Chapter 6 Recurrence Relations

6-1 Linear Recurrence Relations

The 1st-order linear recurrence relation of a_n: $a_{n+1}=c_na_n+d_n$, $c_n\neq 0$

Iteratively solving 1st-order linear recurrence relation $a_{n+1}=c_na_n+d_n$:

$a_1=c_0a_0+d_0$, $a_2=c_1a_1+d_1=c_1(c_0a_0+d_0)+d_1=c_1c_0a_0+c_1d_0+d_1$,

$a_3=c_2a_2+d_2=c_2(c_1c_0a_0+c_1d_0+d_1)+d_2=c_2c_1c_0a_0+c_2c_1d_0+c_2d_1+d_2$, …,

$a_n=c_{n-1}c_{n-2}\cdots c_0a_0+c_{n-1}c_{n-2}\cdots c_1d_0+c_{n-1}c_{n-2}\cdots c_2d_1+\cdots+d_{n-1}$

If $c_n=c$ and $d_n=d$ are independent of n, $a_n=c^na_0+(c^{n-1}+c^{n-2}+c^{n-3}+\ldots+c+1)d$.

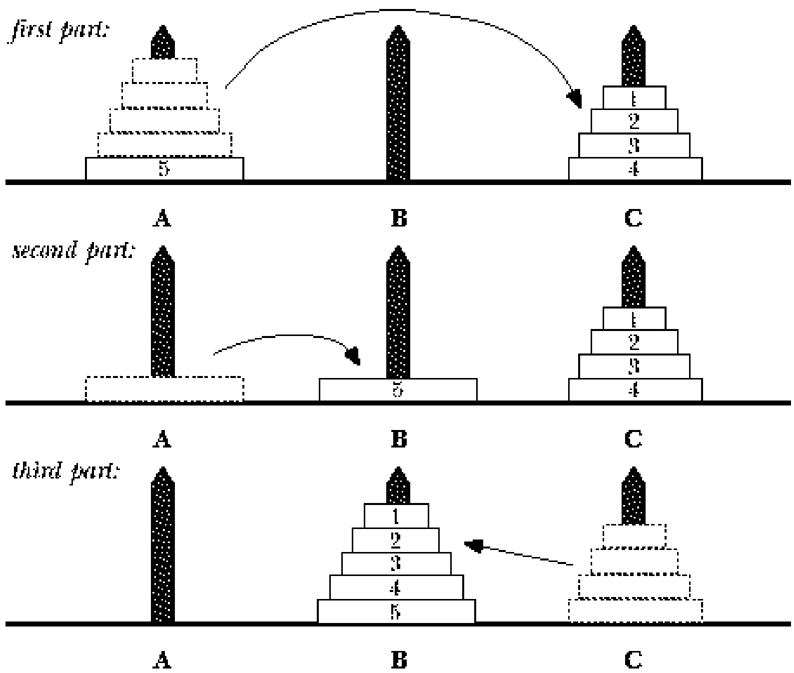
Eg. Solve $a_{n+1}=2a_n-1$.

(Sol.) $a_1=2a_0-1$, $a_2=2a_1-1=4a_0-3$, $a_3=2a_2-1=8a_0-7$, …

By mathematical induction: $a_n=2^na_0-(2^n-1)$

Tower of Hanoi: Transfer n disks to another peg by moving on disk at a time and keep smaller disks being placed on top of the larger ones. Let c_n denote the number of moves. We have the 1st-order recurrence relation: $c_n=2c_{n-1}+1$, $c_1=1$. And we have $c_1=1=2^1-1$, $c_2=2c_1+1=3=2^2-1$, $c_3=2c_2+1=2^3-1=7$, …, $c_n=2^n-1$.

Videos of Hanoi Tower with 2 disks, 3 disks, 4 disks, 5 disks.



Eg. Given recurrence relation S_0=1 and S_n=2S_{n-1}+n. (a) Find an explicit formula for S_n. (b) Prove your answer in (a) by mathematical induction. [台科大資工所]

(Sol.) (a) $S_0=1$, $S_1=2S_0+1=3$, $S_2=2S_1+2=4S_0+1\times2+2=8$,

$S_3=2S_2+3=8S_0+1\times2\times2+2\times2+3=19$, $S_4=2S_3+4=16S_0+1\times2\times2\times2+2\times2\times2+3\times2+4=42$, …, $S_n=2^n+1\times2^{n-1}+2\times2^{n-2}+3\times2^{n-3}+4\times2^{n-4}+\ldots+n$.

(b) $n=1$, $S_0=1=2^{1+1}-1$. If $S_n=2^n+1\times2^{n-1}+2\times2^{n-2}+3\times2^{n-3}+4\times2^{n-4}+\ldots+n$ holds, then
$S_{n+1}=2S_n+n+1=2(2^n+1\times2^{n-1}+2\times2^{n-2}+3\times2^{n-3}+4\times2^{n-4}+\ldots+n)+n+1$
$=2^{n+1}+1\times2^n+2\times2^{n-1}+3\times2^{n-2}+4\times2^{n-3}+\ldots+n\times2+n+1$

The 2nd-order linear homogeneous recurrence relation of a_n: $a_{n+2}+c_na_{n+1}+d_na_n=0$
The 2nd-order linear nonhomogeneous recurrence relation of a_n:
$a_{n+2}+c_na_{n+1}+d_na_n=f_n$

Solutions of 2nd-order linear homogeneous recurrence relation $a_{n+2}+ca_{n+1}+da_n=0$ with constant coefficients:

Suppose $a_n=r^n$, $r^{n+2}+cr^{n+1}+dr^n=0$, $r^2+cr+d=0$

Case 1 $r=r_1,\ r_2\in R$, $r_1\neq r_2\Rightarrow a_n=c_1r_1^n+c_2r_2^n$

Eg. Solve $a_{n+2}+2a_{n+1}-3a_n=0$.

