{\sqrt{a^2 - u}} = -2\pi a^2 \cdot \sqrt{a^2 - u} \Big|_{u=0}^{u=a^2} = 2\pi a^3$

$\vec{F} \times d\vec{A} = \vec{F} \times \hat{N} \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dxdy = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \frac{x}{a} & \frac{y}{a} & \frac{z}{a} \end{vmatrix} \frac{adxdy}{z} = 0 \Rightarrow \iint_S \vec{F} \times d\vec{A} = 0$

Ex. Compute the surface integral $\iint_S (x + y + z) dA$, where $S: z = x + y, 0 \leq y \leq x, 0 \leq x \leq 1$. [1990 台大農工研]

(Sol.) $\sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} = \sqrt{3}$, $\iint_S (x + y + z) dA = \int_0^1 \int_0^x 2(x + y) \cdot \sqrt{3} dy dx = \sqrt{3}$



Eg. $f = \frac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, find $\iint_S \nabla f \cdot \hat{n} dA$ for $S: x^2 + y^2 + z^2 = a^2$. [1990 中山電研]

(Sol.) $S: x^2 + y^2 + z^2 = a^2 \Rightarrow z = \pm \sqrt{a^2 - x^2 - y^2}$

For $z = \sqrt{a^2 - x^2 - y^2} \Rightarrow \vec{N} = -\frac{\partial \phi}{\partial x} \hat{x} - \frac{\partial \phi}{\partial y} \hat{y} + \hat{z} = \frac{x}{z} \hat{x} + \frac{y}{z} \hat{y} + \hat{z}$,

$$|\vec{N}| = \frac{a}{z}, \quad \hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{1}{a}(x\hat{x} + y\hat{y} + z\hat{z}), \quad \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} = \frac{a}{z},$$

$$\nabla f = \nabla\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{-(x\hat{x} + y\hat{y} + z\hat{z})}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{-(x\hat{x} + y\hat{y} + z\hat{z})}{a^3}$$

$$\Rightarrow \nabla(f) \cdot \hat{n} dA = \nabla(f) \cdot \hat{N} \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dxdy = -\frac{x^2 + y^2 + z^2}{a^3 z} dxdy = \frac{-dxdy}{a \cdot \sqrt{a^2 - x^2 - y^2}}$$

$$\iint_{z=\sqrt{a^2-x^2-y^2}} \nabla f \cdot \hat{n} dA = \iint_{x^2+y^2 \leq a^2} \frac{-dxdy}{a \cdot \sqrt{a^2 - x^2 - y^2}} = \frac{-1}{a} \int_0^{2\pi} \int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}} = \frac{-2\pi}{a} \int_0^a \frac{rdr}{\sqrt{a^2 - r^2}}$$

$= -2\pi$. Similarly, $\iint_{z=-\sqrt{a^2-x^2-y^2}} \nabla f \cdot \hat{n} dA = -2\pi$ for $z = -\sqrt{a^2 - x^2 - y^2}$

$\therefore \iint_S \nabla f \cdot \hat{n} dA = (-2\pi) + (-2\pi) = -4\pi$

Green's theorem Let C be a regular, closed, positively-oriented curve enclosing a region D , $\vec{F}(x, y) = F_x(x, y)\hat{x} + F_y(x, y)\hat{y}$.

$$\oint_C \vec{F} \cdot d\vec{R} = \oint_C F_x(x, y)dx + F_y(x, y)dy = \iint_D \left[\frac{\partial F_y(x, y)}{\partial x} - \frac{\partial F_x(x, y)}{\partial y} \right] dxdy$$

Eg. $\hat{F} = \frac{-y\hat{x} + x\hat{y}}{x^2 + y^2}$, find $\oint_C \vec{F} \cdot d\vec{R}$, where C is any closed curve.

(Sol.) $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0, \forall (x, y) \neq 0, \oint_C \vec{F} \cdot d\vec{R} = 0$ if C does not enclose 0.

Else, $\oint \vec{F} \cdot d\vec{r} = 2\pi$

Stokes's theorem Let S be a regular surface with coherently oriented boundary C , $\iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{\ell}$.

