Chapter 15 Difference Equations and the Z-Transforms

15-1 The First-order Linear Difference Equation $y_{k+1}=a_k y_k+b_k,\ a_k\neq 0$

Solution: $$\frac{1}{\prod_{i=0}^{k} a_i} y_{k+1} - \frac{a_k}{\prod_{i=0}^{k} a_i} y_k = \frac{b_k}{\prod_{i=0}^{k} a_i} \Rightarrow \frac{y_{k+1}}{\prod_{i=0}^{k} a_i} - \frac{y_k}{\prod_{i=0}^{k-1} a_i} = \frac{b_k}{\prod_{i=0}^{k} a_i}$$

Let $$A_k = \frac{y_k}{\prod_{i=0}^{k-1} a_i} \Rightarrow A_{k+1} - A_k = \frac{b_k}{\prod_{s=0}^{k} a_s},\quad \sum_{j=0}^{k-1}[A_{j+1} - A_j] = A_k - A_0 = \sum_{j=0}^{k-1} \frac{b_j}{\prod_{s=0}^{j} a_s}$$

$$\Rightarrow y_k = \prod_{i=0}^{k-1} a_i \cdot \left[A_0 + \sum_{j=0}^{k-1} \frac{b_j}{\prod_{s=0}^{j} a_s} \right]$$

Eg. Solve $y_{k+1}+2^k y_k=1$.

(Sol.) $y_{k+1} = -2^k y_k + 1$, $a_k = -2^k$, $a_0 a_1 a_2 \cdots a_k = (-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}$

$$\frac{y_{k+1}}{(-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}} = \frac{y_k}{(-1)^k \cdot 2^{\frac{(k-1)k}{2}}} + \frac{1}{(-1)^{k+1} \cdot 2^{\frac{k(k+1)}{2}}}$$

$$A_k = \frac{y_k}{(-1)^k \cdot 2^{(k-1)k/2}},\quad A_{k+1} = \frac{y_{k+1}}{(-1)^{k+1} \cdot 2^{k(k+1)/2}}$$

$$A_{k+1} - A_k = (-1)^{-1-k} \cdot 2^{-\frac{k(k+1)}{2}},\quad A_k - A_0 = \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{-\frac{i(i+1)}{2}}$$

$$A_k = A_0 + \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{-\frac{i(i+1)}{2}},\quad y_k = (-1)^k \cdot 2^{\frac{k(k-1)}{2}} \cdot \left[A_0 + \sum_{i=0}^{k-1} (-1)^{-1-i} \cdot 2^{-\frac{i(i+1)}{2}} \right]$$

Eg. Solve $y_{k+1} = \frac{k}{k+1} y_k + 4$.

(Sol.) $a_k = \frac{k}{k+1} \neq 0$ for $k\neq 0$, $a_1 a_2 \cdots\cdots a_k = \frac{1}{k+1}$

$$\frac{y_{k+1}}{1/(k+1)} - \frac{k}{k+1} \cdot \frac{y_k}{1/(k+1)} = \frac{4}{1/(k+1)},\quad (k+1)y_{k+1} - k y_k = 4(k+1)$$

Let $A_k = k y_k$, $A_{k+1} = (k+1) y_{k+1}$, $A_{k+1} - A_k = 4(k+1)$

$$\sum_{j=1}^{k-1}[A_{j+1} - A_j] = A_k - A_1 = 4 \cdot \frac{(k-1)(k+2)}{2}$$

$$A_k = A_1 + 2(k-1)(k+2) = k y_k \Rightarrow y_k = \frac{1}{k}[A_1 + 2(k-1)(k+2)]$$

Eg. Solve $y_{k+1}=ky_k$.

(Sol.) $a_k=k$, $a_1a_2\cdots a_k=k!$, $\dfrac{y_{k+1}}{k!}=\dfrac{y_k}{(k-1)!}$. Let $A_k=y_k/(k-1)!$, $A_{k+1}=y_{k+1}/k!$

$$A_{k+1}=A_k\Rightarrow A_k=A_1$$
$$\Rightarrow y_k=A_1(k-1)!$$

Eg. Solve $y_{k+1}=y_k+k^2-k$.

