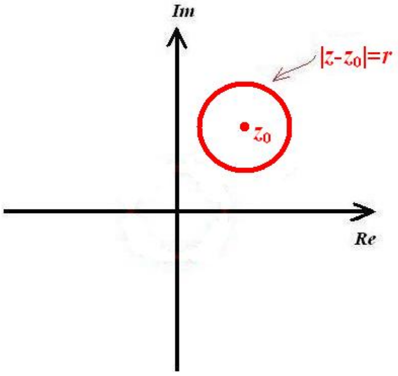
Chapter 10 Integration in the Complex Plane

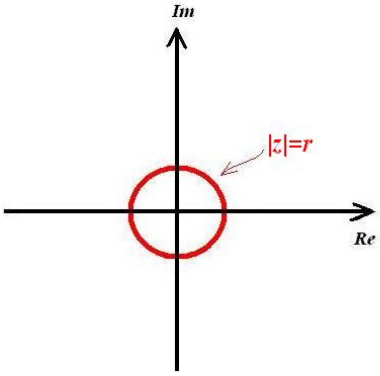
10-1 Complex Line Integrals and Some Integral Theorems

For smooth curve C: $z=z(t)$ for $a \leqq t \leqq b$, then $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$

Special case 1 C: $|z-z_0|=r \Leftrightarrow z(t)=z_0+re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$

Special case 2 C: $|z|=r \Leftrightarrow z(t)=re^{it}$, $dz=z'(t)dt=ire^{it}dt$, and $0 \leqq t \leqq 2\pi$





Eg. Find $\oint_c \frac{1}{z} dz$ **,** ***C*****: |*z*|=1.**

(Sol.) $|z|=1 \Leftrightarrow z=e^{it}$, $0 \leqq t \leqq 2\pi$, $z'(t)= ie^{it}$, $\oint_c \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$

Eg. Evaluate $\oint_C \frac{dz}{z-3i}$ **,** ***C*****:** $|z-3i|=\frac{1}{3}$**.**

(Sol). $z(t)=3i+\frac{1}{3}e^{it}$, $z'(t)=\frac{1}{3}ie^{it}$, $\oint_C \frac{dz}{z-3i} = \int_0^{2\pi} \frac{1}{\frac{1}{3}e^{it}} \cdot \frac{i}{3}e^{it} dt = 2\pi i$

Eg. Evaluate $\oint_c \bar{z} dz$ **,** ***C*****: |*z*|=1.**

(Sol.) $z(t)=e^{it}$, $\bar{z}=e^{-it}$, $z'(t)= ie^{it}$, $\oint_c \bar{z} dz = \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$

Eg. Evaluate $\oint_c [z - R_e(z)]dz$ **,** ***C*****: |*z*|=2.**

(Sol.) $|z|=2 \Leftrightarrow z(t)=2e^{it}$, $z'(t)= i2e^{it}$

$$R_e(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(2e^{it} + 2e^{-it}), \quad z - R_e(z) = \frac{1}{2}(z-\bar{z}) = \frac{1}{2}(2e^{it} - 2e^{-it})$$

$$\oint_c [z - R_e(z)]dz = \int_0^{2\pi} [z(t) - R_e(z(t))] \cdot z'(t)dt$$

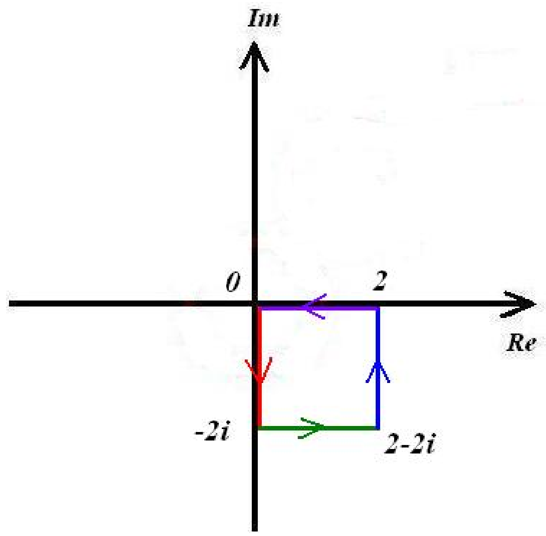
$$= \int_0^{2\pi} \frac{2}{2}(e^{it} - e^{-it}) \cdot 2ie^{it} dt = 2i\int_0^{2\pi} (e^{2it} - 1)dt = -4\pi i$$

Eg. Evaluate $\oint_C [z^2 + I_m(z)]dz$ **, where** ***C*** **is the square with 0, -2*i*, 2-2*i*, and 2.**

(Sol.) $\oint_C [z^2 + I_m(z)]dz = \int_0^{-2} (-t^2 + t)i\,dt + \int_0^{2} [(t-2i)^2 - 2]dt + \int_{-2}^{0} [(2+it)^2 + t]i\,dt$

$$+\int_2^0 (t^2 + 0)dt = i\left(-\frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_0^{-2} + \left(\frac{t^3}{3} - 2it^2 - 6t\right)\Bigg|_0^{2}$$

$$+i\left(4t + 2it^2 - \frac{t^3}{3} + \frac{t^2}{2}\right)\Bigg|_{-2}^{0} + \frac{t^3}{3}\Bigg|_2^0 = -4$$



Cauchy's integral theorem **Let *f*(*z*) be analytic in a simply-connected domain *D*, *C* is a simple closed curve in *D*, then** $\oint_C f(z)dz = 0$.

