# Chapter 11 Complex Sequences and Series

## 11-1 Sequence and Series

**Sequence $\{z_n\}$:** $z_0, z_1, z_2, z_3, \ldots, z_n, \ldots$

**Series:** $\sum_{n=0} z_n = z_0 + z_1 + z_2 + \cdots + z_n + \cdots$

**Absolute convergence:** $\sum_{n=0}^{\infty} |z_n|$ converges. (In this case, $\sum_{n=0}^{\infty} z_n$ converges also.)

**Eg.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ **is an absolutely convergent series because** $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n+1)^2} \right| = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ **and** $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots$ **are both convergent.**

**Conditional convergence:** $\sum_{n=0}^{\infty} z_n$ converges but $\sum_{n=0}^{\infty} |z_n|$ diverges.

**Eg.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ **is conditionally convergent because** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$ **is convergent but** $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{2n+1} \right| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ **is divergent.**

**Radius of convergence for** $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ **:** There exists $R$ (possibly $R=\infty$) such that the series converges absolutely if $|z-z_0|<R$ and diverges if $|z-z_0|>R$.

(Proof) $\frac{|a_{n+1}(z-z_0)^{n+1}|}{|a_n (z-z_0)^n|} < 1 \Rightarrow |z - z_0| < \frac{|a_n|}{|a_{n+1}|}$

Define $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$, so the proof is complete.

**Eg. Find the radius of convergence for** $\sum_{n=0}^{\infty} \frac{n^n}{n!} (z-i)^n$ **.**

(Sol.) $\left| \dfrac{\frac{(n+1)^{n+1}}{(n+1)!}(z-i)^{n+1}}{\frac{n^n}{n!}(z-i)^n} \right| = \left| \left( \frac{n+1}{n} \right)^n (z-i) \right| < 1$

$$|z - i| < \left(1 + \frac{1}{n}\right)^{-n} \to e^{-1},\ R=e^{-1}$$

## 11-2 Complex Taylor's Series & Laurent's Series

**Taylor's series:** $f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)(z-z_0)^n$ . In case $z_0=0$, it is called Maclaurin's series.

**Eg. Find the Maclaurin's expansion of $f(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-w^2}dw$.**

(Sol.) $e^{-w^2}=1-w^2+\frac{(-w^2)^2}{2!}+\frac{(-w^2)^3}{3!}+\frac{(-w^2)^4}{4!}+\frac{(-w^2)^5}{5!}+\cdots$

$$=1-w^2+\frac{w^4}{2!}-\frac{w^6}{3!}+\frac{w^8}{4!}-\frac{w^{10}}{5!}+\cdots+(-1)^n\frac{w^{2n}}{n!}+$$

$$\int_0^z e^{-w^2}dw=z-\frac{z^3}{3}+\frac{z^5}{5\cdot 2!}-+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)n!}+\cdots$$

$$f(z)=\frac{2}{\sqrt{\pi}}\left(z-\frac{z^3}{3}+\frac{z^5}{5\cdot 2!}-+\cdots+(-1)^n\frac{z^{2n+1}}{(2n+1)n!}+\cdots\right)$$

**Laurent's series:**

$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ ,

where $a_n=\frac{1}{2\pi i}\oint_c\frac{f(w)}{(w-z_0)^{n+1}}dw$ , and $C$: $|z-z_0|=\rho$, $r_1<\rho<r_2$

**Eg. Find the Laurent's series of $f(z)=\frac{z-\sin z}{z^3}$ about $z_0=0$, indicate the type of singularity and the region of convergence of the series. 【清大電研】**

(Sol.) $f(z)=\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-\cdots\right)\right]$

$$=\frac{1}{z^3}\left[\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-\frac{z^9}{9!}+-\cdots\right]=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\frac{z^6}{9!}+-\cdots$$

$\therefore$ $z_0=0$ is a removable singularity.

