# Chapter 7 Logics and Other Basic Mathematics

**7-1 Logics**

**Proposition (命題):** A sentence is either true or false, but not both.

**Eg. A man is an animal. (True, T) Eg. A tree is an animal. (False, F)**

**Eg. 3>2 (True, T) Eg.-1+3=18 (False, F)**

**Eg. Give me an apple. (Not a proposition)**

***Conjunction* of *p* and *q*:** $p \wedge q$. ***Disjunction* of *p* and *q*:** $p \vee q$.

***Negation* of *p*:** $\neg p$ or $p'$

**Truth Table:**

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $p \vee q$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | F | T | F | T |
| F | T | T | F | F | T |
| F | F | T | T | F | F |

**De Morgan's Laws:** $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$

**Eg. If "Shin-Huei Bai (白歆惠) and Tang Suei (隋棠) are both beautiful." is not valid, it implies "Shin-Huei Bai is ugly, or Tang Suei is ugly".**

**Eg. If "Shin-Huei Bai or Tang Suei is ugly." is not correct, then "Shin-Huei Bai is pretty, and Tang Suei is pretty".**

**If *p* then *q* :** $p \rightarrow q$ for a **hypothesis** (antecedent) ***p*** and a **conclusion** (conseqent) ***q***

**Eg. If *A* is an orthogonal matrix ($AA^t = I$), then $A^{-1} = A^t$.**

**Truth Table for $p \rightarrow q$:**

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Logically equivalent statements $p \leftrightarrow q$:** $p$ if and only if $q$. It means that $p \rightarrow q$ and $q \rightarrow p$.

**Theorem $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.** (It is usually utilized in the **proof by contradiction**)

**Eg.** "**If the network is down, then you can not access the Internet.**" is logically equivalent to "**If you can access the Internet, then the network is not down.**"

**Eg. "Chi-Ling Lin is beautiful, so some boys like her." is logically equivalent to "No boy likes Chi-Ling Lin, so she looks ugly."**

**Theorem** $\neg(p\rightarrow q)$ **is logically equivalent to** $p \wedge \neg q$.

(Proof) by the Truth Table, we can find that $\neg(p\rightarrow q)$ is logically equivalent to $p \wedge \neg q$.

| $p$ | $q$ | $\neg(p\rightarrow q)$ | $p \wedge \neg q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | F | F |
| F | F | F | F |

**Eg. If "Chi-Ling Lin is beautiful, so some boys like her." is not correct, you can find out "Chi-Ling Lin is beautiful but no boy likes her."**

**Eg. "If a man is good at sport, then he is stupid (若四肢發達，則頭腦簡單)." If you disagree with this statement, you can give an example "A person who is both good at sport and smart."**

**Theorem** $p\rightarrow q$ **is logically equivalent to** $\neg p \vee q$.

(Proof) $\neg(p\rightarrow q)$ is logically equivalent to $p \wedge \neg q$. That is, $p\rightarrow q$ is logically equivalent to $\neg(p \wedge \neg q)=\neg p \vee q$.

**Eg. "Chi-Ling Lin is beautiful, so everybody like her." is logically equivalent to "Chi-Ling Lin is ugly, or (otherwise) somebody likes her."**

**Theorem** $(p \wedge q)\rightarrow r$ **is logically equivalent to** $p\rightarrow(q\rightarrow r)$.

(Proof) It can be proved by the truth table.

**Eg. "Chi-Ling Lin is beautiful and nice, so everybody likes her." is logically equivalent to "Chi-Ling Lin is beautiful, so she is nice to everybody and this personality makes everybody like her."**

**Rules of Inference for propositions:**

| Rule of Inference | Name | Rule of Inference | Name |
|---|---|---|---|
| $p \to q$<br>$p$<br>———<br>$\therefore q$ | Modus ponens | $p$<br>$q$<br>———<br>$\therefore p \wedge q$ | Conjunction |
| $p \to q$<br>$\neg q$<br>———<br>$\therefore \neg p$ | Modus tollens | $p \to q$<br>$q \to r$<br>———<br>$\therefore p \to r$ | Hypothetical syllogism |
| $p$<br>———<br>$\therefore p \vee q$ | Addition | $p \vee q$<br>$\neg p$<br>———<br>$\therefore q$ | Disjunctive syllogism |
| $p \wedge q$<br>———<br>$\therefore p$ | Simplification | $p \to r$<br>$q \to r$<br>———<br>$\therefore (p \vee q) \to r$ | |


**Eg. "Shin-Huei Bai is pretty." and "Ruo-Ya Lin (林若亞) is pretty." can be combined into "Shin-Huei Bai and Ruo-Ya Lin are both pretty." (Conjunction)**

**Eg. "Tom is a man, so he is an animal." and "He is an animal, so he can eat food." can be combined into "Tom is a man, so he can eat food." (Hypothetical syllogism)**