Eg. Evaluate $\oint_C \frac{1}{z}dz$ **, *C*: |z-2|=1.**

(Sol.) *f*(*z*)=$\frac{1}{z}$ is analytic except *z*=0. No poles are within *C*, ∴ $\oint_C \frac{dz}{z} = 0$

Eg. Evaluate $\oint_{|z|=1} \frac{dz}{z^2-4}$**.** 【1991 交大土木所】

(Sol.) *f*(*z*)=$\frac{1}{z^2-4}$ is analytic except *z*=±2. No poles are within *C*, ∴ $\oint_{|z|=1} \frac{dz}{z^2-4}$=0

Eg. Evaluate $\oint_C \frac{z}{\sin(z)(z-2i)^3}dz$ **, *C*: |z-8*i*|=1.**

(Sol.) $f(z) = \frac{z}{\sin(z)(z-2i)^3}$ is analytic except *z*=2*i*, *n*π (*n*=0, ±1, ±2, …)

No poles are within *C*, ∴ $\oint f(z)dz = 0$

Cauchy's integral formulae **Let $f(z)$ be analytic in a simply-connected region D, and let C be a simple curve enclosing z_0 in D, then** $\oint_c \frac{f(z)}{z-z_0}dz = 2\pi i \cdot f(z_0)$ **and** $\oint_c \frac{f(z)dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!}\cdot f^{(n-1)}(z_0)$.

Eg. Evaluate $\oint_{|z|=3} \frac{e^{2z}dz}{z-2}$. **【1991 交大土木所】** (Sol.) $\oint_{|z|=3} \frac{e^{2z}}{z-2} = 2\pi i \cdot e^{2\times 2} = 2\pi i e^4$

Eg. Evaluate $\oint_c \frac{e^{iz}}{z^3}dz$**, C: $|z|=3$.** (Sol.) $\oint_c \frac{e^{iz}}{z^3}dz = i2\pi \cdot \frac{(e^{iz})''}{(3-1)!}\Big|_{z_0=0} = -i\pi$

Eg. Evaluate $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz$ **and** $\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz$ **if** $C: \left|z-\frac{\pi}{6}\right| = \delta > 0$.

(Sol.) Let $f(z)=\sin^6(z)$ and $n=3$, $\oint_c \frac{\sin^6(z)}{z-\frac{\pi}{6}}dz = 2\pi i \cdot \sin^6\left(\frac{\pi}{6}\right) = \frac{\pi i}{32}$,

$\oint_c \frac{\sin^6(z)}{\left(z-\frac{\pi}{6}\right)^3}dz = \frac{2\pi i}{2!}\cdot[\sin^6(z)]''\Big|_{z_0=\frac{\pi}{6}} = \frac{21\pi i}{16}$

Eg. Let z_0 be within C, find $\oint_c \frac{dz}{z-z_0}$ **and** $\oint_c \frac{dz}{(z-z_0)^n}$**, $n \geqq 2$.**

(Sol.) Let $f(z)=1$, $f^{(n-1)}(z_0)=0 \Rightarrow \oint \frac{dz}{z-z_0} = 2\pi i$ and $\oint_c \frac{dz}{(z-z_0)^n} = 0$.

Eg. Evaluate $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz$**, C is a closed curve not passing 1.**

(Sol.) If C does not enclose 1, $\frac{2\sin(z^2)}{(z-1)^4}$ is analytic within C, $\therefore$ $\oint_c \frac{2\sin(z^2)}{(z-1)^4}dz = 0$

If C encloses 1, let $f(z) = 2\sin(z^2) \Rightarrow \frac{2\sin(z^2)}{(z-1)^4} = \frac{f(z)}{(z-1)^4}$, $n=4$, $n-1=3$

$f^{(3)}(z) = -24z\sin(z^2) - 16z^3\cos(z^2)$

$\therefore \oint_c \frac{2\sin(z^2)}{(z-1)^4}dz = \frac{2\pi i}{3!}[-24\sin(1) - 16\cos(1)] = \frac{\pi i}{3}[-24\sin(1) - 16\cos(1)]$

10-2 Laurent's Theorem & the Residue Theorem

If $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \underbrace{\frac{a_{-3}}{(z-z_0)^3}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{(z-z_0)}}_{(principal\ part)} \underbrace{+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots}_{(analytic\ part)}$