$$f(z)=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\frac{z^6}{9!}+-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot z^{2n-2}}{(2n+1)!}=\sum_{n=1}^{\infty}a_n$$

$$\frac{|a_{n+1}|}{|a_n|}=\left|\frac{(-1)^{n+1}\cdot z^{2n}}{(2n+3)!}\right|\Big/\left|\frac{(-1)^n\cdot z^{2n-2}}{(2n+1)!}\right|<1$$

$z^2<(2n+3)(2n+2)\Rightarrow z<R\to\infty$

**Eg. Expand $f(z)=\frac{z}{(z-1)(2-z)}$ by Laurent's series in case: (a) $|z|<1$, (b) $1<|z|<2$, (c) $|z|>2$, (d) $|z-1|>1$, and (e) $0<|z-2|<1$.**

(Sol.) $f(z)=\frac{z}{(z-1)(2-z)}=-\frac{z}{(z-1)(z-2)}=\frac{1}{z-1}-\frac{2}{z-2}$

(a) $|z|<1 \Rightarrow 1>\frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1}=\frac{-1}{1-z}=(-1)(1+z+z^2+z^3+\cdots) \\ \quad =-1-z-z^2-z^3-z^4-\cdots \\ \frac{-2}{z-2}=\frac{2}{2-z}=\frac{1}{1-\frac{z}{2}}=1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots \end{cases}, \therefore\ f(z)=-\frac{1}{2}z-\frac{3}{4}z^2-\frac{7}{8}z^3-\frac{15}{16}z^4-\cdots$$

(b) $1<|z|<2 \Rightarrow 1>\frac{1}{|z|},\ 1>\frac{|z|}{2}$

$$\Rightarrow \begin{cases} \frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots\right) \\ \quad =\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots \\ \frac{-2}{z-2}=\frac{1}{1-\frac{z}{2}}=1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots \end{cases}, \therefore\ f(z)=\cdots+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+\frac{z}{2}+\frac{z^2}{4}+\cdots$$

(c) $|z|>2 \Rightarrow \frac{2}{|z|}<1 \Rightarrow 1>\frac{1}{|z|}$

$$\Rightarrow \begin{cases} \frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) \\ \quad =\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots \\ \frac{-2}{z-2}=-\frac{2}{z}\cdot\frac{1}{1-\frac{2}{z}}=-\frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right) \\ \quad =-\frac{2}{z}-\frac{4}{z^2}-\frac{8}{z^3}-\cdots \end{cases}, \therefore\ f(z)=-\frac{1}{z}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}-\cdots$$

(d) $|z-1|>1 \Rightarrow \frac{1}{|z-1|}<1$

$$\frac{-2}{z-2}=\frac{-2}{(z-1)-1}=-\frac{2}{z-1}\cdot\frac{1}{1-\frac{1}{z-1}}=-\frac{2}{z-1}\cdot\left[1+(z-1)^{-1}+(z-1)^{-2}+\cdots\right]=-\frac{2}{z-1}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}-\cdots$$

$$\therefore\ f(z)=\frac{1}{z-1}-\frac{2}{z-2}=\frac{-1}{z-1}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}-\frac{2}{(z-1)^4}-\cdots$$

(e) $0<|z-2|<1 \Rightarrow \frac{1}{z-1}=\frac{1}{1+(z-2)}=1-(z-2)+(z-2)^2-(z-2)^3+-\cdots$

$$\therefore\ f(z)=\frac{1}{z-1}-\frac{2}{z-2}=-\frac{2}{z-2}+1-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4-+\cdots$$