(Sol.) $y_{k+1}-y_k=k^2-k$, $y_k-y_0=\sum_{n=0}^{k-1}(n^2-n)=\sum_{n=0}^{k-1}n^2-\sum_{n=0}^{k-1}n$

$$=\frac{k(k-1)(2k-1)}{6}-\frac{(k-1)k}{2}=\frac{k(k-1)(k-2)}{3}$$

$$\therefore\ y_k=y_0+\frac{k(k-1)(k-2)}{3}$$

Notations: $Dy_k=y_{k+1}-y_k$, $D^2y_k=D[Dy_k]=D[y_{k+1}-y_k]=y_{k+2}-2y_{k+1}+y_k$, …,

$$D^ny_k=y_{k+n}-C_1^ny_{k+n-1}+C_2^ny_{k+n-2}-+\cdots+y_k$$

15-2 The Second-order Constant-coefficient Difference Equations

$y_{k+2}+ay_{k+1}+by_k=f_k$

Homogeneous form: $y_{k+2}+ay_{k+1}+by_k=0$

Suppose $y_k=r^k$, $r^{k+2}+ar^{k+1}+br^k=0$, $r^2+ar+b=0$

Case 1 $r=r_1,\ r_2\in R$, $r_1\neq r_2\Rightarrow y_k=c_1r_1^{\,k}+c_2r_2^{\,k}$

Eg. Solve $y_{k+2}+2y_{k+1}-3y_k=0$.

(Sol.) $r^2+2r-3=0$, $r=1,-3\Rightarrow y_k=c_1(1)^k+c_2(-3)^k=c_1+c_2(-3)^k$

Case 2 $r=r_1=r_2\Rightarrow y_k=c_1r_1^{\,k}+c_2kr_1^{\,k}$

Eg. Solve $y_{k+2}-8y_{k+1}+16y_k=0$.

(Sol.) $r^2-8r+16=0$, $r=4,4\Rightarrow y_k=c_14^k+c_2k4^k$

Case 3 $r=\alpha\pm i\beta=\rho e^{\pm i\theta}\Rightarrow y_k=c_1(\alpha+i\beta)^k+c_2(\alpha-i\beta)^k$

$$=\rho^k[d_1\cos(k\theta)+d_2\sin(k\theta)]$$

Eg. Solve $y_{k+2}-2y_{k+1}+2y_k=0$.

(Sol.) $r^2-2r+2=0$, $r=1\pm i=\sqrt{2}e^{\pm j\frac{\pi}{4}}\Rightarrow y_k=\left(\sqrt{2}\right)^k\left[c_1\cos\left(\frac{k\pi}{4}\right)+c_2\sin\left(\frac{k\pi}{4}\right)\right]$

Eg. Given the following recurrence equation: $r_n=r_{n-1}+2r_{n-2}$ with $r_0=1$, $r_1=1$. (a) Please answer the values of r_2,and r_{10}. (b) Please find the general expression for r_n in terms of n. [台大電研]

(Sol.) (b) Set n-2=m, n=m+2, and then reset m=n. We have $r_{n+2}-r_{n+1}-2r_n=0$.

$r^2-r-2=0$, r=2, -1$\Rightarrow r_n=c_1 2^n+c_2(-1)^n$

$r_0=1$, $r_1=1\Rightarrow c_1=2/3$, $c_2=1/3\Rightarrow r_n=\frac{2}{3}\cdot 2^n+\frac{1}{3}\cdot(-1)^n$

(a) $r_2=\frac{2}{3}\cdot 2^2+\frac{1}{3}\cdot(-1)^2=3$ and $r_{10}=\frac{2}{3}\cdot 2^{10}+\frac{1}{3}\cdot(-1)^{10}=683$

Method of undetermined coefficients:

A particular solution $y_k^{(p)}$ is chosen by these rules:

1. f_k is a polynomial, $y_k^{(p)}$ may be a polynomial.

2. f_k is a combination of sinusoidal functions, so is $y_k^{(p)}$.

3. f_k is an exponential function, so is $y_k^{(p)}$.