**Eg. A pitcher can throw only a slider (滑球) or a fork (指叉球). It is known that he did not throw a slider, then he throw a fork. (Disjunctive syllogism)**

**Resolution proof:** If $p \vee q$ and $\neg p \vee r$ are both true, then $q \vee r$ is true.
**Another form of resolution proof:** If $p \vee q$ and $p \to r$ are both true, then $q \vee r$ is true.
**Eg. "Mary will bear a girl *or* a boy." and "If she bears a girl, she will dislike this baby." are logically equivalent to "Mary will bear a boy, or she dislike this baby."**

**Eg. Show that $p \to q$ and $p \vee q$, then $q$.**

(Proof) 1. $p \to q \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$

2. $p \vee q$

$\therefore q$

**Eg. Show that $a \vee \neg(b \wedge c)$ and $\neg(a \vee d)$, then $\neg b$.**

(Proof) 1. $a \vee \neg b$

2. $a \vee \neg c$

3. $\neg a$

4. $\neg d$

$\therefore$ $\neg b$ by combining (1) and (3), or $\neg c$ by combining (2) and (3).

**Quantifiers:** $\forall x P(x)$, $\exists x P(x)$, $\forall x \forall y\ P(x,y)$, $\exists x \exists y\ P(x,y)$, $\forall x \exists y\ P(x,y)$, etc.

**Eg. $\forall x \forall y\ ((x<y) \to (x^2<y^2))$ is true if $x>0$. However, if $x=-3$ and $y=1$, we have $x<y$ but $x^2=9>y^2=1$.**

**Eg. $\forall n\ (2^n+1$ is prime) is false if $n=3$, $2^3+1=9$ is not prime.**

**Rules for negating statement with one quantifier**

$\neg[\forall x P(x)] \Leftrightarrow \exists x \neg P(x)$

$\neg[\exists x P(x)] \Leftrightarrow \forall x \neg P(x)$

$\neg[\forall x \neg P(x)] \Leftrightarrow \exists x P(x)$

$\neg[\exists x \neg P(x)] \Leftrightarrow \forall x P(x)$

**Eg. "Each integer is greater than 0" ($\forall x,\ x>0$) is wrong. That is, "There exist some integers are not more than 0" ($\exists x,\ x \leqq 0$).**

**Eg. "There exist some squares of integers less than 0" ($\exists x,\ x^2<0$) is wrong. That is, "Each square of integer is greater than or equal to 0" ($\forall x,\ x^2 \geq 0$).**

**Proof by Mathematical Induction:** $S(1)$ is true; $\forall n$, if $S(n)$ is true, then $S(n+1)$ is also true.

**Eg. Show that $n! \geqq 2^{n-1}$.**

(Proof) $n=1$, $1!=1=2^0$

For a certain $n$, $n! \geqq 2^{n-1}$, $(n+1)!=(n+1)\cdot(n!) \geqq (n+1)\cdot 2^{n-1} \geqq 2\cdot 2^{n-1}=2^n$, we have completed the proof.

## 7-2 Set Theory and Relations

**Union of sets:** $A \cup B$, **Intersection of sets:** $A \cap B$

**Some theorems of sets:**

$A \cap \overline{A} = \phi$, $A \cup \overline{A} = U$, $A \cap A = A$, $A \cup A = A$, $A \subset B$ **is equivalent to** $\overline{B} \subset \overline{A}$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$ **(De Morgan's first Law),** $\overline{A \cup B} = \overline{A} \cap \overline{B}$ **(De Morgan's second Law)**

**Eg. Let $A$={$x$| $x$ is a white horse}, $B$={$x$| $x$ is a horse}, and then we have $A \subset B$. On the other hand, $\overline{A}$={$x$| $x$ is not a white horse}={brown horse, yellow horse, black horse, pig, monkey, lion, ⋯, etc.}, $\overline{B}$={$x$| $x$ is not a horse}={pig, monkey, lion, …, etc.}, and then we have $\overline{B} \subset \overline{A}$.**

**The principle of inclusion and exclusion:** $N(A \cup B)=N(A)+N(B)-N(A \cap B)$

**Cartesian product:** $A \times B=\{(a,b) \mid a \in A \text{ and } b \in B\}$

**Eg. $A$={1,2}, $B$={-1,0,1}, then $A \times B$={(1,-1),(1,0),(1,1),(2,-1),(2,0),(2,1)}**

***R* is a relation from *A* to *B*:** $R \subseteq A \times B$.

**Eg. $A$={1,2}, $B$={-1,0,1}, then $R$={(1,0),(2,-1),(2,0)} is a relation from $A$ to $B$. But $R$'={(1,-1),(0,5)} is not a relation from $A$ to $B$.**

***a R b*:** $(a,b) \in R$, $a$ is related to $b$.

**Binary relation:** Any subset of $A \times A$ is called a binary relation on $A$.

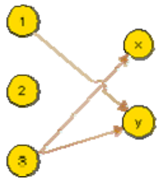
**Eg. $A$={4,5}, $A \times A$={(4,4), (4,5), (5,4), (5,5)}. {(4,4), (4,5), (5,4)}, and {(4,4), (5,5)}, {(5,4), (5,5)} are all the binary relations on $A$.**

**Matrices of Relations:** Given $M_{\alpha}=[m_{ij}]_{m \times n}$, where $m_{ij}=\begin{cases} 1, a_i \ R \ b_j \\ 0, \ \ else \end{cases}$.