Divergence theorem Let S be a regular, positive-oriented closed surface, enclosing a region V , $\oiint_S \vec{F} \cdot d\vec{A} = \iiint_V (\nabla \cdot \vec{F}) dV$.

Eg. Compute $\oiint_S \vec{F} \cdot d\vec{A}$, where $\vec{F} = (y^2 + z^2)^{\frac{2}{3}} \hat{x} + \sin(x^2 + z) \hat{y} + e^{x^2 - y^2} \hat{z}$,

$S: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [1990 成大工程科學所]

(Sol.) $\nabla \cdot \vec{F} = 0 \Rightarrow \oiint_S \vec{F} \cdot d\vec{A} = \iiint_V (\nabla \cdot \vec{F}) dx dy dz = 0$

Eg. $f = \frac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, find $\iint_S \nabla f \cdot \hat{n} dA$ for $S: x^2 + y^2 + z^2 = a^2$. [1990 中山電研]

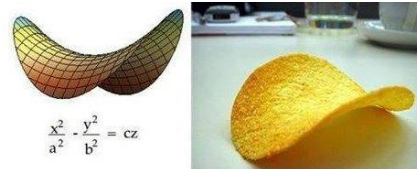
(Sol.) $\vec{F} = \nabla f = \nabla \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{-1}{(\sqrt{x^2 + y^2 + z^2})^3} (x\hat{x} + y\hat{y} + z\hat{z}) = \frac{-1}{a^3} (x\hat{x} + y\hat{y} + z\hat{z})$,

$\nabla \cdot \vec{F} = \frac{-3}{a^3}$, $\iint_S \nabla f \cdot \hat{n} dA = \oiint_S \vec{F} \cdot d\vec{A} = \iiint_V (\nabla \cdot \vec{F}) dx dy dz = \frac{-3}{a^3} \cdot \frac{4a^3 \pi}{3} = -4\pi$

Green's identities

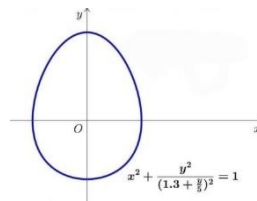
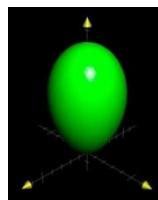
ϕ, ψ are scalars, $\left[\begin{array}{l} (a) \iiint_V [\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi] dx dy dz = \iint_S (\phi \nabla \psi) \cdot d\vec{A} \\ (b) \iiint_V [\phi \nabla^2 \psi - \psi \nabla^2 \phi] dx dy dz = \iint_S [\phi \nabla \psi - \psi \nabla \phi] \cdot d\vec{A} \end{array} \right.$