(Sol.) $r^2+2r-3=0$, $r=1,-3\Rightarrow a_n=c_1(1)^n+c_2(-3)^n=c_1+c_2(-3)^n$

Case 2 $r=r_1=r_2\Rightarrow a_n=c_1r_1^n+c_2nr_1^n$

Eg. Solve $a_{n+2}-8a_{n+1}+16a_n=0$.

(Sol.) $r^2-8r+16=0$, $r=4,4\Rightarrow a_n=c_14^n+c_2n4^n$

Case 3

$r=\alpha\pm i\beta=\rho e^{\pm i\theta}\Rightarrow a_n=c_1(\alpha+i\beta)^n+c_2(\alpha-i\beta)^n=\rho^n[d_1\cos(n\theta)+d_2\sin(n\theta)]$

Eg. Solve $a_{n+2}-2a_{n+1}+2a_n=0$.

(Sol.) $r^2-2r+2=0$, $r=1\pm i=\sqrt{2}e^{\pm j\frac{\pi}{4}}\Rightarrow a_n=\left(\sqrt{2}\right)^n\left[c_1\cos\left(\frac{n\pi}{4}\right)+c_2\sin\left(\frac{n\pi}{4}\right)\right]$

Eg. Given the following recurrence equation: $r_n=r_{n-1}+2r_{n-2}$ with $r_0=1$, $r_1=1$. (a) Please answer the values of r_2, and r_{10}. (b) Please find the general expression for r_n in terms of n. [台大電研]

(Sol.) (b) Set n-2=m, m=n+2, and then reset m=n. We have $r_{n+2}-r_{n+1}-2r_n=0$.

$r^2-r-2=0$, r=2, -1 $\Rightarrow r_n=c_1 2^n+c_2(-1)^n$

$r_0=1$, $r_1=1 \Rightarrow c_1=2/3$, $c_2=1/3 \Rightarrow r_n=\frac{2}{3}\cdot 2^n+\frac{1}{3}\cdot(-1)^n$

(a) $r_2=\frac{2}{3}\cdot 2^2+\frac{1}{3}\cdot(-1)^2=3$ and $r_{10}=\frac{2}{3}\cdot 2^{10}+\frac{1}{3}\cdot(-1)^{10}=683$

Fibonacci sequence: $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, n=3, 4, 5, ….

We can obtain the Fibonacci sequence as 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ….

Set n-2=m, m=n+2, and then reset m=n. We have $f_{n+2}-f_{n+1}-f_n=0$.

$r^2-r-1=0$, $r=\frac{1\pm\sqrt{5}}{2}$, $f_n=c(\frac{1+\sqrt{5}}{2})^n+d(\frac{1-\sqrt{5}}{2})^n$, $f_1=f_2=1 \Rightarrow c=-d=\frac{1}{\sqrt{5}}$

$f_n=\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n-\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Eg. Show that $f_{n+1}\geq(\frac{1+\sqrt{5}}{2})^{n-1}$, $n\geq 1$.

Recursive Expressions

Eg. Rewrite $s_n=1+2+3+4+\ldots+n$ into a recursive expression.

(Sol.) $s_1=1$, $s_n=s_{n-1}+n$ for all $n\geq 1$.

6-2 Nonlinear Recurrence Relations

Eg. $a_n=3a_{n-1}a_{n-2}$ **and** $a_n=\sqrt{\frac{a_{n-2}}{a_{n-1}}}$ **are both the 2nd-order nonlinear homogeneous recurrence relations of a_n.**

Eg. Solve $\sqrt{a_n}=\sqrt{a_{n-1}}+2\sqrt{a_{n-2}}$.

(Sol.) Let $b_n=\sqrt{a_n}$, $b_n=b_{n-1}+2b_{n-2}$, $b_n-b_{n-1}-2b_{n-2}=0$, $r^2-r-2=0$, r=-1, 2

$b_n=\sqrt{a_n}=c_1(-1)^n+c_2 2^n$, $a_n=[c_1(-1)^n+c_2 2^n]^2$

6-3 Double-index Recurrence Relations

Ackermann's function: $A(m,n)=A(m-1,A(m,n-1))$, $A(m,0)=A(m-1,1)$, and $A(0,n)=n+1$.

Eg. Find $A(1,1)$.

(Sol.) $A(1,1)=A(0,A(1,0))=A(0,A(0,1))=A(0,2)=3$

Eg. Show that $C_r^n + C_{r+1}^n = C_{r+1}^{n+1}$ if $C_r^n = \frac{n!}{r!(n-r)!}$.

(Proof) $C_r^n + C_{r+1}^n = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} = n! \cdot [\frac{1}{r!(n-r)!} + \frac{1}{(r+1)!(n-r-1)!}]$

$= n! \cdot [\frac{r+1}{(r+1)!(n-r)!} + \frac{n-r}{(r+1)!(n-r)!}] = n! \cdot \frac{n+1}{(r+1)!(n-r)!} = \frac{(n+1)!}{(r+1)!(n-r)!} = C_{r+1}^{n+1}$