Eg. Solve $y_{k+2}-7y_{k+1}+12y_k=-16+12k$.

(Sol.) $y_{k+2}-7y_{k+1}+12y_k=0\Rightarrow y_k^{(h)}=c_1 3^k+c_2 4^k$

Let $y_k^{(p)}=A+Bk$, $y_{k+1}=A+B(k+1)$, $y_{k+2}=A+B(k+2)$

$$y_{k+2}-7y_{k+1}+12y_k=A+B(k+2)-7[A+B(k+1)]+12[A+Bk]=-16+12k$$

$$6A-5B+6Bk=-16+12k$$

$$\Rightarrow \begin{matrix}6A-5B=-16\\6B=12\end{matrix}\Rightarrow\begin{matrix}A=-1\\B=2\end{matrix}\Rightarrow y_k^{(p)}=-1+2k$$

$$\therefore\ y_k=c_1 3^k+c_2 4^k-1+2k$$

Eg. Solve $y_{k+2}+8y_{k+1}+12y_k=e^k$.

(Sol.) $y_k^{(h)}=c_1(-2)^k+c_2(-6)^k$

Let $y_k{}^{(p)}=Ae^k$, $y_{k+1}=Ae^{k+1}=Ae\cdot e^k$, $y_{k+2}=Ae^{k+2}=Ae^2\cdot e^k$

$$y_{k+2}^{(p)}+8y_{k+1}^{(p)}+12y_k^{(p)}=A(e^2+8e+12)e^k=e^k$$

$$\Rightarrow A=\frac{1}{e^2+8e+12}\Rightarrow y_k^{(p)}=\frac{e^k}{e^2+8e+12}$$

$$\therefore\ y_k=c_1(-2)^k+c_2(-6)^k+\frac{e^k}{e^2+8e+12}$$

Eg. Solve y_{k+2}-4y_k=sin(k).

(Sol.) $y_k^{(h)}=c_1 2^k+c_2(-2)^k$. Let $y_k^{(p)}=A\sin(k)+B\cos(k)$

$$\Rightarrow A\sin(k+2)+B\cos(k+2)-4A\sin(k)-4B\cos(k)$$

$$=[A\cos(2)-B\sin(2)-4A]\sin(k)+[A\sin(2)+B\cos(2)-4B]\cos(k)=\sin(\mathrm{k})$$

$$\Rightarrow \begin{cases} A\cos(2)-B\sin(2)-4A=1 \\ A\sin(2)+B\cos(2)-4B=0 \end{cases} \Rightarrow \begin{matrix} A=\dfrac{\cos(2)-4}{17-8\cos(2)} \\ B=\dfrac{-\sin(2)}{17-8\cos(2)} \end{matrix}$$

$$\therefore\ y_k=c_1 2^k+c_2(-2)^k+\frac{1}{17-8\cos(2)}[(\cos(2)-4)\sin(k)-\sin(2)\cos(k)]$$

Eg. Solve y_{k+2}+3y_{k+1}-4y_k=20k.

(Sol.) $y_{k+2}+3y_{k+1}-4y_k=0\Rightarrow y_k^{(h)}=c_1\cdot 1^k+c_2(-4)^k=c_1+c_2(-4)^k$

1. $y_k^{(p)}=A+Bk$, A and B are independent of k.

   $\Rightarrow A+B(k+2)+3A+3B(k+1)-4A-4Bk=20k$

   $\Rightarrow B=4k$ is not a constant.

2. $y_k{}^{(p)}=A+Bk+Ck^2$

$$\Rightarrow A+B(k+2)+C(k+2)^2+3A+3B(k+1)+3C(k+1)^2-4A-4Bk-4Ck$$

$$=(5B+7C)+10Ck=20k$$

$$\Rightarrow \begin{matrix} 5B+7C=0 \\ 10C=20 \end{matrix} \Rightarrow \begin{matrix} B=-\dfrac{14}{5} \\ C=2 \end{matrix} \Rightarrow \mathrm{y_k}^{(\mathrm{p})}=-\frac{14}{5}\mathrm{k}+2\mathrm{k}^2$$

$$\therefore\ y_k=c_1+c_2(-4)^k-\frac{14}{5}k+2k^2$$

Eg. Solve y_{k+2}-4y_{k+1}+4y_k=3 · 2^k+5 · 4^k. [清大化工所]