Hyperbolic Paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$

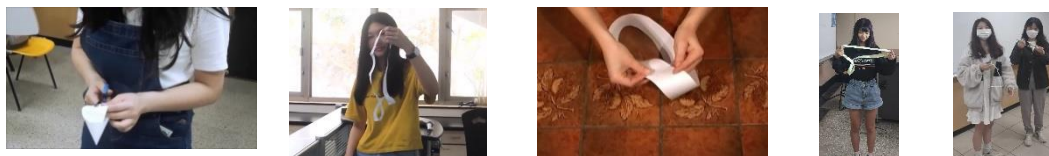


Egg Equations

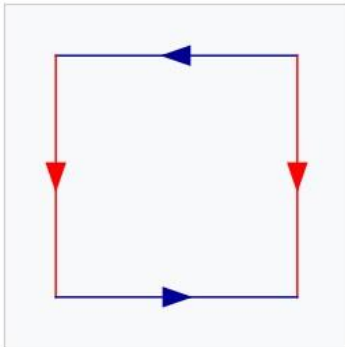
$x^2 + y^2 + z^2 = z^{1.5}$



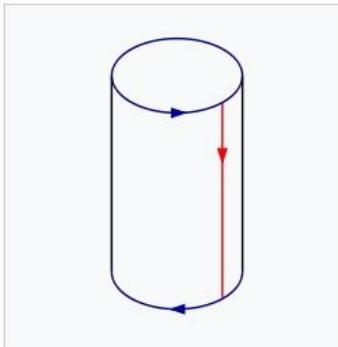
Mobius Belt/Strip



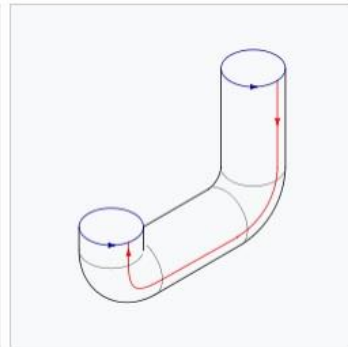
Klein Bottle



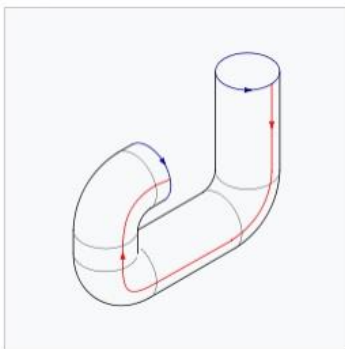
Schritt 1



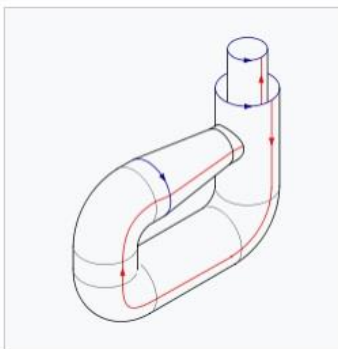
Schritt 2



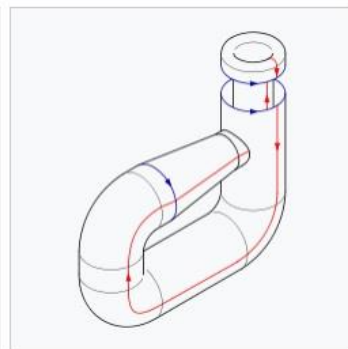
Schritt 3



Schritt 4



Schritt 5



Schritt 6

7-5 Potential Theory

Potential: ϕ is called a potential for the vector field \vec{F} if $\vec{F} = \nabla\phi$ or $\vec{F} = -\nabla\phi$

Test for a potential: If \vec{F} and $\nabla \cdot \vec{F}$ are continuous in a simply-connected domain Ω , then \vec{F} has a potential function. $\Leftrightarrow \nabla \times \vec{F} = 0$

(Proof) " \Rightarrow " : $\vec{F} = \pm \nabla\phi \Rightarrow \nabla \times \vec{F} = 0$.

$$" \Leftarrow " : \nabla \times \vec{F} = 0, \oint_C \vec{F} \cdot d\vec{\ell} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{A} = 0$$

Let C be the boundary of $\phi(x,y,z)=\text{constant}$, then we choose $\vec{F} = \pm \nabla\phi$, then $\int_C \vec{F} \cdot d\vec{\ell} = 0$ is always valid. $\Rightarrow \vec{F}$ has a potential function.

Eg. Check (a) $\vec{F} = 2xy\hat{x} + z^2\hat{y} + (x - y + z)\hat{z}$,

and (b) $\vec{F} = (yze^{xyz} - 4x)\hat{x} + (xze^{xyz} + z)\hat{y} + (xye^{xyz} + y)\hat{z}$, which does have a potential?
[1991 成大機研]

(Sol.)

$$(a) \quad \nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & z^2 & x - y + z \end{vmatrix} = (-2z - 1)\hat{x} - \hat{y} - 2xz\hat{z} \neq 0, \quad \therefore \text{no potential.}$$

