

Chapter 13 Special Functions

13-1 Gamma & Beta Functions

Gamma function: $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$

Theorem $\Gamma(x+1) = x\Gamma(x)$, $\Gamma'(x) = \int_0^{\infty} \ln(t) \cdot t^{x-1} e^{-t} dt$

(Proof)
$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} t^x e^{-t} dt = -\int_0^{\infty} t^x de^{-t} \\ &= -t^x e^{-t} \Big|_{t=0}^{t=\infty} + \int_0^{\infty} e^{-t} dt^x = x \int_0^{\infty} t^{x-1} e^{-t} dt = x\Gamma(x) \end{aligned}$$

Theorem $\Gamma(n) = (n-1)!$, $\Gamma(1) = 0! = 1$.

Theorem $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} = \int_0^{\infty} \frac{t^{x-1}}{1+t} dt$

Theorem $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma'(1) = \int_0^{\infty} e^{-t} \ln t dt = -\gamma = -0.5772156\dots$

$(\gamma = \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right])$

(Proof)
$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 - \frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Stirling's formula: $\Gamma(n+1) = n! \approx \sqrt{2\pi n} n^n e^{-n}$

Beta function: $\beta(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Eg. Evaluate $\int_0^1 \frac{dz}{\sqrt{1-z^4}}$. 【1991 清大材研】

(Sol.) Let $t=z^4$, $dt=4z^3 dz$, $\int_0^1 \frac{dz}{\sqrt{1-z^4}} = \int_0^1 \frac{\frac{1}{4}t^{-\frac{3}{4}} dt}{\sqrt{1-t}} = \frac{1}{4} \int_0^1 t^{\frac{1}{4}-1} \cdot (1-t)^{\frac{1}{2}-1} dt = \frac{1}{4} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$

Theorem $\beta(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}(\theta) \cos^{2n-1}(\theta) d\theta$

Theorem $\beta(x,y) = \beta(y,x)$

13-2 Bessel Functions

Bessel differential equation of order ν : $x^2y''+xy'+(x^2-\nu^2)y=0$, its general solution is $y(x)=c_1J_\nu(x)+c_2Y_\nu(x)$.

Bessel differential equation of order ν with parameter λ : $x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0$, its solution is $y(x)=c_1J_\nu(\lambda x)+c_2Y_\nu(\lambda x)$.

Bessel function of the 1st kind: $J_\nu(x)=\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{\nu+2k}}{k! \Gamma(\nu+k+1)}$

Note: $J_{-n}(x)=(-1)^n J_n(x)$, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, But $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent for $\nu \notin \mathbb{N}$.

Orthogonality of $J_n(x)$:

If $\lambda \neq \mu$ and $J_n(\lambda)=J_n(\mu)=0$, then $\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = 0$

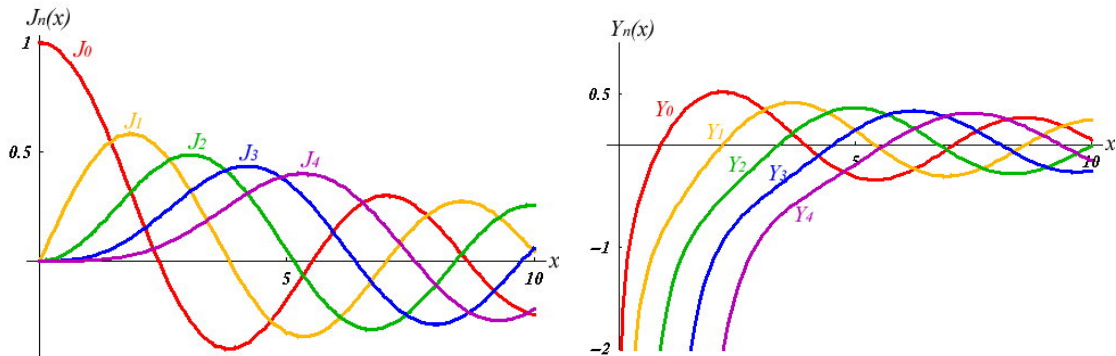
Else if $\lambda \neq \mu$ but $J_n(\lambda) \neq 0 \neq J_n(\mu)$, then

$$\int_0^1 x J_n(\lambda x) J_n(\mu x) dx = \frac{\mu J_n(\lambda) J_n'(\mu) - \lambda J_n(\mu) J_n'(\lambda)}{\lambda^2 - \mu^2}$$

Else if $\lambda = \mu$, $J_n(\lambda) \neq 0$, then $\int_0^1 x J_n^2(\lambda x) dx = \frac{1}{2} \left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda) \right]$

Bessel function of the 2nd kind:

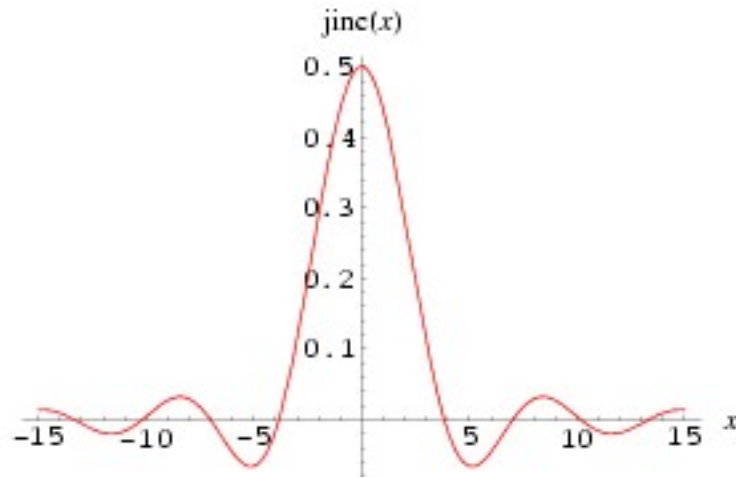
$$Y_\nu(x) = \begin{cases} \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, & \nu \text{ is not an integer} \\ \lim_{\nu \rightarrow n} \frac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, & \nu \text{ is an integer } n. \end{cases}$$



Hankel functions: $\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) : \text{the 1st kind} \\ H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) : \text{the 2nd kind} \end{cases}$

Note: $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are linearly independent.

Jinc function: $jinc(x) = \frac{J_1(x)}{x}$ and $jinc'(x) = -\frac{J_2(x)}{x}$



Properties of Bessel functions:

1. $B'_\nu(x) = \frac{1}{2}[B_{\nu-1}(x) - B_{\nu+1}(x)]$

2. $[x^\nu B_\nu(x)]' = -x^\nu B_{\nu-1}(x)$

3. $[x^{-\nu} B_\nu(x)]' = -x^{-\nu} B_{\nu+1}(x)$

4. $B_{\nu+1}(x) = \frac{2\nu}{x} B_\nu(x) - B_{\nu-1}(x)$

5. $x B'_\nu(x) = -\nu B_\nu(x) + x B_{\nu-1}(x) = \nu B_\nu(x) - x B_{\nu+1}(x)$

6. $J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y)$

7. **Generating function $g(x,t)$ for $J_n(x)$:** $g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n$

$$g(x,t) = g\left(x, \frac{-1}{t}\right) \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

8. $\cos(x \sin \theta) = J_0(x) + 2 \sum_{m=1}^{\infty} J_{2m}(x) \cdot \cos(2m\theta)$

$$\sin(x \sin \theta) = 2 \sum_{m=1}^{\infty} J_{2m-1}(x) \cdot \sin((2m-1)\theta)$$

$$1 = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) = J_0^2(x) + 2 \sum_{n=1}^{\infty} J_n^2(x)$$

$$\cos(x \cos \theta) = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) \cos(2n\theta)$$

$$\sin(x \cos \theta) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(x) \cos((2n-1)\theta)$$

$$9. J_{2n}(x) = \frac{1}{\pi} \int_0^\pi \cos(2n\theta) \cdot \cos(x \sin \theta) d\theta = \frac{(-1)^n}{\pi} \int_0^\pi \cos(2n\theta) \cdot \cos(x \cos \theta) d\theta$$

$$J_{2n+1}(x) = \frac{1}{\pi} \int_0^\pi \sin((2n+1)\theta) \cdot \sin(x \sin \theta) d\theta = \frac{(-1)^n}{\pi} \int_0^\pi \cos((2n+1)\theta) \cdot \sin(x \cos \theta) d\theta$$

$$10. \quad J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \quad \text{as } x \rightarrow \infty$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

Spherical Bessel functions:

$$J_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2}{\pi}} x^{n+\frac{1}{2}} \left(\frac{1}{x} \frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \quad J_{\frac{1}{2}-n}(x) = \sqrt{\frac{2}{\pi}} x^{n-\frac{1}{2}} \left(\frac{1}{x} \frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

$$Y_{n+\frac{1}{2}}(x) = (-1)^{n+1} J_{-\frac{1}{2}-n}(x), \quad Y_{\frac{1}{2}-n}(x) = (-1)^n J_{n+\frac{1}{2}}(x)$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Modified Bessel functions:

$$I_\nu(x) = (i)^{-\nu} J_\nu(ix): \text{ the 1st kind}$$

$$K_\nu(x) = \frac{\pi}{2 \sin(\nu\pi)} [I_{-\nu}(x) - I_\nu(x)]: \text{ the 2nd kind, where } \nu \text{ is non-integer}$$

13-3 Legendre Differential Equations and Legendre Functions

Legendre equation: $(1-x^2)y''-2xy'+n(n+1)y=0$, its general solution is $y(x)=c_1P_n(x)+c_2Q_n(x)$.