(Ans.) $y_k=c_1 2^k+c_2 k2^k+\dfrac{3}{8}k^2 2^k+\dfrac{5}{4}4^k$

Variation-of-parameter method to find the particular solutions:

For a homogeneous second-order linear difference equation, if $y_k^{(1)}$ and $y_k^{(2)}$ are solution of it, then $c_1 y_k^{(1)} + c_2 y_k^{(2)}$ is also a solution for arbitrary c_1 and c_2.

Wronskian determinant: $W_k = \begin{vmatrix} y_k^{(1)} & y_k^{(2)} \\ y_{k+1}^{(1)} & y_{k+1}^{(2)} \end{vmatrix} = y_k^{(1)} y_{k+1}^{(2)} - y_k^{(2)} y_{k+1}^{(1)}$

Determine the particular solution of $y_{k+2} + a_k y_{k+1} + b_k y_k = f_k$.

Suppose $y_k^{(h)} = c_1 y_k^{(1)} + c_2 y_k^{(2)}$, then let $y_k^{(p)} = A_k y_k^{(1)} + B_k y_k^{(2)}$

$$\begin{aligned} y_{k+1}^{(p)} &= A_{k+1} y_{k+1}^{(1)} + B_{k+1} y_{k+1}^{(2)} \\ &= A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)} + [y_{k+1}^{(1)}(A_{k+1} - A_k) + y_{k+1}^{(2)}(B_{k+1} - B_k)] \\ &= A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)} + y_{k+1}^{(1)} DA_k + y_{k+1}^{(2)} DB_k \end{aligned}$$

Impose the condition $y_{k+1}^{(1)} DA_k + y_{k+1}^{(2)} DB_k = 0 \cdots\cdots(1)$

$$\Rightarrow y_{k+1}^{(p)} = A_k y_{k+1}^{(1)} + B_k y_{k+1}^{(2)}$$

$$\begin{aligned} y_{k+2}^{(p)} &= A_{k+1} y_{k+2}^{(1)} + B_{k+1} y_{k+2}^{(2)} \\ &= y_{k+2}^{(1)}[A_{k+1} - A_k] + y_{k+2}^{(2)}[B_{k+1} - B_k] + A_k y_{k+2}^{(1)} + B_k y_{k+2}^{(2)} \\ &= y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k + A_k y_{k+2}^{(1)} + B_k y_{k+2}^{(2)} \end{aligned}$$

$$\begin{aligned} &\Rightarrow y_{k+2}^{(p)} + a_k y_{k+1}^{(p)} + b_k y_k^{(p)} \\ &= y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k + A_k[y_{k+2}^{(1)} + a_k y_{k+1}^{(1)} + b_k y_k^{(1)}] + B_k[y_{k+2}^{(2)} + a_k y_{k+1}^{(2)} + b_k y_k^{(2)}] \\ &= f_k \end{aligned}$$

$$\Rightarrow y_{k+2}^{(1)} DA_k + y_{k+2}^{(2)} DB_k = f_k \cdots\cdots(2)$$

by (1) & (2): $\begin{cases} DA_k = A_{k+1} - A_k = -\dfrac{f_k y_{k+1}^{(2)}}{W_{k+1}} \\ DB_k = B_{k+1} - B_k = \dfrac{f_k y_{k+1}^{(1)}}{W_{k+1}} \end{cases}$

$\Rightarrow \begin{cases} A_k - A_0 = \displaystyle\sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \\ B_k - B_0 = \displaystyle\sum_{j=0}^{k-1} \frac{f_j j_{j+1}^{(1)}}{W_{j+1}} \end{cases} \Rightarrow \begin{cases} A_k = \displaystyle\sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \\ B_k = \displaystyle\sum_{j=0}^{k-1} \frac{f_j y_{j+1}^{(1)}}{W_{j+1}} \end{cases}$ if we choose $A_0 = B_0 = 0$.

$$\Rightarrow y_k = c_1 y_k^{(1)} + c_2 y_k^{(2)} + \left(\sum_{j=0}^{k-1} \frac{-f_j y_{j+1}^{(2)}}{W_{j+1}} \right) \cdot y_k^{(1)} + \left(\sum_{j=0}^{k-1} \frac{f_j y_{j+1}^{(1)}}{W_{j+1}} \right) \cdot y_k^{(2)}$$

Eg. Solve $y_{n+2}+8y_{n+1}+7y_n=ne^n$.