Laurent's theorem $a_n=\frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$

Residue: $a_{-1}=\frac{1}{2\pi i}\oint_c f(z)dz$

Residue theorem **Let $f(z)$ be analytic in D except $z_1, z_2, \ldots, z_n$ and C encloses $z_1, z_2, \ldots, z_n$ within D. Then we have** $\oint_c f(z)dz = 2\pi i \cdot \sum_{j=1}^{n} \operatorname{Re}_{z_j} s(f)$ **and**

$\operatorname{Re}_{z_j} s(f)=\frac{1}{(m-1)!}\lim_{z\to z_j}\frac{d^{m-1}}{dz^{m-1}}[(z-z_j)^m\cdot f(z)]$, **where m is the order of a pole $z=z_j$.**

In case of m=1, $\operatorname{Re}_{z_j} s(f)=\lim_{z\to z_j}[(z-z_j)\cdot f(z)]$.

Eg. Find the residue of $f(z)=\frac{\sin(z)}{(z-i)^3}$ **and evaluate** $\oint_c \frac{\sin(z)}{(z-i)^3}dz$ **, C: $|z-i|=2$.**

(Sol.) m=3, $\operatorname{Re}_{i} s(f)=\frac{1}{(3-1)!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\cdot\frac{\sin(z)}{(z-i)^3}]=\frac{-1}{2}\sin(i)=-\frac{1}{2}i\sinh(1)$

$$\therefore \oint \frac{\sin(z)}{(z-i)^2}dz = 2\pi i\cdot\left(-\frac{1}{2}i\sinh(1)\right)=\pi\sinh(1)$$

Eg. Evaluate $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ **for (a) C: $|z|=\frac{1}{3}$, (b) C: $|z-1|=\frac{1}{3}$, (c) C: $|z|=2$. 【1991 台大機研】**

(Sol.) (a) There is only one pole 0 within C.

$\operatorname{Re}_{0} s(f)=\frac{1}{(2-1)!}\lim_{z\to 0}\frac{d}{dz}[z^2\cdot\frac{\cos(z)}{z^2(z-1)}]=\left.\frac{-(z-1)\sin(z)-\cos(z)}{(z-1)^2}\right|_{z=0}$ =-1,

$\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =-2πi

(b) There is only one pole 1 within C.

$\operatorname{Re}_{1} s(f)=\lim_{z\to 1}[(z-1)\cdot\frac{\cos(z)}{z^2(z-1)}]$=cos(1), $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =2πi · cos(1)

(c) There are two poles 0 and 1 within C. $\oint_c \frac{\cos(z)}{z^2(z-1)}dz$ =2πi · [-1+cos(1)]

Eg. Evaluate $\oint_c \frac{z^2+1}{z^2-1}dz$**, *C*: |*z*-1|=1. 【1991 交大科管所】**

(Sol.) There is only one pole 1 within *C*.

$$\operatorname{Re}_1 s(f)=\lim_{z\to 1}[(z-1)\cdot\frac{z^2+1}{z^2-1}]=\lim_{z\to 1}\frac{z^2+1}{z+1}=1,\ \therefore\ \oint_c \frac{z^2+1}{z^2-1}dz=2\pi i\cdot 1=2\pi i$$

Eg. Evaluate $\oint_c \tan z dz$**, *C*: |*z*|=2. [1993 中山電研]**

(Sol.) There are two poles ±π/2 within *C*.

$$\operatorname{Re}_{\frac{\pi}{2}} s(f)=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to\frac{\pi}{2}}[(z-\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to\frac{\pi}{2}}\frac{z\sin(z)-\frac{\pi}{2}\sin(z)}{\cos(z)}=$$

$$\lim_{z\to\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)-\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$$\operatorname{Re}_{-\frac{\pi}{2}} s(f)=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\tan(z)]=\lim_{z\to-\frac{\pi}{2}}[(z+\frac{\pi}{2})\cdot\frac{\sin(z)}{\cos(z)}]=\lim_{z\to-\frac{\pi}{2}}\frac{z\sin(z)+\frac{\pi}{2}\sin(z)}{\cos(z)}=$$

$$\lim_{z\to-\frac{\pi}{2}}\frac{\sin(z)+z\cos(z)+\frac{\pi}{2}\cos(z)}{-\sin(z)}=-1$$

$$\oint_c \tan z dz=2\pi i\cdot[(-1)+(-1)]=-4\pi i$$

Eg. Evaluate $\oint_c \frac{\sin(z)}{z^2(z^2+4)}dz$**, *C* is any piecewise-smooth curve enclosing 0, 2*i*, and –2*i*.**

(Sol.) $f(z)=\frac{\sin(z)/z}{z(z+2i)(z-2i)}$, $\because \lim_{z\to 0}[\sin(z)/z]=1$, $\therefore$ *f*(*z*) has a removable singularity at $0\Rightarrow m$ of the pole *z*=0 in *f*(*z*) is 1.