(b)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{xyz} - 4x & xze^{xyz} + z & xye^{xyz} + y \end{vmatrix} = (1 + xe^{xyz} + x^2 yze^{xyz} - 1 - xe^{xyz} - x^2 yze^{xyz})\hat{x} \\ + (ye^{xyz} + xy^2 ze^{xyz} - ye^{xyz} - xy^2 ze^{xyz})\hat{y} + (ze^{xyz} + xyz^2 - ze^{xyz} - xyz^2)\hat{z} = 0, \\ \therefore \text{there exists a potential.}$$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z} = (yze^{xyz} - 4x)\hat{x} + (xze^{xyz} + z)\hat{y} + (xye^{xyz} + y)\hat{z}$$

$$\frac{\partial\phi}{\partial x} = yze^{xyz} - 4x \Rightarrow \phi = \int (yze^{xyz} - 4x)dx = e^{xyz} - 2x^2 + I(y, z)$$

$$\frac{\partial\phi}{\partial y} = xze^{xyz} + z \Rightarrow \phi = \int (xze^{xyz} + z)dy = e^{xyz} + yz + J(x, z)$$

$$\frac{\partial\phi}{\partial z} = xye^{xyz} + y \Rightarrow \phi = \int (xye^{xyz} + y)dz = e^{xyz} + yz + K(x, y), \quad \therefore \phi = e^{xyz} - 2x^2 + yz + C$$

Eg. $\vec{F} = (x^2 + y^2 + z^2)^n (x\hat{x} + y\hat{y} + z\hat{z})$, find a scalar potential $\phi(x, y, z)$ so that

$$\vec{F} = -\nabla\phi. \text{ [1990 台大材研] (Ans.) } \phi(x, y, z) = \frac{-(x^2 + y^2 + z^2)^{n+1}}{2(n+1)} + C$$

Theorem If \vec{F} has a potential, then the line integral of \vec{F} is independent of path in Ω . That is, $\int_C \vec{F} \cdot d\vec{\ell} = \int_K \vec{F} \cdot d\vec{\ell}$, whenever C and K are regular curves in Ω with the same initial point and the same terminal point.

$$\text{(Proof)} \quad \int_C \vec{F} \cdot d\vec{\ell} - \int_K \vec{F} \cdot d\vec{\ell} = \oint_{C'} \vec{F} \cdot d\vec{\ell}$$

$$\text{If } \vec{F} \text{ has a potential, then } \oint_{C'} \vec{F} \cdot d\vec{\ell} = 0, \therefore \int_C \vec{F} \cdot d\vec{\ell} = \int_K \vec{F} \cdot d\vec{\ell}.$$

Theorem Let \vec{F} be a 2-D vector field of a simply-connected domain Ω . Then

$$\vec{F} \text{ has a potential on } \Omega. \Leftrightarrow \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}. \text{ (In this case, } \vec{F} = F_x(x, y)\hat{x} + F_y(x, y)\hat{y}\text{)}$$

$$\text{(Proof) If } \vec{F} = \nabla\phi, \text{ then } F_x = \frac{\partial\phi(x, y)}{\partial x}, F_y = \frac{\partial\phi(x, y)}{\partial y}, \frac{\partial F_x}{\partial y} = \frac{\partial^2\phi}{\partial y\partial x} = \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial F_y}{\partial x}$$

By Green's theorem, $\oint_C F_x dx + F_y dy = \iint_D \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy = 0$, $\therefore \vec{F}$ is conservative on Ω .

Eg. Evaluate $\int_{\Gamma} \vec{v} \cdot d\vec{R}$, where $\vec{v} = 2x^2\hat{x} - 2yz\hat{y} - (y^2 + 3)\hat{z}$ and Γ is some complicated path from $(0,0,0)$ to $(0,0,4)$. [2016 成大電研]

(Sol.) $\nabla \times \vec{v} = 0$, $\therefore \vec{v}$ is conservative. $\therefore \exists \phi(x, y, z)$ such that $\vec{v} = \nabla\phi$

$$\frac{\partial\phi}{\partial x} = 2x^2 \Rightarrow \phi = \int 2x^2 dx = \frac{2x^3}{3} + I(y, z)$$

$$\frac{\partial\phi}{\partial y} = -2yz \Rightarrow \phi = \int -2yz dy = -y^2 z + J(x, z)$$

$$\frac{\partial\phi}{\partial z} = -y^2 - 3 \Rightarrow \phi = \int (-y^2 - 3) dz = -y^2 z - 3z + K(x, y)$$

$$\therefore \phi = -y^2 z - 3z + \frac{2x^3}{3} + C \Rightarrow \int \vec{v} \cdot d\vec{r} = \phi(x, y, z) \Big|_{(0,0,0)}^{(0,0,4)} = -12$$