(Sol.) $y_n^{(h)} = c_1(-1)^n + c_2(-7)^n$

$$W_{n+1} = \begin{vmatrix} (-1)^{n+1} & (-7)^{n+1} \\ (-1)^{n+2} & (-7)^{n+2} \end{vmatrix} = -6\cdot 7^{n+1}$$

$$-f_j y_{j+1}^{(2)} = -je^j\cdot(-7)^{j+1},\quad f_j y_{j+1}^{(1)} = je^j\cdot(-1)^{j+1}$$

$$-f_j y_{j+1}^{(2)}\Big/W_{j+1} = \frac{-je^j(-7)^{j+1}}{-6\cdot 7^{j+1}} = \frac{je^j(-7)^{j+1}}{6\cdot 7^{j+1}}$$

$$f_j y_{j+1}^{(1)}\Big/W_{j+1} = \frac{je^j(-1)^{j+1}}{-6\cdot 7^{j+1}} = \frac{-je^j(-1)^{j+1}}{6\cdot 7^{j+1}}$$

$$\therefore\quad y_n = c_1(-1)^n + c_2(-7)^n + \left[\sum_{j=0}^{n-1}\frac{je^j(-7)^{j+1}}{6\cdot 7^{j+1}}\right]\cdot(-1)^n + \left[\sum_{j=0}^{n-1}\frac{-je^j(-1)^{j+1}}{6\cdot 7^{j+1}}\right]\cdot(-7)^n$$

15-3 Cauchy-Euler Difference Equation $k(k+1)D^2y_k+akDy_k+by_k=0$

Solution: Let $y_k = \dfrac{\Gamma(k+r)}{\Gamma(k)} = k(k+1)\cdots\cdots(k+r-1)$, where $\Gamma(n+1)=n!$ and n is an integer.

$$Dy_k = \frac{\Gamma(k+r+1)}{\Gamma(k+1)} - \frac{\Gamma(k+r)}{\Gamma(k)} = \frac{(k+r)\Gamma(k+r)}{k\Gamma(k)} - \frac{\Gamma(k+r)}{\Gamma(k)} = \frac{r}{k}\frac{\Gamma(k+r)}{\Gamma(k)}$$

$$D^2y_k = Dy_{k+1} - Dy_k = y_{k+2} - 2y_{k+1} + y_k$$

$$= \frac{\Gamma(k+r+2)}{\Gamma(k+2)} - 2\cdot\frac{\Gamma(k+r+1)}{\Gamma(k+1)} + \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$= \left[\frac{(k+r+1)(k+r)}{(k+1)k} - 2\cdot\frac{(k+r)}{k} + 1\right]\cdot\frac{\Gamma(k+r)}{\Gamma(k)} = \frac{r(r-1)}{k(k+1)}\cdot\frac{\Gamma(k+r)}{\Gamma(k)}$$

$$\Rightarrow k(k+1)D^2y_k + akDy_k + by_k = (r^2 - r + ar + b)\frac{\Gamma(k+r)}{\Gamma(k)} = 0$$

$$\Rightarrow r^2 + (a-1)r + b = 0 \Rightarrow \text{ Solve } r.$$

Eg. Solve $k(k+1)D^2y_k-5kDy_k+8y_k=0$.