$$\operatorname{Re}_0 s(f)=\lim_{z\to 0} z\cdot\frac{\sin(z)}{z^2(z^2+4)}=\frac{1}{4}$$

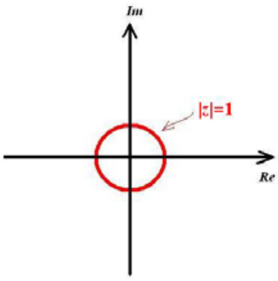
$$\operatorname{Re}_{2i} s(f)=\lim_{z\to 2i}(z-2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\operatorname{Re}_{-2i} s(f)=\lim_{z\to -2i}(z+2i)\cdot\frac{\sin(z)}{z^2(z+2i)(z-2i)}=\frac{i}{16}\sin(2i)=-\frac{1}{16}\sinh(2)$$

$$\oint_c f(z)dz=2\pi i\left[\frac{1}{4}-\frac{1}{16}\sinh(2)-\frac{1}{16}\sinh(2)\right]=\frac{\pi i}{2}-\frac{\pi i}{4}\sinh(2)$$

Eg. Evaluate $\oint_c \frac{z^3 e^{\frac{1}{z}}}{z^3+1}dz$**, *C*: |*z*|=3. [2013 中山電研]**

10-3 Evaluation of Real Integrals



Case 1 $\int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta$

Choose C: $|z|=1$, $z=e^{i\theta}$ $\Rightarrow \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$,

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $d\theta = \frac{dz}{iz}$

$$\Rightarrow I = \int_0^{2\pi} K[\cos\theta, \sin\theta]d\theta = \oint_c K\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]$$

Eg. Evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\cos(\theta)}$.

(Sol.) $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \oint_c \frac{dz/iz}{5-3\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)} = \oint_c \frac{2idz}{3z^2-10z+3} = \oint_c \frac{2idz}{3(z-\frac{1}{3})(z-3)}$

$= 2\pi i \cdot \operatorname{Res}_{\frac{1}{3}}(f) = 2\pi i \cdot \lim_{z\to\frac{1}{3}} (z-\frac{1}{3}) \cdot \frac{2i}{3(z-\frac{1}{3})(z-3)} = 2\pi i \cdot \frac{2i}{-8} = \frac{\pi}{2}$.

Eg. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ **if $a>|b|$.**

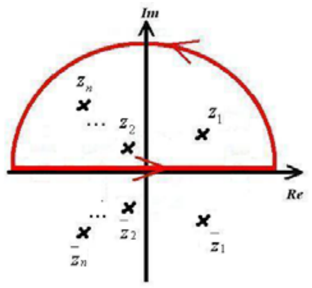
(Proof) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_c \frac{dz/iz}{a+b\cdot\frac{1}{2i}\left(z-\frac{1}{z}\right)} = \oint_c \frac{2dz}{bz^2+2iaz-b} = \oint_c \frac{2dz}{b(z-z_1)(z-z_2)}$

Poles: $z_1 = \frac{i}{b}\left(-a+\sqrt{a^2-b^2}\right)$ is within C: $|z|=1$, but $z_2 = \frac{i}{b}\left(-a-\sqrt{a^2-b^2}\right)$ is not.

$\operatorname{Res}_{z_1}(f) = \lim_{z\to z_1}(z-z_1)\cdot\frac{2}{b(z-z_1)(z-z_2)} = \frac{2}{b}\cdot\frac{1}{z_1-z_2} = \frac{2}{b}\cdot\frac{b}{2i\sqrt{a^2-b^2}} = \frac{1}{i\sqrt{a^2-b^2}}$

$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = 2\pi i \cdot \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Eg. Evaluate $\int_0^{2\pi} \frac{dt}{a^2\cos^2(t)+b^2\sin^2(t)}$. **[2011 成大電研]**



Case 2 $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$

Choose C as a semi-circle with infinite radius enclosing the upper half-plane. Poles: z_1, z_2, …, z_n are in the upper half-plane, $\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n$ are in the lower half-plane. Assume *deg*(q) ≥ *deg*(p)+2, then

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \oint_c f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z_j}(f)$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{xdx}{(x^2-2x+2)^2}$ **[1991中山電研]**

(Sol.) Poles: 1+i (upper half-plane), 1-i (lower half-plane)

$$\operatorname*{Res}_{1+i}\left[\frac{z}{(z^2-2z+1)^2}\right] = \frac{1}{(2-1)!} \lim_{z\to z_j} \frac{d^{2-1}}{dz^{2-1}} \left\{[z-(1+i)]^2 \cdot \frac{z}{[z-(1+i)]^2 \cdot [z-(1-i)]^2}\right\}$$

$$= \lim_{z\to z_j} \frac{d}{dz}\left\{\frac{z}{[z-(1-i)]^2}\right\} = \left.\frac{[z-(1-i)]^2 - 2z[z-(1-i)]}{[z-(1-i)]^4}\right|_{1+i} = \frac{-i}{4},$$

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2-2x+2)^2} = 2\pi i \cdot \operatorname*{Res}_{1+i}\left[\frac{z}{(z^2-2z+1)^2}\right] = 2\pi i \cdot \left(\frac{-i}{4}\right) = \pi/2.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+64}dx$.