**Eg. Find the Laurent's series of $f(z)=\dfrac{1}{z^2-3z+2}$ for $1<|z|<2$. 【台大機研】**

(Sol.) $$\frac{1}{z^2-3z+2}=\frac{-1}{z-1}+\frac{1}{z-2}$$

$$1<|z|<2\Rightarrow 1>\frac{1}{|z|},\ \frac{-1}{z-1}=\frac{-1}{z}\cdot\frac{1}{1-\frac{1}{z}}=\frac{-1}{z}\cdot\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$\Rightarrow 1>\frac{|z|}{2},\quad \frac{1}{z-2}=\frac{-1}{2}\cdot\frac{1}{1-\frac{z}{2}}=\frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right]$$

$$\Rightarrow f(z)=-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

**Eg. (a) $\dfrac{1}{(z-1)(z-3)}$ for $1<|z|<3$ about $z_0=0$, (b) $\dfrac{e^z}{(z-1)^2}$ for $|z-1|>0$ about $z_0=1$. Find their respective Laurent's series. 【交大電信】**

(Sol.) (a) $$\frac{1}{(z-1)(z-3)}=\frac{-\frac{1}{2}}{z-1}+\frac{\frac{1}{2}}{z-3}$$

$$1<|z|<3\Rightarrow\frac{1}{|z|}<1,\ \frac{-\frac{1}{2}}{z-1}=\frac{-\frac{1}{2}}{z}\cdot\frac{1}{(1-\frac{1}{z})}=\frac{-\frac{1}{2}}{z}\cdot\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$1<|z|<3\Rightarrow 1>\frac{|z|}{3},\quad \frac{\frac{1}{2}}{z-3}=\frac{-\frac{1}{2}}{3}\cdot\frac{1}{(1-\frac{z}{3})}=\frac{-1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

$$\Rightarrow f(z)=-\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)-\frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)$$

(b) $$\frac{e^z}{(z-1)^2}=\frac{e\cdot e^{z-1}}{(z-1)^2}=\frac{e}{(z-1)^2}\cdot[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\ldots]$$

$$=e\cdot[\frac{1}{(z-1)^2}+\frac{1}{(z-1)}+\frac{1}{2!}+\frac{(z-1)}{3!}+\ldots]$$

## 11-3 Summation of Series by the Residue Theorem

**Theorem Let $z_1, \ldots, z_m$ be the poles of $f(z)$, then** $\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{j=1}^{m} \operatorname{Res}_{z_j}[\cot(\pi z)\cdot f(z)]$

**and** $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{j=1}^{m} \operatorname{Res}_{z_j}[\csc(\pi z)\cdot f(z)]$**.**

**Eg. (a)** $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = ?$ **(b)** $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = ?$

(Sol.) (a) $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\pi \cdot \left[\operatorname{Res}_{ai}\left(\frac{\cot(\pi z)}{z^2+a^2}\right) + \operatorname{Res}_{-ai}\left(\frac{\cot(\pi z)}{z^2+a^2}\right)\right]$

$$= -\pi \cdot \left[\frac{\cot(\pi ai)}{2ai} + \frac{\cot(-\pi ai)}{-2ai}\right] = -\frac{\pi\cot(\pi ai)}{ai} = \frac{\pi}{a}\coth(\pi a)$$

(b) $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth(\pi a)$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2}\left[\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right]$$

**Eg. Evaluate** $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$**.**

(Sol.) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = (-1)\cdot\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n-1)} = \sum_{n=-\infty}^{0} \frac{(-1)^n}{2n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \left(\cdots + \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1\right) + \left(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - + \cdots\right)$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\cdots = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{2}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\csc(\pi z)}{2z-1}\right]$$

$$= \frac{\pi}{4}\operatorname{Res}_{\frac{1}{2}}\left[\frac{\csc(\pi z)}{z-\frac{1}{2}}\right] = \frac{\pi}{4}\lim_{z\to\frac{1}{2}}\left[\frac{\csc \pi z}{z-\frac{1}{2}} \times \left(z-\frac{1}{2}\right)\right] = \frac{\pi}{4}$$

$$\therefore\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$$

**Eg. Evaluate** $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$.

(Sol.) $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n^2}=-\frac{\pi}{2}\operatorname*{Res}_0\left(\frac{\cot(\pi z)}{z^2}\right)$