(Sol.) $a=-5$, $b=8 \Rightarrow r^2 - 6r + 8 = 0 \Rightarrow r = 2, 4$

$$\frac{\Gamma(k+2)}{\Gamma(k)} = k(k+1),\quad \frac{\Gamma(k+4)}{\Gamma(k)} = k(k+1)(k+2)(k+3),$$

$$\therefore\quad y_k = c_1k(k+1) + c_2k(k+1)(k+2)(k+3)$$

15-4 The Z-transform and Its Application

Discrete Laplace transform: $L[f^*(t)] = L[f(nT)] = \sum_{n=0}^{\infty} f(nT)e^{-nTs}$

Let $Z=e^{Ts}$, and define Z-transform as $F(Z)=z[f(t)]= z[f^*(t)]=\sum_{n=0}^{\infty} f(nT)Z^{-n}$

Properties of Z-transform $z[f(t)]=F(Z)$:

1. $z[c_1 f_1(t)+c_2 f_2(t)]=c_1 F_1(Z)+c_2 F_2(Z)$

2. $z[f(t+T)]=Z[F(Z)-f(0^+)]$, $z[f(t+2T)]=Z^2F(Z)-Z^2f(0^+)-Zf(T)$

$z[f(t+nT)]= Z^n\left[F(Z)-\sum_{k=0}^{n-1} f(kT)Z^{-k}\right] = Z^nF(Z)-Z^nf(0)-Z^{n-1}f(T)-Z^{n-2}f(2T)$
$-Z^{n-3}f(3T)-\cdots-Zf((n-1)T)$

(Proof) $z[f(t+nT)] = \sum_{m=0}^{\infty} f(mT+nT)Z^{-m} = Z^n\sum_{m=0}^{\infty} f(mT+nT)Z^{-(m+n)}$

$$= Z^n\sum_{k=n}^{\infty} f(kT)Z^{-k} = Z^n\left[\sum_{k=0}^{\infty} f(kT)Z^{-k} - \sum_{k=0}^{n-1} f(kT)Z^{-k}\right] = Z^n\left[F(Z)-\sum_{k=0}^{n-1} f(kT)\cdot Z^{-k}\right]$$

3. $z[f(t-T)]=Z^{-1}F(Z)$, $z[f(t-nT)u(t-nT)]=Z^{-n}F(Z)$

(Proof) $z[f(t-nT)u(t-nT)] = \sum_{m=0}^{\infty} f(mT-nT)u(mT-nT)Z^{-m}$

$$= Z^{-n}\sum_{m=0}^{\infty} f(mT-nT)u(mT-nT)Z^{-(m-n)} = Z^{-n}\sum_{k=0}^{\infty} f(kT)Z^{-k} = Z^{-n}F(Z)$$

4. $z[e^{-at}f(t)]=F(e^{aT}Z)$

(Proof) $z[e^{-at}f(t)]=\sum_{n=0}^{\infty} e^{-naT}f(nT)Z^{-n} = \sum_{n=0}^{\infty} f(nT)(e^{+aT}Z)^{-n} = F(e^{aT}Z)$

5. $z\left[\sum_{k=0}^{n} f(kT)\right] = \frac{Z}{Z-1}F(Z)$ **for $|Z|>1$**

6. $z[tf(t)]=-TZ\frac{d}{dZ}F(Z)$, $z[t^kf(t)] = -TZ\frac{d}{dZ}\{z[t^{k-1}f(t)]\}$

(Proof)

$$z[tf(t)] = \sum_{n=0}^{\infty} nTf(nT)Z^{-n} = -TZ\sum_{n=0}^{\infty} f(nT)(-nZ^{-n-1}) = -TZ\frac{d}{dZ}\left[\sum_{n=0}^{\infty} f(nT)\cdot Z^{-n}\right] = -TZ\frac{d}{dZ}F(Z)$$

7. $f(0^{+})=\lim_{Z\to\infty} F(Z)$: the initial-value theorem

$\lim_{n\to\infty} f(nT)=\lim_{Z\to 1}(Z-1)F(Z)$ **: the final-value theorem**

(Proof) $F(Z)=\sum_{n=0}^{\infty} f(nT)Z^{-n}=f(0T)+\frac{f(T)}{Z}+\frac{f(2T)}{Z^2}+\cdots,\ \therefore\ \ell im_{Z\to\infty} F(Z)=f(0)$

$$z[f(t+T)-f(t)]=\ell im_{n\to\infty}\sum_{k=0}^{n}[f((k+1)T)-f(kT)]Z^{-k}=ZF(Z)-Zf(0)-F(Z)$$