(Sol.) (a) Poles: 8i (upper half-plane), -8i (lower half-plane)

$$\operatorname*{Res}_{8i}(f) = \lim_{z\to 8i}(z-8i) \cdot \frac{1}{(z+8i)(z-8i)} = \frac{1}{16i}, \quad \int_{-\infty}^{\infty} \frac{1}{x^2+64}dx = 2\pi i \cdot \frac{1}{16i} = \frac{\pi}{8}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

(Sol.) Poles: $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$ (upper half-plane), $e^{\frac{5\pi i}{4}}$, $e^{\frac{7\pi i}{4}}$ (lower half-plane)

$$\operatorname*{Res}_{e^{i\pi/4}}(f) = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{z-e^{\frac{i\pi}{4}}}{1+z^4}\right] = \lim_{z\to e^{\frac{i\pi}{4}}}\left[\frac{1}{4z^3}\right] = \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4}e^{-3\pi i/4} = \frac{1}{4}\left[-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right],$$

$$\operatorname*{Res}_{e^{i3\pi/4}}(f) = \frac{1}{4\left(e^{i3\pi/4}\right)^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}\left[\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right], \quad \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i\left[\frac{1}{4}\cdot(-i\sqrt{2})\right] = \frac{\pi\sqrt{2}}{2}.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx$ **and** $\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx$.

(Sol.) Poles: 2*i*, 3*i* (upper half-plane), -2*i*, -3*i* (lower half-plane)

$$f(z)=\frac{e^{iz}}{(z^2+4)(z^2+9)},\quad \operatorname*{Res}_{2i}(f)=\frac{e^{-2}}{20i},\quad \operatorname*{Res}_{3i}(f)=\frac{-e^{-3}}{30i}$$

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x^2+9)}dx=2\pi i\left(\frac{e^{-2}}{20i}-\frac{e^{-3}}{30i}\right)=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx=\frac{\pi}{5}\left(\frac{e^{-2}}{2}-\frac{e^{-3}}{3}\right),\quad \int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx=0.$$

Eg. Evaluate $\int_{-\infty}^{\infty}\frac{x\sin(\sqrt{3}x)}{x^2+16}dx$.

(Sol.) Poles: 4*i* (upper half-plane), -4*i* (lower half-plane)

$$f(z)=\frac{ze^{i\sqrt{3}z}}{z^2+16},\quad \operatorname*{Res}_{4i}(f)=\frac{4ie^{-4\sqrt{3}}}{8i}=\frac{e^{-4\sqrt{3}}}{2},\quad \int_{-\infty}^{\infty}\frac{xe^{i\sqrt{3}x}}{x^2+16}dx=2\pi i\cdot\frac{e^{-4\sqrt{3}}}{2}=\pi ie^{-4\sqrt{3}}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x\cos(\sqrt{3}x)}{x^2+16}dx=0 \quad\text{and}\quad \int_{-\infty}^{\infty}\frac{x\sin(\sqrt{3}x)}{x^2+16}dx=\pi e^{-4\sqrt{3}}.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx$.

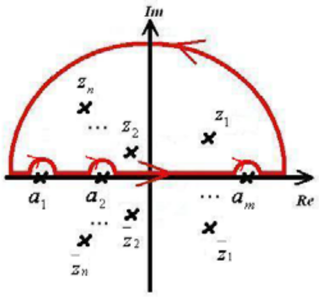
(Sol.) $\int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx$. Poles: *i* (upper half-plane), -*i* (lower half-plane). $f(z)=\frac{z^2}{(z^2+1)^2}=\frac{z^2}{(z+i)^2(z-i)^2}$, *m* of $(z-i)^2$ is 2.

$$\lim_{z\to i}\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-i)^2\cdot\frac{z^2}{(z+i)^2(z-i)^2}\right]=\frac{2z(z+i)^2-2z^2(z+i)}{(z+i)^4}\Big|_{z=i}=\frac{-8i+4i}{16}=-\frac{i}{4}$$

$$\therefore \int_{0}^{\infty}\frac{x^2}{(x^2+1)^2}dx=\frac{2\pi i}{2}\left(-\frac{i}{4}\right)=\frac{\pi}{4}.$$

Eg. Evaluate $\int_{0}^{\infty}\frac{x\sin(x)}{x^2+4}dx$. **【1991 交大電信所】** (Ans.) $\frac{\pi e^{-2}}{2}$

Eg. Evaluate $\int_{0}^{\infty}\frac{\cos(ax)}{x^2+b^2}dx$, $\begin{matrix}a\ge 0\\ b>0\end{matrix}$. **【1991 台大機研】**



Case 3 $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}dx$ **or** $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cdot \begin{Bmatrix} \cos ax \\ \sin ax \end{Bmatrix} dx$.