**Eg. The relation $R$ from {2,3,4} to {5,6,7,8} is defined by $(x,y) \in R$={(2,6),(3,6),(2,8),(4,8)} if $x$ divides $y$. And then the matrix of the relation $R$ from {2.3,4} to {5,6,7,8} is**

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

**Eg. The matrix of the relation $R$ from {1,2,3} to {$x$,$y$} is** $\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$**, then it is represented by the following diagram.**




**Symmetric relation:** If $(x,y) \in R$, then $(y,x) \in R$.

**Antisymmetric relation:** If $(x,y) \in R$, then $(y,x) \notin R$ for $x \neq y$.

**7-3 Analysis of Some Algorithms**

**Examples of recursive algorithms**

**Eg. Computing *n* factorial.**

```
Input: n
Output n!
factorial (n) {
    if (n==0) return 1
    return n*factorial (n-1)
}
```

**Execution:** Input $n$=3. We have factorial(3)=3*factorial(2)=3*2*factorial(1) = 3*2*1*factorial(0)=6

**Eg. Robot walking**

```
Input: n
Output walk (n)
walk (n) {
    if (n==1 ∨ n==2) return n
    return walk(n-1)+walk(n-2)
}
```

**Execution:** Input $n$=4. We have walk(3)+walk(2)=walk(2)+walk(1)+walk(2) =2+1+2=5

**Examples of sorting and searching**

**Eg. Selection sort**

Input: $s_1, s_2, \ldots, s_n$ and the length $n$ of the sequence

Output: $s_1, s_2, \ldots, s_n$ arranged in nondecreasing order

```
selection_sort(s, n) {
   // base case
   if (n == 1)
      return
   // find largest
   max_index = 1 // assume initially that s_1 is largest
   for i = 2 to n
      if (s_i > s_max_index) // found larger, so update
         max_index = i
   // move largest to end
   swap(s_n, s_max_index)
   selection_sort(s, n − 1)
}
```

**Execution:** Input $s$=3, 1, 5, 4 and $n$=4. We have 3, 1, 5, 4, and then 3, 1, 4, 5, and then 1, 3, 4, 5

**Eg. Binary search**

Input: A sequence $s_i, s_{i+1}, \ldots, s_j$, sorted in nondecreasing order, a value *key*, $i$, and $j$

Output: The output is an index $k$ for which $s_k = key$, or if *key* is not in the sequence, the output is the value 0.

```
binary_search(s, i, j, key) {
   if (i > j) // not found
      return 0
   k = ⌊(i + j)/2⌋
   if (key == s_k) // found
      return k
   if (key < s_k) // search left half
      j = k − 1
   else // search right half
      i = k + 1
   return binary_search(s, i, j, key)
}
```

**Execution:** Input $s$=1, 3, 4, 6, $i$=1, $j$=4, and key=6. We have $k$=2 and key=6>$s_2$=3, and then input $s$=4, 6, $i$=3, $j$=4. We have $k$=3 and then key=6= $s_4$=6.

**Time Complexity or Computational Complexity, $O(f(n))$:** The execution time of an algorithm is dominated by $f(n)$.

**Eg. Show that $a_m n^m + a_{m-1}n^{m-1} + a_{m-2}n^{m-2} + a_{m-3}n^{m-3} + \ldots + a_1 n + a_0$ is $O(n^m)$.**

(Proof) Choose $a=\max(a_0, a_1, a_2, \ldots, a_m)$

$$\Rightarrow \left|a_m n^m + a_{m-1}n^{m-1} + a_{m-2}n^{m-2} + \cdots + a_0\right| \le |a_m|n^m + |a_{m-1}|n^{m-1} + |a_{m-2}|n^{m-2} + \cdots + |a_0|$$

$\le an^m + an^m + an^m + \cdots + an^m \le (m+1)an^m = Cn^m$, so the time complexity is $O(n^m)$.

**Eg. Show that $1^m+2^m+3^m+4^m+\ldots+n^m$ is $O(n^{m+1})$.**

(Proof) $1^m + 2^m + 3^m + \cdots + n^m \le n^m + n^m + n^m + \cdots + n^m = n \cdot n^m = n^{m+1}$, so the time complexity is $O(n^{m+1})$.