Eg. (a) Is $\vec{F}(x, y, z) = -\frac{y}{z}\hat{x} - \frac{x}{z}\hat{y} + \frac{xy}{z^2}\hat{z}$ conservative in the region $z > 0$? (b)

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = -\frac{y}{z}\hat{x} - \frac{x}{z}\hat{y} + \frac{xy}{z^2}\hat{z}$, C is a piecewise smooth curve from $(1,1,1)$ to $(2,-1,3)$ and not crossing the xy -plane. [1990 成大化工所]

(Sol.) (a) $\nabla \times \vec{F} = 0$ in the region $z > 0$, $\therefore \vec{F}$ is conservative.

(b) $\nabla \times \vec{F} = 0$, $\therefore \exists \phi(x, y, z)$ such that $\vec{F} = \nabla \phi$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{z} \Rightarrow \phi = \int -\frac{y}{z} dx = \frac{-xy}{z} + I(y, z)$$

$$\frac{\partial \phi}{\partial y} = -\frac{x}{z} \Rightarrow \phi = \int -\frac{x}{z} dy = \frac{-xy}{z} + J(x, z)$$

$$\frac{\partial \phi}{\partial z} = \frac{xy}{z^2} \Rightarrow \phi = \int \frac{xy}{z^2} dz = \frac{-xy}{z} + K(x, y)$$

$$\Rightarrow \phi(x, y, z) = -\frac{xy}{z} + C \Rightarrow \int \vec{F} \cdot d\vec{r} = \phi(x, y, z) \Big|_{(1,1,1)}^{(2,-1,3)} = \frac{5}{3}$$

7-6 Curvilinear Coordinates

Coordinate transformation:
$$\begin{cases} x = x(q_1, q_2, q_3) \\ y = y(q_1, q_2, q_3) \\ z = z(q_1, q_2, q_3) \end{cases} \Leftrightarrow \begin{cases} q_1 = q_1(x, y, z) \\ q_2 = q_2(x, y, z) \\ q_3 = q_3(x, y, z) \end{cases}$$

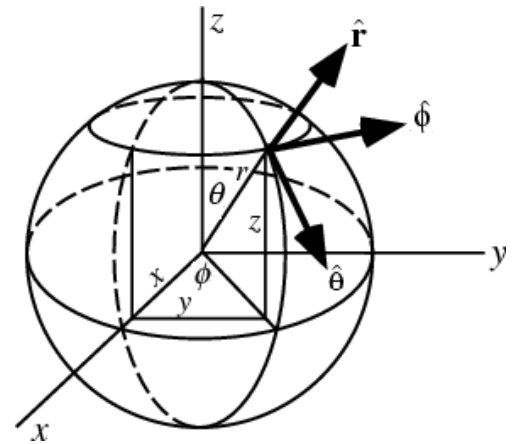
Scalar factors: If (q_1, q_2, q_3) are orthogonal system, then

$$h_{ij} = \begin{cases} 0, i \neq j \\ \left[\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2 \right]^{1/2}, i = j \end{cases}$$

Eg. Rectangular coordinate \Leftrightarrow Spherical coordinate

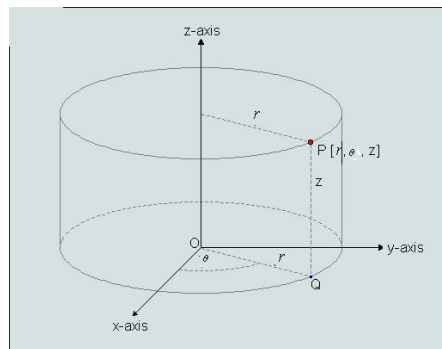
$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \sin^{-1} \left(\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \right) \\ \phi = \cos^{-1} \left(\frac{x}{\sqrt{x^2 + y^2}} \right) \end{cases}$$

$$\Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 \right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 \right]^{1/2} = r \\ h_\phi = h_{33} = \left[\left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2 \right]^{1/2} = r \sin \theta \end{cases}$$



Differential length vector in the spherical coordinate:

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r \sin \theta d\phi$$



Eg. Rectangular coordinate \Leftrightarrow Cylindrical coordinate

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}(y/x) \\ z = z \end{cases}$$

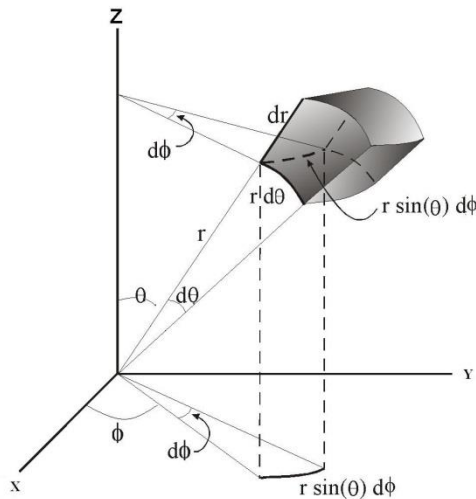
$$\Rightarrow \begin{cases} h_r = h_{11} = \left[\left(\frac{\partial x}{\partial r} \right)^2 + \left(\frac{\partial y}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial r} \right)^2 \right]^{1/2} = 1 \\ h_\theta = h_{22} = \left[\left(\frac{\partial x}{\partial \theta} \right)^2 + \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 \right]^{1/2} = r \\ h_z = h_{33} = \left[\left(\frac{\partial x}{\partial z} \right)^2 + \left(\frac{\partial y}{\partial z} \right)^2 + \left(\frac{\partial z}{\partial z} \right)^2 \right]^{1/2} = 1 \end{cases}$$

Differential length vector in the cylindrical coordinate: $d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_z dz$

Differential arc length: $ds = [(h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2]^{1/2}$

Differential elements of area on the $q_i q_j$ -plane: $dA_{ij} = ds_i ds_j = h_i h_j dq_i dq_j$

Differential element of volume: $dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$



Eg. Spherical coordinate:

$$\begin{cases} ds = [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2]^{1/2} \\ dA_{\theta, \phi} = r^2 \sin \theta d\theta d\phi, dA_{r, \theta} = r dr d\theta, \text{ etc.} \\ dV = r^2 \sin \theta dr d\theta d\phi \end{cases}$$

Eg. Cylindrical coordinate:

$$\begin{cases} ds = [dr^2 + r^2 d\theta^2 + dz^2]^{1/2} \\ dA_{\theta, z} = rd\theta dz, dA_{r, \theta} = r dr d\theta, dA_{r, z} = dr dz \\ dV = r dr d\theta dz \end{cases}$$

Jacobian determinants: $(p_1, p_2, p_3) \Leftrightarrow (q_1, q_2, q_3)$

$$dp_i dp_j = \det \begin{bmatrix} \frac{\partial p_i}{\partial q_i} & \frac{\partial p_j}{\partial q_i} \\ \frac{\partial p_i}{\partial q_j} & \frac{\partial p_j}{\partial q_j} \end{bmatrix} \cdot dq_i dq_j$$

$$\text{and } dp_1 dp_2 dp_3 = \det \begin{bmatrix} \frac{\partial p_1}{\partial q_1} & \frac{\partial p_2}{\partial q_1} & \frac{\partial p_3}{\partial q_1} \\ \frac{\partial p_1}{\partial q_2} & \frac{\partial p_2}{\partial q_2} & \frac{\partial p_3}{\partial q_2} \\ \frac{\partial p_1}{\partial q_3} & \frac{\partial p_2}{\partial q_3} & \frac{\partial p_3}{\partial q_3} \end{bmatrix} \cdot dq_1 dq_2 dq_3$$

Eg. Let $I = \iiint_V f(x, y, z) dx dy dz$. Transform the integral from (x, y, z) into (r, θ, ϕ) .