$$\lim_{Z\to 1}\lim_{n\to\infty}\sum_{k=0}^{n}[f(k+1)T)-f(kT)]Z^{-k}=\lim_{n\to\infty}\{[f(T)-f(0)]+[f(2T)-f(T)]+\cdots\cdots+[f(n+1)T)-f(nT)]\}$$

$$=\lim_{n\to\infty}[-f(0)+f(n+1)T)]\Rightarrow \lim_{n\to\infty} f(nT)=\lim_{n\to\infty} f((n+1)T)=\lim_{Z\to 1}(Z-1)F(Z)$$

8. $z\left[\frac{\partial}{\partial a}f(t,a)\right]=\frac{\partial}{\partial a}F(Z,a)$, $z\left[\int_{a_1}^{a_2} f(t,a)da\right]=\int_{a_1}^{a_2} F(Z,a)da$

Eg. Determine $z[1]$, $z[t]$, $z[a^{t}]$, $z[e^{-at}]$, $z[\sin(at)]$, and $z[\cos(at)]$.

(Sol.) $f(t)=1\Rightarrow f(nT)=1,\ \forall n$

$$z[1]=\sum_{n=0}^{\infty} f(nT)Z^{-n}=\sum_{n=0}^{\infty} Z^{-n}=\frac{1}{1-\frac{1}{Z}}=\frac{Z}{Z-1}$$

$$z[t]=z[nT]=\sum_{n=0}^{\infty} nTZ^{-n}=(-TZ)\sum_{n=0}^{\infty}(-nZ^{-n-1})=-TZ\sum_{n=0}^{\infty}\frac{d}{dZ}(Z^{-n})=-TZ\frac{d}{dZ}\left(\sum_{n=0}^{\infty} Z^{-n}\right)$$

$$=-TZ\frac{d}{dZ}\left(\frac{Z}{Z-1}\right)=-TZ\cdot\frac{Z-1-Z}{(Z-1)^2}=\frac{TZ}{(Z-1)^2}$$

$$z[a^{t}]=z[a^{nT}]=\sum_{n=0}^{\infty} a^{nT}\cdot Z^{-n}=\sum_{n=0}^{\infty}\left(\frac{a^{T}}{Z}\right)^{n}=\frac{1}{1-\frac{a^{T}}{Z}}=\frac{Z}{Z-a^{T}}$$

$$z[e^{-at}]=z[e^{-anT}]=\sum_{n=0}^{\infty} e^{-naT}\cdot Z^{-n}=\sum_{n=0}^{\infty}(e^{aT}Z)^{-n}=\frac{1}{1-(e^{aT}Z)^{-1}}=\frac{Z}{Z-e^{-aT}}$$

$$z[e^{+jat}]=z[e^{+janT}]=z[\cos(anT)]+jz[\sin(anT)]$$

$$=\frac{Z}{Z-e^{+jaT}}=\frac{Z(Z-e^{-jaT})}{(Z-e^{+jaT})(Z-e^{-jaT})}$$

$$=\frac{Z^2-Z\cos(aT)+jZ\sin(aT)}{Z^2-(e^{jaT}+e^{-jaT})Z+1}=\frac{Z^2-Z\cos(aT)+jZ\sin(aT)}{Z^2-2Z\cos(aT)+1}$$

$$\therefore\ z[\sin(at)]=z[\sin(naT)]=\frac{Z\sin(aT)}{Z^2-2Z\cos(aT)+1}$$

$$z[\cos(at)]=z[\cos(naT)]=\frac{Z^2-Z\cos(aT)}{Z^2-2Z\cos(aT)+1}$$

Table of the Z-transform pairs: $T\neq1$

$f(nT)$	$F(Z)$
$\delta(nT)$	1
1 or $u(t)$	$\frac{Z}{Z-1}$
nT	$\frac{TZ}{(Z-1)^2}$
e^{-naT}	$\frac{Z}{Z-e^{-aT}}$
a^{nT}	$\frac{Z}{Z-a^T}$
$\sin(naT)$	$\frac{Z\sin(aT)}{Z^2-2Z\cos(aT)+1}$
$\cos(naT)$	$\frac{Z^2-Z\cos(aT)}{Z^2-2Z\cos(aT)+1}$