Some poles of $q(z)$ are located on the real axis.

Choose C as a semi-circle with infinite radius enclosing the upper half-plane, but excluding the poles on the real axis. Let z_k $(1 \le k \le n)$ be the pole on the upper half-plane and a_j $(1 \le j \le m)$ be the pole on the real axis. Then we have $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \cdot \sum_{k=1}^{n} \operatorname{Res}_{z_k}(f) + \pi i \cdot \sum_{j=1}^{m} \operatorname{Res}_{a_j}(f)$.

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x(x^2 - 4x + 5)} dx$. **[2003交大電信所]**

(Sol.) $f(z) = \frac{1}{z(z^2 - 4z + 5)} = \frac{1}{z[z - (2+i)][z - (2-i)]}$ has 3 poles:

0 (on the real axis), $2+i$ (upper half-plane), $2-i$ (lower half-plane)

$$\int_{-\infty}^{\infty} \frac{1}{x(x^2 - 4x + 5)} dx = 2\pi i \cdot \operatorname{Res}_{2+i}(f) + \pi i \cdot \operatorname{Res}_{0}(f)$$

$$= 2\pi i \cdot \lim_{z \to 2+i} [[z - (2+i)] \cdot \frac{1}{z[z - (2+i)][z - (2-i)]}] + \pi i \cdot \lim_{z \to 0} [z \cdot \frac{1}{z(z^2 - 4z + 5)}]$$

$$= \frac{2\pi i}{(2+i) \cdot 2i} + \frac{\pi i}{5} = \frac{\pi(2-i)}{5} + \frac{\pi i}{5} = \frac{2\pi}{5}.$$

Eg. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^3 + 1}$.

(Sol.) Poles: $e^{\frac{\pi i}{3}}$ (upper half-plane), -1 (on the real axis), $e^{\frac{5\pi i}{3}}$ (lower half-plane)

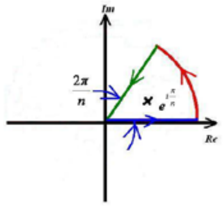
$$\operatorname{Res}_{e^{\frac{i\pi}{3}}}(f) = \lim_{z \to e^{\frac{i\pi}{3}}} [\frac{z - e^{\frac{i\pi}{3}}}{z^3 + 1}] = \lim_{z \to e^{\frac{i\pi}{3}}} [\frac{1}{3z^2}] = \frac{1}{3(e^{\pi i/3})^2} = \frac{1}{3} e^{-2\pi i/3} = \frac{1}{3}\left[-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right],$$

$$\operatorname{Res}_{-1}(f) = \frac{1}{3(-1)^2} = \frac{1}{3}, \quad \int_{-\infty}^{\infty} \frac{dx}{x^3 + 1} = 2\pi i \left[\frac{1}{6}(-1 - i\sqrt{3})\right] + \pi i \left[\frac{1}{3}\right] = \frac{\pi\sqrt{3}}{3}.$$

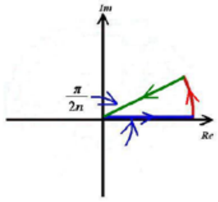
Eg. Evaluate $\int_{-\infty}^{\infty} \frac{\cos(\frac{\pi x}{2})}{x - 2} dx$ **and** $\int_{-\infty}^{\infty} \frac{\sin(\frac{\pi x}{2})}{x - 2} dx$.

(Sol.) Pole: 2 (on the real axis), $\oint_c \frac{e^{i\frac{\pi z}{2}}}{z - 2} dz = \pi i \cdot \operatorname{Res}_{2}[\frac{e^{i\frac{\pi z}{2}}}{z - 2}] = \pi i \cdot e^{i\frac{\pi}{2} \cdot 2} = \pi i \cdot e^{i\pi} = -\pi i$,

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(\frac{\pi x}{2})}{x - 2} dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{2} x)}{x - 2} dx = -\pi.$$



Case 4 $\int_0^\infty \begin{Bmatrix} \sin(x^n) \\ \cos(x^n) \end{Bmatrix} dx$ **or** $\int_0^\infty G(x^n)dx$



Choose C as a sector with angle $\frac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$ or a sector with angle $\frac{\pi}{2n}$ enclosing no poles.