$$|J| = \left| \det \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{bmatrix} \right| = |r^2 \sin \theta \cos^2 \theta \cos^2 \phi$$

$$+ r^2 \sin^3 \theta \sin^2 \phi + r^2 \sin \theta \cos^2 \theta \sin^2 \phi + r^2 \sin^3 \theta \cos^2 \phi| = |r^2 \sin \theta \cos^2 \theta + r^2 \sin^3 \theta| = r^2 \sin \theta = h_1 h_2 h_3$$

$$\therefore I = \iiint_V f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \cdot r^2 \sin \theta dr d\theta d\phi$$

Del operators in (q_1, q_2, q_3) coordinate system:

$$\nabla \psi(q_1, q_2, q_3) = \frac{1}{h_1} \frac{\partial \psi}{\partial q_1} \hat{u}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial q_2} \hat{u}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial q_3} \hat{u}_3$$

$$\nabla \cdot \vec{F}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1} (F_1 h_2 h_3) + \frac{\partial}{\partial q_2} (F_2 h_1 h_3) + \frac{\partial}{\partial q_3} (F_3 h_1 h_2) \right)$$

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}$$

$$\nabla^2 \psi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial q_3} \right) \right],$$

where $\vec{F} = F_1 \hat{u}_1 + F_2 \hat{u}_2 + F_3 \hat{u}_3$

Eg. Spherical coordinate system: $h_r=1, h_\theta=r, h_\phi=r \sin \theta$

$$\vec{F} = F_r \hat{a}_r + F_\theta \hat{a}_\theta + F_\phi \hat{a}_\phi$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \cdot \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \theta} (\sin \theta \cdot F_\theta) + \frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \phi} (F_\phi)$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{a}_r & r \hat{a}_\theta & r \sin \theta \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin \theta \cdot F_\phi \end{vmatrix}, \quad \nabla \psi = \frac{\partial \psi}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \hat{a}_\phi$$

$$\nabla^2 \psi = \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \cdot \frac{\partial^2 \psi}{\partial \phi^2}$$

Eg. Cylindrical coordinate system: $h_r=1, h_\theta=r, h_z=1$

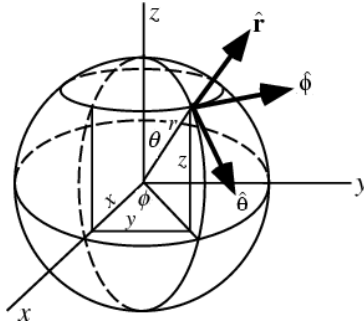
$$\vec{F} = F_r \hat{a}_r + F_\theta \hat{a}_\theta + F_z \hat{a}_z$$

$$\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{a}_r + \left(\frac{\partial F_\theta}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{a}_\theta + \left(\frac{1}{r} \frac{\partial}{\partial r} (r F_\theta) - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right) \hat{a}_z$$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{a}_\theta + \frac{\partial \psi}{\partial z} \hat{a}_z, \quad \nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Unit vector conversions between distinct coordinates:



Eg. Rectangular coordinate \Leftrightarrow Spherical coordinate

(Proof) $\hat{x} \cdot \hat{r} = \sin \theta \cos \phi, \quad \hat{y} \cdot \hat{r} = \sin \theta \sin \phi,$

$\hat{z} \cdot \hat{r} = \cos \theta, \therefore \hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$

Similarly, $\hat{x} \cdot \hat{\theta} = \cos \theta \cos \phi, \quad \hat{y} \cdot \hat{\theta} = \cos \theta \sin \phi,$

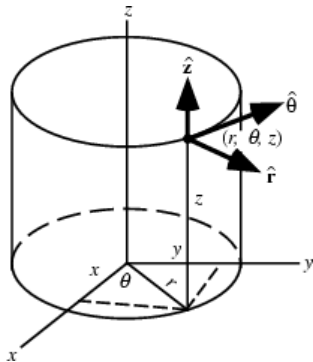
$\hat{z} \cdot \hat{\theta} = -\sin \theta, \therefore \hat{\theta} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta,$

And $\hat{x} \cdot \hat{\phi} = -\sin \phi, \quad \hat{y} \cdot \hat{\phi} = \cos \phi, \quad \hat{z} \cdot \hat{\phi} = 0,$

$\therefore \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix},$$

or $\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$



Eg. Rectangular coordinate \Leftrightarrow Cylindrical coordinate

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$