Table of the Z-tranasform pairs: $T=1$

$f(n)$	$F(Z)$
$\delta(n)$	1
1	$\frac{Z}{Z-1}$
n	$\frac{Z}{(Z-1)^2}$
e^{-na}	$\frac{Z}{Z-e^{-a}}$
a^n	$\frac{Z}{Z-a}$

Inverse Z-transform: If $F(Z)/Z=\frac{A_1}{Z-p_1}+\frac{A_2}{Z-p_2}+\cdots+\frac{A_k}{Z-p_k}$, and then we have $f(n)=A_1p_1^n+A_2p_2^n+\ldots+A_kp_k^n$

Eg. For $F(Z)=\frac{Z^2}{(Z-1)(Z-0.5)}$, $f(n)=?$

(Sol.) $\frac{F(Z)}{Z}=\frac{Z}{(Z-1)(Z-0.5)}=\frac{2}{Z-1}-\frac{1}{Z-0.5}\Rightarrow f(n)=2\cdot1^n-\left(\frac{1}{2}\right)^n=2-\left(\frac{1}{2}\right)^n$

Eg. Solve a_{n+1}-0.5a_n =1, a_0=1.

(Sol.) Let $a_n=f(n)$, $z[a_n]=F(Z)$, $z[a_{n+1}]=ZF(Z)-Zf(0)=ZF(Z)-Z$, $a_0=f(0)=1$

$$ZF(Z)-Z-\frac{1}{2}F(Z)=\frac{Z}{Z-1},\quad \left(Z-\frac{1}{2}\right)F(Z)=\frac{Z^2}{Z-1}$$

$$\frac{F(Z)}{Z}=\frac{Z}{(Z-1)\left(Z-\frac{1}{2}\right)}=\frac{2}{Z-1}-\frac{1}{Z-\frac{1}{2}}\Rightarrow f(n)=2-\left(\frac{1}{2}\right)^n=a_n$$

Eg. Solve a_n-0.5a_{n-1} =1, a_0=2.

(Sol.) Let $m=n-1$, $n=m+1\Rightarrow a_{m+1}-\frac{1}{2}a_m=1\Rightarrow a_{n+1}-\frac{1}{2}a_n=1$,

$$z(a_{n+1})-\frac{1}{2}z(a_n)=z(1)\Rightarrow ZF(Z)-2Z-\frac{1}{2}F(Z)=\frac{Z}{Z-1},$$

$$\left(Z-\frac{1}{2}\right)F(Z)=\frac{Z+2Z^2-2Z}{Z-1}=\frac{2Z^2-Z}{Z-1},\quad \frac{F(Z)}{Z}=\frac{2}{Z-1}\Rightarrow f(n)=a_n=2$$

Eg. Solve $r_n=r_{n-1}+2r_{n-2}$ with $r_0=1$, $r_1=1$.

(Sol.) Let $m=n-2$, $n=m+2$ $\Rightarrow$ $r_{m+2}-r_{m+1}-2r_m=0$ $\Rightarrow$ $r_{n+2}-r_{n+1}-2r_n=0$, $z(r_{n+2})-z(r_{n+1})-2z(r_n)=0$.

By $z(r_n)=R(Z)$, $z(r_{n+1})=Z[R(Z)-f(0^+)]$, $z(r_{n+2})=Z^2R(Z)-Z^2f(0^+)-Zf(T)$, and define $r_0=f(0^+)$, $r_1=f(T)\Rightarrow[Z^2R(Z)-Z^2r_0-Zr_1]-[ZR(Z)-Zr_0)]-2R(Z)=0$

$$\Rightarrow[Z^2-Z-2]R(Z)=Z^2,\quad \frac{R(Z)}{Z}=\frac{Z}{Z^2-Z-2}=\frac{\frac{2}{3}}{Z-2}+\frac{\frac{1}{3}}{Z+1},\quad \therefore\ r_n=\frac{2}{3}\cdot 2^n+\frac{1}{3}\cdot(-1)^n$$