Eg. Evaluate $\int_0^\infty \frac{dx}{1+x^n}$**,** $n>1$**.**

(Sol.) Choose C as a sector with angle $\frac{2\pi}{n}$ enclosing only one pole at $e^{i\frac{\pi}{n}}$.

$$\oint_c \frac{dz}{1+z^n} = 2\pi i \cdot \operatorname*{Res}_{e^{i\frac{\pi}{n}}}(f) = 2\pi i \cdot \lim_{z\to e^{i\frac{\pi}{n}}}\left[\left(z-e^{i\frac{\pi}{n}}\right)\frac{1}{1+z^n}\right] = \frac{2\pi i}{nz^{n-1}}\bigg|_{z=e^{i\frac{\pi}{n}}} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$= \int_0^R \frac{dx}{1+x^n} + \int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} + \int_R^0 \frac{e^{i\frac{2\pi}{n}}dR}{1+R^n e^{i2\pi}}$$

As $R\to\infty$, $\int_0^{\frac{2\pi}{n}} \frac{iRe^{i\theta}\,d\theta}{1+R^n e^{in\theta}} \to 0$ ($\because$ $n>1$)

$$\therefore \int_0^\infty \frac{dx}{1+x^n} + \int_\infty^0 \frac{e^{i\frac{2\pi}{n}}dR}{1+R^n} = \left(1-e^{i\frac{2\pi}{n}}\right)\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}e^{i\frac{\pi}{n}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^n} = \frac{e^{i\frac{\pi}{n}}}{1-e^{i\frac{2\pi}{n}}}\cdot\frac{-2\pi i}{n} = \frac{1}{\frac{e^{i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}}}{2i}}\cdot\frac{\pi}{n} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$

Eg. Show that $\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$.

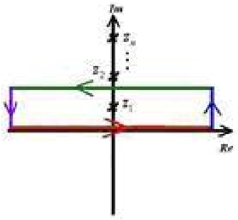
(Proof) Choose C as a sector with angle $\frac{\pi}{4}$ enclosing no poles.

$$\oint_c e^{iz^2}dz = 0 = \int_0^R e^{ix^2}dx + \int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta + \int_R^0 e^{iR^2e^{i\frac{\pi}{2}}}\cdot e^{i\frac{\pi}{4}}dR$$

As $R\to\infty$, $\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}\cdot iRe^{i\theta}\,d\theta = iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos2\theta+i\sin2\theta)}e^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin2\theta}e^{i(\theta+R^2\cos2\theta)}d\theta \to 0$,

$$\int_0^\infty [\cos(x^2)+i\sin(x^2)]dx = \int_0^\infty\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}i}{2}\right)e^{-R^2}dR = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{\pi}}{2}+\frac{\sqrt{2}}{2}\frac{\sqrt{\pi}}{2}i$$

$$\therefore \int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$



Case 5 $\int_{-\infty}^{\infty} G(e^x)dx$, **where** $G(x)=\dfrac{p(x)}{x^n+q_{n-1}(x)}$

Choose C as an infinitely-wide rectangle and there is one pole on the imaginary axis.

Eg. Evaluate $\int_{-\infty}^{\infty} \dfrac{e^{mx}}{1+e^x}dx$, **0<*m*<1.** **【1991台大機研】**

(Sol.) Pole: $i\pi$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = \int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx + \int_0^{2\pi}\frac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy + \int_{\infty}^{-\infty}\frac{e^{mx}\cdot e^{i2m\pi}}{1+e^x\cdot e^{i2\pi}}dx + \int_{2\pi}^{0}\frac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy$$

$$= 2\pi i\cdot \underset{\pi i}{\mathrm{Re}\,s}(f) = 2\pi i\cdot \lim_{z\to\pi i}\frac{(z-\pi i)e^{mz}}{1+e^z}$$

$$= 2\pi i\cdot \lim_{z\to\pi i}\frac{e^{mz}+m(z-\pi i)e^{mz}}{e^z} = 2\pi i(-1)^{m-1} = 2\pi i e^{i(m-1)\pi}$$

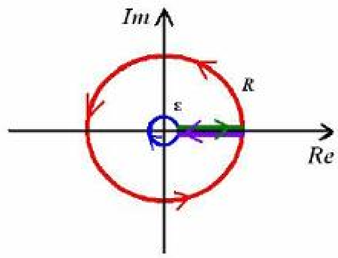
$\because$ 0<*m*<1, $\therefore R\to\infty$, $\int_0^{2\pi}\dfrac{e^{mR}\cdot e^{imy}}{1+e^R\cdot e^{iy}}idy \to 0$ and $\int_{2\pi}^{0}\dfrac{e^{-mR}\cdot e^{imy}}{1+e^{-R}\cdot e^{iy}}idy \to 0$

$$\oint_c \frac{e^{mz}}{1+e^z}dz = 2\pi i e^{i(m-1)\pi} = \int_{-\infty}^{\infty}(1-e^{i2m\pi})\cdot\frac{e^{mx}}{1+e^x}dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{mx}}{1+e^x}dx = \frac{2\pi i}{1-e^{i2m\pi}}\cdot e^{im\pi}\cdot(-1) = \frac{\pi}{\dfrac{e^{im\pi}-e^{-im\pi}}{2i}} = \frac{\pi}{\sin(m\pi)}$$

***Case 6* Other types**

Eg. For 0<*p*<1, $\int_0^\infty \frac{x^p\,dx}{x(1+x)}=?$



(Sol.) ∵ $0<p<1$, ∴ Poles are 0 and -1.