$$\frac{\cot(\pi z)}{z^2}=\frac{\cos(\pi z)}{z^2\sin(\pi z)}=\frac{\left(1-\frac{\pi^2 z^2}{2!}+\frac{\pi^4 z^4}{4!}-+\cdots\right)}{z^2\left(\pi z-\frac{\pi^3 z^3}{3!}+-\cdots\right)}$$

$$=\frac{1}{\pi z^3}\left(1-\frac{\pi^2 z^2}{2}+\cdots\right)\left(1+\frac{\pi^2 z^2}{6}+\cdots\right)=\cdots-\frac{\pi}{3}\cdot\frac{1}{z}+\cdots$$

$$\therefore\ \operatorname*{Res}_0\left(\frac{\cot(\pi z)}{z^2}\right)=-\frac{\pi}{3}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\left(-\frac{\pi}{2}\right)\left(-\frac{\pi}{3}\right)=\frac{\pi^2}{6}$$

**11-4 Infinite products**

**Theorem If** $\prod_{n=1}^{\infty} a_n \ (\neq 0)$ **converges, then** $\lim_{n\to\infty} a_n = 1$.

(Proof) Let $p_N = \prod_{n=1}^{N} a_n$, $1 = \lim_{N\to\infty} p_N \Big/ \lim_{N\to\infty} p_{N-1} = \lim_{N\to\infty}\left(\prod_{n=1}^{N} a_n \Big/ \prod_{n=1}^{N-1} a_n\right) = \lim_{N\to\infty} a_N$

**Theorem** $\prod_{n=1}^{\infty}(1+a_n)$ **converges if and only if** $\sum_{n=1}^{\infty}\log(1+a_n)$ **converges.**

**Theorem** $\prod_{n=1}^{\infty}(1+a_n)$ **is absolutely convergent if and only if** $\sum_{n=1}^{\infty} a_n$ **is absolutely convergent.**

**Eg. Determine the domain *D* of convergence for** $\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2}\cdot z^n\right]$.

(Sol.) $\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2} z^n\right]$ diverges if $\ell im_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^{n^2} z^n\right| > 1$

$$\left(1+\frac{1}{n}\right)^{n^2} = e^{n^2\ln\left(1+\frac{1}{n}\right)} = e^{n\ln\left(1+\frac{1}{n}\right)^n} \to e^{n\ln e} = e^n$$

$$\lim_{n\to\infty}\left|e^n\cdot z^n\right| > 1 \Rightarrow |z| > e^{-1},\ \prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{n^2} z^n\right] \text{ diverges}$$

$\therefore\ D = \{z : |z| < e^{-1}\}$ for convergence.

**Weierstrass' theorem Let *f*(*z*) be analytic. All zeros $a_1, a_2, a_3, \ldots, a_n, \ldots$ are single and $0<|a_1|<|a_2|<|a_3|<\ldots$,** $\ell im_{n\to\infty}|a_n| = \infty$**, then *f*(*z*)=** $f(0)e^{zf'(0)/f(0)}\cdot\prod_{k=1}^{\infty}\left[(1-z/a_k)e^{z/a_k}\right]$.

**Eg. Show that cos(z)=**$\left[1-\frac{z^2}{(\pi/2)^2}\right]\left[1-\frac{z^2}{(3\pi/2)^2}\right]\left[1-\frac{z^2}{(5\pi/2)^2}\right]\cdots$.

(Proof) Zeros of $\cos(z) = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\cdots$

$$\cos(0) = 1,\ \ z[\cos(z)]'/\cos(z)_{|z=0} = 0,\ \ e^{z/a_k}\cdot e^{z/(-a_k)} = 1$$

$$\therefore\ \cos(z) = \left[1-\frac{z^2}{(\pi/2)^2}\right]\cdot\left[1-\frac{z^2}{(3\pi/2)^2}\right]\cdots$$

**Note:** $\sin(z) = z\left[1-\frac{z^2}{\pi^2}\right]\cdot\left[1-\frac{z^2}{(2\pi)^2}\right]\cdot\left[1-\frac{z^2}{(3\pi)^2}\right]\cdots$