Legendre functions (polynomial) of degree n of the 1st kind:

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k! (n-2k)! (n-k)!}$$

$$= \frac{(2n-1)(2n-3)\cdots 1}{n!} \left\{ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} + \cdots \right\}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

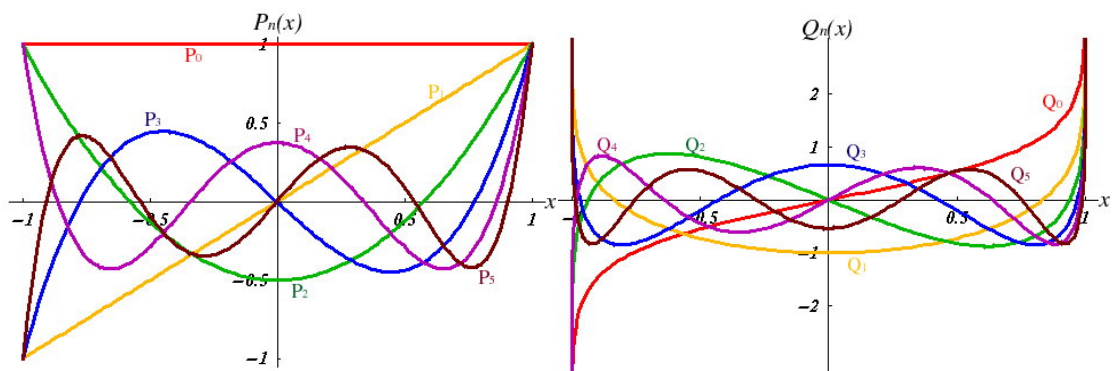
Orthogonality of $P_n(x)$: $\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1} \delta_{mn}$

Legendre functions (polynomial) of degree n of the 2nd kind:

For $|x| < 1$,

$$Q_n(x) = \begin{cases} \frac{(-1)^{n/2} 2^n \left[\left(\frac{n}{2} \right)! \right]^2}{n!} \cdot \left[x + \sum_{k=1}^{\infty} \frac{(-1)^k (n-1)\cdots(n-2k+1)(n+2)\cdots(n+2k)x^{2k+1}}{(2k+1)!} \right], & n : \text{even} \\ \frac{(-1)^{\frac{n+1}{2}} \cdot 2^{n+1} \left[\left(\frac{n-1}{2} \right)! \right]^2}{n!} \cdot \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k n(n-2)\cdots(n-2k+2)(n+1)\cdots(n+2k-1)x^{2k}}{(2k)!} \right], & n : \text{odd} \end{cases}$$

$Q_n(x) = \sum_{k=0}^{\infty} \frac{2^n (n+k)!(n+2k)!}{k!(2n+2k+1)!} x^{-n-2k-1}$ if $|x| > 1$, but $Q_n(x)$ is divergent at $x = \pm 1$.



Properties of Legendre polynomials:

1. $(n+1)L_{n+1}(x) - (2n+1)xL_n(x) + nL_{n-1}(x) = 0, n > 0$

2. $L'_{n+1}(x) - (2n+1)L_n(x) - L'_{n-1}(x) = 0$

3. $(x^2 - 1)L'_n(x) - nxL_n(x) + nL_{n-1}(x) = 0$

4. Generating functions for $P_n(x)$ and $Q_n(x)$:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=-\infty}^{\infty} P_n(x)t^n, \quad \frac{1}{\sqrt{1-2xt+t^2}} \cdot \cosh^{-1}\left(\frac{t-x}{\sqrt{x^2-1}}\right) = \sum_{n=-\infty}^{\infty} Q_n(x)t^n$$

Associated Legendre equations and functions: $(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0,$

its general solution is $y(x) = c_1 P_n^m(x) + c_2 Q_n^m(x),$

where $L_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} L_n(x).$

Properties of the associated Legendre polynomials:

1. $L_{n+1}^m(x) - (2n+1)\sqrt{1-x^2}L_n^{m-1}(x) - L_{n-1}^m(x) = 0$

2. $xL_n^m(x) - L_{n-1}^m(x) + (m-n-1)\sqrt{1-x^2}L_n^{m-1}(x) = 0$

3. $\int_{-1}^1 P_m^k(x)P_n^k(x)dx = \frac{2}{2n+1} \cdot \frac{(n+k)!}{(n-k)!} \delta_{mn}$

13-4 Applications of Bessel and Legendre Functions

Electrostatic potentials in the cylindrical coordinate:

In source-free region: $\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\rho^2 \partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Set $V(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z) \Rightarrow \frac{1}{R} \left(\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial R}{\partial \rho} \right) + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0$

Set $\frac{\partial^2 Z}{\partial z^2} = k^2 Z, \quad \frac{\partial^2 \Phi}{\partial \phi^2} = -n^2 \Phi, \quad \rho = \frac{x}{k} \Rightarrow \frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0$

$\Rightarrow \Phi(\phi) = A \cos(n\phi) + B \sin(n\phi)$

$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \Rightarrow Z(z) = c_1 e^{kz} + c_2 e^{-kz}$

$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{n^2}{x^2}\right)R = 0 \Rightarrow x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + (x^2 - n^2)R = 0$ (Bessel differential

equation) $\Rightarrow R(x) = R(k\rho) = e_1 J_n(k\rho) + e_2 Y_n(k\rho)$

(If V is finite at $\rho=0 \Rightarrow e_2=0$)

Special case 1: V is independent of z , $\frac{\partial^2}{\partial z^2} = 0$

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \Rightarrow \Phi(\phi) = A\cos(n\phi) + B\sin(n\phi)$$

$$\rho^2 \frac{d^2R(\rho)}{d\rho^2} + \rho \frac{dR(\rho)}{d\rho} - n^2R(\rho) = 0 \quad (\text{Euler's equation}) \Rightarrow R(\rho) = c_1\rho^n + c_2\rho^{-n}$$

Special case 2: V is independent of φ and z

$$\frac{d}{d\rho} \left[\rho \frac{dR(\rho)}{d\rho} \right] = 0 \Rightarrow V(\rho) = R(\rho) = d_1 \ln \rho + d_2$$

Electrostatic potentials in the spherical coordinate:

In source-free region: $\nabla^2 V = \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial V}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$

Set $V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\Rightarrow \frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \cdot \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0$$

Set $\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} = \ell(\ell+1)$

$$\Rightarrow r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \ell(\ell+1)R = 0 \quad (\text{Euler's equation}) \Rightarrow R(r) = Ar^\ell + Br^{-(\ell+1)}$$

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi \Rightarrow \Phi(\phi) = C \cos(m\phi) + D \sin(m\phi)$$

Let $x = \cos \theta \Rightarrow (1-x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] \Theta = 0$ (Associated Legendre equation)

$$\Rightarrow \Theta = \Theta(x) = EP_\ell^m(x) + FQ_\ell^m(x) = EP_\ell^m(\cos \theta) + FQ_\ell^m(\cos \theta)$$

Special case: $m=0 \Rightarrow (1-x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + \ell(\ell+1)\Theta = 0$ (Legendre equation)

$$\Rightarrow \Theta = EP_\ell(\cos \theta) + FQ_\ell(\sin \theta)$$

13-5 Elliptic Integral Functions

If we set $v = \sin \theta$, $x = \sin \phi$, and $0 < k < 1$,

the 1st-kind elliptic integral: $F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^x \frac{dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}}$.

1. $\phi = \frac{\pi}{2}$, complete integral, $F\left(k, \frac{\pi}{2}\right) \equiv F(k)$. 2. $0 < \phi < \frac{\pi}{2}$, incomplete integral.

the 2nd-kind elliptic integral of: $E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^x \sqrt{\frac{1 - k^2 v^2}{1 - v^2}} dv$.

1. $\phi = \frac{\pi}{2}$, complete integral, $E\left(k, \frac{\pi}{2}\right) = E(k)$. 2. $0 < \phi < \frac{\pi}{2}$, incomplete integral.

the 3rd-kind elliptic integral of:

$\Pi(k, n, \theta) = \int_0^\theta \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} = \int_0^x \frac{dv}{(1 + nv^2) \sqrt{(1 - v^2)(1 - k^2 v^2)}}$.

1. $\phi = \frac{\pi}{2}$, complete integral, $\Pi\left(k, \frac{\pi}{2}\right) \equiv \Pi(k)$. 2. $0 < \phi < \frac{\pi}{2}$, incomplete integral.

Eg. Evaluate $\int_0^{\pi/2} \sqrt{1 + 4 \sin^2 x} dx$.

(Sol.) $\int_0^{\pi/2} \sqrt{1 + 4 \sin^2 x} dx = \int_0^{\pi/2} \sqrt{5 - 4 \cos^2 x} dx$ (set $x = \frac{\pi}{2} - \theta$)

$$= \sqrt{5} \int_0^{\pi/2} \sqrt{1 - \frac{4}{5} \sin^2 \theta} d\theta = \sqrt{5} E\left(\sqrt{\frac{4}{5}}\right)$$