$$\oint \frac{z^p\,dz}{z(1+z)} = 2\pi i\cdot \operatorname*{Res}_{-1}(f) = 2\pi i\cdot \lim_{z\to -1}(z+1)\cdot\frac{z^p}{z(z+1)} = 2\pi i\cdot e^{i\pi(p-1)}\text{, and}$$

$$\oint \frac{z^p\,dz}{z(1+z)} = \int_\varepsilon^R \frac{x^p\,dx}{x(1+x)} + \int_0^{2\pi}\frac{i(\mathrm{R}e^{i\theta})^p\,d\theta}{1+\mathrm{R}e^{i\theta}} + \int_R^\varepsilon \frac{(xe^{i2\pi})^{p-1}d(xe^{i2\pi})}{1+xe^{i2\pi}} + \int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p\,d\theta}{1+\varepsilon e^{i\theta}}$$

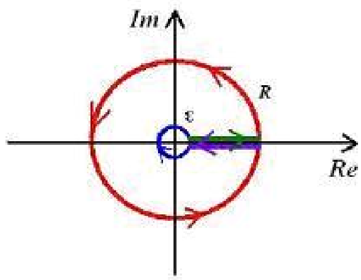
∵ $0<p<1$, $R\to\infty$, ∴ $\int_0^{2\pi}\frac{i(\mathrm{R}e^{i\theta})^p\,d\theta}{1+\mathrm{R}e^{i\theta}}\to 0$

∵ $\varepsilon\to 0$, ∴ $\int_{2\pi}^0 \frac{i(\varepsilon e^{i\theta})^p\,d\theta}{1+\varepsilon e^{i\theta}}\to 0$

$$\Rightarrow \oint \frac{z^p\,dz}{z(1+z)} = \int_0^\infty \frac{x^p\,dx}{x(1+x)} + \int_\infty^0 \frac{(xe^{i2\pi})^{p-1}d(xe^{i2\pi})}{1+xe^{i2\pi}} = \int_0^\infty \frac{[1-e^{i2\pi(p-1)}]\cdot x^p\,dx}{x(1+x)}$$

$$\therefore \int_0^\infty \frac{x^p\,dx}{x(1+x)} = \frac{2\pi i\cdot e^{i\pi(p-1)}}{1-e^{i2\pi(p-1)}} = \frac{\pi}{(e^{ip\pi}-e^{-ip\pi})/2i} = \frac{\pi}{\sin(p\pi)}$$

Eg. For -1<*a*<1, $\int_0^\infty \frac{x^a\,dx}{(1+x)^2}=?$ **【交大電信研究所、2003 中央光電所】**



(Sol.) −1 is a multiple-order pole.

$$\oint \frac{z^a\,dz}{(1+z)^2} = 2\pi i\cdot \operatorname*{Res}_{-1}(f) = 2\pi i\cdot \lim_{z\to -1}\frac{1}{1!}d[(z+1)^2\cdot\frac{z^a}{(z+1)^2}]/dz = 2\pi i\cdot a e^{i\pi(a-1)}\text{, and}$$

$$\oint \frac{z^a\,dz}{(1+z)^2} = \int_\varepsilon^R \frac{x^a\,dx}{(1+x)^2} + \int_0^{2\pi}\frac{(\mathrm{R}e^{i\theta})^a\,i\mathrm{R}e^{i\theta}\,d\theta}{(1+\mathrm{R}e^{i\theta})^2} + \int_R^\varepsilon \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a\,i\varepsilon e^{i\theta}\,d\theta}{(1+\varepsilon e^{i\theta})^2}$$

∵ $-1<a<1$, $R\to\infty$, ∴ $\int_0^{2\pi}\frac{(\mathrm{R}e^{i\theta})^a\,i\mathrm{R}e^{i\theta}\,d\theta}{(1+\mathrm{R}e^{i\theta})^2}\to 0$

∵ $\varepsilon\to 0$, ∴ $\int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^a\,i\varepsilon e^{i\theta}\,d\theta}{(1+\varepsilon e^{i\theta})^2}\to 0$

$$\Rightarrow \oint \frac{z^a\,dz}{(1+z)^2} = \int_0^\infty \frac{x^a\,dx}{(1+x)^2} + \int_\infty^0 \frac{(xe^{i2\pi})^a d(xe^{i2\pi})}{(1+xe^{i2\pi})^2} = \int_0^\infty \frac{[1-e^{i2\pi(a+1)}]\cdot x^a\,dx}{(1+x)^2}$$

$$\therefore \int_0^\infty \frac{x^a\,dx}{(1+x)^2} = \frac{2\pi i\cdot a e^{i\pi(a-1)}}{1-e^{i2\pi(a+1)}} = \frac{\pi a}{(e^{i\pi a}-e^{-i\pi a})/2i} = \frac{a\pi}{\sin(a\pi)}$$